PHYS 541 Handout 1

Basic Equations of Electromagnetism:

These are Maxwell's Equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

The Lorentz Force Law:

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$

and the Continuity Equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

In most undergraduate courses, these equations are gradually introduced over the course of the semester, and primacy is usually given to things like Coloumb's Law, which for two point charges is:

$$\mathbf{F_{12}} = \frac{q_1 q_2}{4\pi\epsilon_0 |\mathbf{r_1} - \mathbf{r_2}|^3}(\mathbf{r_1} - \mathbf{r_2})$$

But can be generalized to give the force on a point charge q_1 due to an arbitrary static charge distribution $\rho(\mathbf{r_2})$:

$$\mathbf{F_{12}} = q_1 \int \frac{\rho(\mathbf{r_2})(\mathbf{r_1} - \mathbf{r_2})}{4\pi\epsilon_0 |\mathbf{r_1} - \mathbf{r_2}|^3} d^3 r_2$$

and the integral in this case is the thing identified as the Electric field.

In symmetric systems, rather than doing the above integral, you learned that you could use Gauss' Law to compute the electric field. Typically this is done with the integral form of the law:

$$\int \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \int \rho dV$$

Similarly, for currents there is something called the Biot-Savart Law that gives the force on a moving charge due to an arbitrary static current distribution:

$$\mathbf{F_{12}} = q_1\mathbf{v}_1 \times \int \frac{\mu_0 \mathbf{j}(\mathbf{r_2}) \times (\mathbf{r_1} - \mathbf{r_2})}{4\pi|\mathbf{r_1} - \mathbf{r_2}|^3} d^3 r_2$$

Where again the integral is identified as the magnetic field. This formula is not often used in practice. Instead you instead learn Ampere's Law, which in integral form is:

$$\oint \mathbf{B} \cdot d\ell = \mu_0 \int \mathbf{j} \cdot d\mathbf{S}$$

Later, you may have learned about Faraday's Law, which describes how changing magnetic fields can produce circulating electric field. In integral form, this equation is:

$$\oint \mathbf{E} \cdot d\ell = \frac{\partial}{\partial t} \int \mathbf{B} \cdot d\mathbf{S}$$

In all these cases, the integral versions of the above equations appear as "tricks" for evaluating fields in specific cases, but in fact, Maxwell's equations are more general than the Coloumb and Biot-Savart laws, which only are clearly defined in cases where the charge and current distributions are static.

By contrast, the *differential* forms of Maxwell's equations are completely local and general, providing a unique description of how any charge will move in any given situation. One quick example of this is how the version of ampere's law without $\partial\mathbf{E}/\partial t$ is incomplete because it is not consistent with the continuity equation, which expresses conservation of charge.

Boundary conditions and solutions to Maxwell's Equations

A basic question is whether Maxwell's equations have *one and only one* solution. If this is not the case, then the theory described by these equations is incomplete.

It turns out that specifying $\nabla \cdot \mathbf{E}$ and $\nabla \times \mathbf{E}$ is enough to uniquely determine $\mathbf{E}$ within a volume, provided that one also specifies $\mathbf{E}$ on the surface of that volume.

To prove that this is the case, let us consider a case where both $\mathbf{E}$ and $\mathbf{E}' = \mathbf{E} + \mathbf{W}$ are solutions to Maxwell's equation in some volume. We want to show that if this is the case, then $\mathbf{W} = \mathbf{0}$ and so $\mathbf{E} = \mathbf{E}'$. Maxwell's Equations just tell us that

$$\nabla \cdot \mathbf{W} = 0$$

and

$$\nabla \times \mathbf{W} = 0$$

But this does not necessarily mean $\mathbf{W} = 0$, for example if $\mathbf{W}$ a constant vector then all derivatives are zero but the vector itself is not zero.

