PHYS 542 Handout 1

Basic Equations of Electromagnetism:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$
$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

To start off this semester, we will consider Quasi-Electrostatic and Quasi-Magnetostatic systems:

Quasi-electrostatic systems have $\partial \mathbf{B}/\partial t \simeq 0$ (this means the equations of electrostatics hold)

Quasi-magnetostatic systems have $\partial \mathbf{E}/\partial t \simeq 0$ (this means the equations of magnetostatics hold)

Example of truly Quasi-electrostatic system:

Magnetic field must be constant, this means that electric field can only change linearly in time. Thus currents must be constant, but charges can change linearly.

Consider a long wire along the z-axis that ends at $z = 0$.

In this case, the current is constant, but the charge at the end of the wire $Q = It$

Assume the magnetic field is constant, then using the laws of Electrostatics, we can say the electric field is (Equation 14.10):

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0}\frac{It}{r^2}\hat{\mathbf{r}}$$

This changing electric field produces a displacement current:

$$\frac{1}{\mu_0 c^2}\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{4\pi}\frac{I}{r^2}\hat{\mathbf{r}}$$

Note this current flows in all directions away from the charge.

We can then find the magnetic field using Ampere's law in integral form. Since this system is still azimuthally symmetric, we can reasonably assume that the field points in the ϕ direction, and so we can consider azimuth loops with radii ρ and location z. The explicit

integral is done in the book to yield Equation 14.13. The critical thing to note is that the displacement current is purely radial, as is the real current, so rather than using a flat surface bounded by the loop, it is simpler to use a spherical surface (although both will give the same answer). For a given value of ρ and z, the corresponding spherical coordinate is given by $\cos\theta = fraczz^2 + \rho^2$, and the total fraction of the displacement current captured by the cap is

$$\int_\theta^\pi d\cos\theta = 1 - \cos\theta$$

, and since the real current only flows at $\theta = -\pi$, it never contributes, and so t the magnetic field is (Equation 14.13):

$$\mathbf{B} = \frac{\mu_0 I}{4\pi\rho}\left(1 - \frac{z}{\sqrt{z^2+\rho^2}}\right)\hat{\phi}$$

Example of truly Quasi-Magnetostatic System: Electric field must be constant, this means that magnetic field can only change linearly in time. Thus charges must be constant, but currents can change linearly.

Consider a solenoid of radius R with a steadily increasing surface current density $\mathbf{K} = \mathbf{K_0}(t/T)$.

Assume the electric field is constant, then at any given time the magnetic field is just that for a normal solenoid:

$$\mathbf{B} = \mu_0 K_0 \frac{t}{T}\hat{\mathbf{z}}$$

The electric field in this case must now satisfy

$$\nabla\cdot\mathbf{E} = 0, \nabla\times\mathbf{E} = -\mu_0 K_0/T\hat{\mathbf{z}}$$

, so we get constant circulating electric fields:

$$\mathbf{E} = -\tfrac{\mu_0}{2T}K_0\rho\hat{\phi} \text{ if } \rho < R$$
$$\mathbf{E} = -\tfrac{\mu_0}{2T}K_0\tfrac{R^2}{\rho}\hat{\phi} \text{ if } \rho > R$$

Approximately Electrostatic Systems

"Perfect" Quasi-static systems are rare, but in other cases, we can approximate the system as quasi-electrostatic or quasi-magnetostatic.

Quasi-electrostatic means $\partial\mathbf{B}/\partial\mathbf{t}$ is approximately zero.

From looking at Maxwell's Equations, this basically means $|\partial\mathbf{B}/\partial\mathbf{t}| << \rho/\epsilon_0$. However, in practice it is always a good idea to check whether this is true.

In a quasi-electrostatic case, we can use the standard solution for electrostatics (14.54):

$$\mathbf{E} = -\nabla\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}d^3r'$$

and it turns out the rules of magnetostatics work as well (14.55). You can show that this works using the same basic logic as in the first homework from PHYS541, just now $\nabla \cdot \mathbf{j} = d\rho/dt$ instead of zero.

$$\mathbf{B} = \nabla \times \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d^3 r'$$

Example of an approximately Quasi-Electrostatic System

Consider a single charge q moving at a constant velocity $\mathbf{v}$, in which case the charge distribution is:

$$\rho = q\delta(\mathbf{r} - \mathbf{v}t)$$

and the current distribution is:

$$j = q\mathbf{v}\delta(\mathbf{r} - \mathbf{v}t)$$

In this case, the quasi-static electric field is:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{v}t}{|\mathbf{r} - \mathbf{v}t|^3}$$

and the quasi-static magnetic field is:

$$\mathbf{B} = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E})$$

Note that this field is not completely time-invariant, but you can show:

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{q}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{v}t|^2} \frac{(3\mathbf{v} \times \mathbf{r})(\mathbf{r} \cdot \mathbf{v} - v^2 t)}{c^2 |\mathbf{r} - \mathbf{v}t|^3}$$

Which means that the electric field induced by this changing magnetic field is of order $(v/c)^2$ times less than the quasi-static field. Thus, so long as long as $v << c$, the quasi-static assumption is reasonable.

Approximately Magnetostatic Systems

If we can assume that $\partial \mathbf{E}/\partial t$ is approximately zero, then the equations of magnetostatics hold, and so the Magnetic field is given by the standard expression (14.68):

$$\mathbf{B} = \nabla \times \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} d^3 r'$$

If we further stipulate the charge density $\rho = 0$, then it turns out that the electric field is given by the expression (14.69):

$$\mathbf{E} = -\frac{\mu_0}{4\pi} \int \frac{\partial \mathbf{j}(\mathbf{r}', t)/\partial t}{|\mathbf{r} - \mathbf{r}'|} d^3 r'$$

We can verify this is the case by computing the curl and gradient of this expression, and showing that with the above expression:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and

$$\nabla \cdot \mathbf{E} = 0$$

(The latter uses the continuity equation and the fact that current does not flow away to infinity).

Example of approximately Quasi-Magnetostatic System

Consider a Solenoid with radius R carrying an oscillation surface current $\mathbf{K} = K_0 \hat{\phi} \cos \omega t$.

In the Quasi-Magnetostatic limit, the magnetic field is

$$\mathbf{B} = \mu_0 K \cos(\omega t) \hat{\mathbf{z}}$$

inside the solenoid and zero outside.

Now note that in this case the electric field is given by

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = \mu_0 K \omega \sin \omega t \hat{\mathbf{z}}$$

Thus inside the solenoid the electric field is:

$$\mathbf{E} = \frac{1}{2} \mu_0 K \rho \omega (\sin \omega t) \hat{\phi}$$

and outside the solenoid it is:

$$\mathbf{E} = \frac{1}{2} \mu_0 K \frac{R^2}{\rho} \omega (\sin \omega t) \hat{\phi}$$

This Electric field is not strictly constant in time, and so this electric field produces displacement currents that generate additional magnetic fields. Inside the solenoid this extra part of the magnetic field is

$$\mathbf{B}' = \frac{1}{2}\mu_0 K \frac{\omega^2 \rho^2}{c^2} \cos(\omega t)\hat{\mathbf{z}}$$

which can be ignored so long as $\omega << c/R$.

However, outside the solenoid, this excess magnetic field is

$$\mathbf{B}' = \frac{1}{2}\mu_0 K \frac{\omega^2 \rho^2}{c^2} \cos(\omega t)[1/2 + \ln(\rho/R)]\hat{\mathbf{z}}$$

So if $\rho >> R$ we cannot assume this term is small for any wavelength, this helps define the radiation zone.