However, we can prove that if $\mathbf{W} = 0$ on the surface of the volume then $\mathbf{W} = 0$ throughout the volume using **Green's First Identity**, which is just a specific application of the Divergence theorem.

$$\int \nabla \cdot \mathbf{F} d^3r = \int \mathbf{F} \cdot d\mathbf{S}$$

Where we assume $\mathbf{F} = \phi \nabla \psi$ and so:

$$\int \nabla \cdot (\phi \nabla \psi) d^3r = \int \phi \nabla \psi \cdot d\mathbf{S}$$

$$\int (\nabla \phi \cdot \nabla \psi + \phi \nabla^2 \psi) d^3r = \int \phi \nabla \psi \cdot d\mathbf{S}$$

Consider the case where $\phi = \psi = W_x$, where W_x is one component of $\mathbf{W}$.

then this gives:

$$\int (\nabla W_x)^2 + W_x \nabla^2 W_x d^3r = \int W_x \nabla W_x \cdot d\mathbf{S}$$

Now note that $W_x = 0$ everywhere on the surface by construction:

$$\int (\nabla W_x)^2 + W_x \nabla^2 W_x d^3r = 0$$

Next, we can note that since $\nabla \times \mathbf{W} = 0$ we can use:

$$0 = \nabla \times (\nabla \times \mathbf{W}) = \nabla(\nabla \cdot \mathbf{W}) - \nabla^2 \mathbf{W}$$

This means that $\nabla^2 W_x = 0$, and so we are left with:

$$\int (\nabla W_x)^2 d^3r = 0$$

This means that $\nabla W_x = 0$ throughout the volume, which means that W_x must be a constant in a volume, and since $W_x = 0$ on the surface, it must be zero everywhere in the volume.

This shows that if a solution exists, it is unique, but now we need to show that Maxwell's equations always have a solution, which we can do using Helmholtz decomposition that any field can be written as follows:

$$\mathbf{X} = -\nabla Y + \nabla \times \mathbf{Z}$$

where Y and $\mathbf{Z}$ are functions that can be set separately, as defined by Maxwell's Equation.

To prove this, we can re-write $\mathbf{X}$ as follows:

$$\mathbf{X}(\mathbf{r}) = \int \mathbf{X}(\mathbf{r}') \delta^3(\mathbf{r} - \mathbf{r}') d^3r'$$

And then we note that the delta function can be written in the following form:

$$\delta^3(\mathbf{r}-\mathbf{r}') = \frac{1}{4\pi}\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

which means:

$$\mathbf{X}(\mathbf{r}) = \nabla^2 \int \frac{\mathbf{X}(\mathbf{r}')}{4\pi(\mathbf{r}-\mathbf{r}')} d^3r'$$

and now using the vector identify $\nabla^2\mathbf{a} = \nabla(\nabla\cdot\mathbf{a}) - \nabla\times(\nabla\times\mathbf{a})$ this means:

$$\mathbf{X}(\mathbf{r}) = \nabla\left[\int \nabla\cdot\frac{\mathbf{X}(\mathbf{r}')}{4\pi(\mathbf{r}-\mathbf{r}')} d^3r'\right] + -\nabla\times\left[\int \nabla\times\frac{\mathbf{X}(\mathbf{r}')}{4\pi(\mathbf{r}-\mathbf{r}')} d^3r'\right]$$

This shows that we can write the field in the desired form, but now we need to show that the two integrals are separate functions:

To do this, we use the following useful identify:

$$\nabla\cdot\frac{\mathbf{X}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} = \mathbf{X}(\mathbf{r}')\cdot\nabla\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} = -\mathbf{X}(\mathbf{r}')\cdot\nabla'\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

and since:

$$\nabla'\cdot\frac{\mathbf{X}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} = \frac{\nabla'\cdot\mathbf{X}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} + \mathbf{X}(\mathbf{r}')\cdot\nabla'\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

this means:

$$\nabla\cdot\frac{\mathbf{X}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} = \frac{\nabla'\cdot\mathbf{X}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} - \nabla'\cdot\frac{\mathbf{X}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

A similar rule holds for curls, allowing us to re-write the field in the following form:

$$\begin{aligned}\mathbf{X}(\mathbf{r}) = -\nabla&\left[\int \frac{\nabla'\cdot\mathbf{X}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}d^3r' - \int\nabla'\cdot\frac{\mathbf{X}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}d^3r'\right]\\ +\nabla\times&\left[\int \frac{\nabla'\times\mathbf{X}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}d^3r' - \int\nabla'\times\frac{\mathbf{X}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}d^3r'\right]\end{aligned}$$

The last two terms can be re-written as follows:

$$\int\nabla'\cdot\frac{\mathbf{X}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}d^3r' = \int\frac{\mathbf{X}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}\cdot d\mathbf{S}'$$

$$\int\nabla'\times\frac{\mathbf{X}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}d^3r' = -\int\frac{\mathbf{X}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}\times d\mathbf{S}'$$

Note the first depends only on the components of $\mathbf{X}$ perpendicular to the boundary of the volume, while the second term only depends on the term parallel to the boundary, so these functions are indeed separate things. At the same time if we say $\mathbf{X} = \mathbf{X_1} + \mathbf{X_2}$ such that $\nabla\cdot\mathbf{X_2} = 0$ and $\nabla\times\mathbf{X_1} = 0$ then only X_1 contributes to the first integral and only $\mathbf{X_2}$ to the second.

Boundary Conditions from Maxwell's Equations

For each of Maxwell's equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

there is a corresponding boundary condition:

$$\mathbf{n}_2 \cdot (\mathbf{E_1} - \mathbf{E_2}) = \sigma/\epsilon_0$$

$$\mathbf{n}_2 \times (\mathbf{E_1} - \mathbf{E_2}) = 0$$

$$\mathbf{n}_2 \cdot (\mathbf{B_1} - \mathbf{B_2}) = 0$$

$$\mathbf{n}_2 \times (\mathbf{B_1} - \mathbf{B_2}) = \mu_0 \mathbf{K}$$

Each of these is basically a special case of Maxwell's Equations for a surface. For flat surfaces, this can be demonstrated with simple pillboxes, etc. However, it can be shown more generally using the differential equations. Say we have a surface carrying a current $\mathbf{K}$ and a surface charge σ. If we place the local normal of the surface at the $z = 0$ plane, then we can say the fields are:

$$\mathbf{B} = \mathbf{B_1}\Theta(z) + \mathbf{B_2}\Theta(-z)$$

$$\mathbf{E} = \mathbf{E_1}\Theta(z) + \mathbf{E_2}\Theta(-z)$$

and the current is and the charge distribution is:

$$\mathbf{j} = \mathbf{j_1}\Theta(z) + \mathbf{j_2}\Theta(-z) + \mathbf{K}\delta(z)$$

$$\rho = \rho_1\Theta(z) + \rho_2\Theta(-z) + \sigma\delta(z)$$

Now we can note:

$$\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{E_1}\Theta(z) + \nabla \cdot \mathbf{E_2}\Theta(-z) + \mathbf{E_1} \cdot \nabla\Theta(z) + \mathbf{E_2} \cdot \nabla\Theta(-z)$$

and since $\frac{\partial}{\partial z}\Theta(z) = \delta(z)$ we get

$$\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{E_1}\Theta(z) + \nabla \cdot \mathbf{E_2}\Theta(-z) + \hat{\mathbf{z}} \cdot (\mathbf{E_1} - \mathbf{E_2})\delta(z)$$

in which case if we use the equation:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

and identify each term, we find:

$$\nabla \cdot \mathbf{E_1} = \frac{\rho_1}{\epsilon_0}$$

$$\nabla \cdot \mathbf{E_2} = \frac{\rho_2}{\epsilon_0}$$

$$\mathbf{z} \cdot (\mathbf{E_1} - \mathbf{E_2}) = \frac{\sigma}{\epsilon_\mathbf{0}}$$

and the last gives the relevant boundary condition.