PHYS 542 Handout 13

Moment expansion of radiation

Just like the static electric and magnetic fields of a compact distribution of charges and currents can be expressed as the sum of a series of moments. The radiation from a time-variable compact distribution of charges and currents can also be expressed as a series of moments.

To first order, a distribution of charges and currents can produce **electric dipole radiation**, so named because the integral over the current in a system can be re-expressed as the time derivative of the electric dipole moment $\mathbf{p}$

$$\int \mathbf{j} d^3 r_1 = \sum q \mathbf{v}_n = \frac{d}{dt} \sum q \mathbf{r}_n = \frac{d\mathbf{p}}{dt}$$

Hence the vector potential in the radiation limit is proportional to $\dot{\mathbf{p}}$:

$$\mathbf{A}(\mathbf{r_2}, t) = \frac{\mu_0}{4\pi r} \int \mathbf{j}(\mathbf{r_1}, t - r_2/c) d^3) r_1 = \frac{\mu_0}{4\pi r} \dot{\mathbf{p}}(t - r_2/c)$$

This means that the magnetic and electric fields are proportional to the second time derivative of the electric dipole moment.

$$\mathbf{B} = -\frac{\mu_0}{4\pi c} \frac{\hat{\mathbf{r}} \times \ddot{\mathbf{p}}}{r}$$

$$\mathbf{E} = -\frac{\mu_0}{4\pi} \frac{\hat{r}(\hat{\mathbf{r}} \cdot \ddot{\mathbf{p}}) - \ddot{\mathbf{p}}}{r}$$

Note that if $\mathbf{p} \propto \mathbf{p_0} e^{i\omega t}$ then the electric field is polarized parallel to the dipole moment. Also note that the power radiated by a current system at this level of approximation is:

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c} |\hat{\mathbf{r}} \times \ddot{\mathbf{p}}|^2$$

However, even if the system never has a net electric dipole moment, it can still emit radiation. For example, imagine a small current loop where the magnitude of the current oscillates. This does not have an electric dipole moment, but does have a time-variable magnetic dipole moment and so can emit **magnetic dipole radiation**. This radiation comes from recognizing that the retarded time is strictly speaking a function of $\mathbf{r_1}$ and so:

$$\int \mathbf{j}(\mathbf{r_1}, t - |\mathbf{r_2} - \mathbf{r_1}|/c) d^3 r_1 \simeq \int \mathbf{j}(\mathbf{r_1}, t - r_2/c + \hat{\mathbf{r}}_2 \cdot \mathbf{r_1}/c) d^3 r_1$$

Treating the last term as a small change to the retarded time, we can use the standard taylor expansion: $y(x + \delta x) = y(x) + (\partial y/\partial x)\delta x$ to say:

$$\int \mathbf{j}(\mathbf{r_1}, t - |\mathbf{r_2} - \mathbf{r_1}|/c) d^3 r_1 \simeq \int \left[\mathbf{j}(\mathbf{r_1}, t - r_2/c) + \dot{\mathbf{j}}(\mathbf{r_1}, t - r_2/c) \hat{\mathbf{r}}_2 \cdot \mathbf{r_1}/c \right] d^3 r_1$$

Next, we can split the second term into two halves, and then use the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ to re-write everything as:

$$\int \mathbf{j}(\mathbf{r_1}, t - |\mathbf{r_2} - \mathbf{r_1}|/c) d^3 r_1 \simeq \int \left[\mathbf{j}(\mathbf{r_1}, t - r_2/c) - \frac{1}{2c} \hat{\mathbf{r}}_2 \times (\mathbf{r_1} \times \dot{\mathbf{j}}) + \frac{1}{2c} \left(\dot{\mathbf{j}} \hat{\mathbf{r}}_2 \cdot \mathbf{r_1} + (\dot{\mathbf{j}} \cdot \hat{\mathbf{r}}_2) \mathbf{r_1} \right) \right] d^3 r_1$$

remember, the integral of the current distribution is the time derivative of the electric dipole moment, so the first term gives the electric dipole radiation.The last term turns out to correspond to electric quadrupole radiation, and the middle term is related to the magnetic dipole moment of the source, which can be seen as follows:

$$\int \frac{1}{2c} \hat{\mathbf{r}}_2 \times (\mathbf{r_1} \times \dot{\mathbf{j}}) d^3 r_1 = \frac{\hat{\mathbf{r}}_2}{c} \times \frac{\partial}{\partial t} \int \frac{1}{2} (\mathbf{r_1} \times \mathbf{j}) d^3 r_1$$

The integral is now explicitly the magnetic dipole moment $\mathbf{m}$, so we have:

$$\int \frac{1}{2c} \hat{\mathbf{r}}_2 \times (\mathbf{r_1} \times \dot{\mathbf{j}}) d^3 r_1 = \frac{\hat{\mathbf{r}}_2}{c} \times \dot{\mathbf{m}}$$

Thus the full inegral of the current distribution not only contains a piece that is proportional to the time derivative of the electric dipole moment, but also the time derivative of the magnetic dipole moment.

In this case, the vector potential is

$$\mathbf{A}(\mathbf{r_2}, t) = -\frac{\mu_0}{4\pi r c} \hat{\mathbf{r}}_2 \times \dot{\mathbf{m}}(t - r_2/c)$$

This meanst the magnetic and electric fields are proportional to the second time derivative of the magnetic dipole moment.

$$\mathbf{E} = \frac{\mu_0}{4\pi c} \frac{\hat{\mathbf{r}} \times \ddot{\mathbf{m}}}{r}$$

$$\mathbf{B} = -\frac{\mu_0}{4\pi c^2} \frac{\hat{r}(\hat{\mathbf{r}} \cdot \ddot{\mathbf{m}}) - \ddot{\mathbf{m}}}{r}$$

Note that if $\mathbf{m} \propto \mathbf{m_0} e^{i\omega t}$ then the magnetic field is polarized parallel to the dipole moment. Also note that the power radiated by a current system at this level of approximation is:

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c^3} |\hat{\mathbf{r}} \times \ddot{\mathbf{m}}|^2$$

Interference in arrays of electric dipoles

Say we have an array of N electric dipoles. Each one has the moment $\mathbf{p} = p_0\hat{\mathbf{z}}e^{i\omega t}$, and they are all evenly spaced by distances d along the x axis, how would the radiation of this collection of dipoles differ from that of a single dipole?

The vector potential of the nth dipole is:

$$\mathbf{A}_n = \frac{\mu_0}{4\pi r_n}\dot{\mathbf{p}}(t - r_n/c) = \frac{\mu_0 p_0}{4\pi}\hat{\mathbf{z}}e^{i\omega t}\frac{e^{i\omega r_n/c}}{r_n}$$

where r_n is the distance from that dipole, and the total vector potential is:

$$\mathbf{A} = \sum \mathbf{A}_n = \frac{\mu_0 p_0}{4\pi}\hat{\mathbf{z}}e^{i\omega t}\sum_n \frac{e^{-i\omega r_n/c}}{r_n}$$

Now we can note that $r_n = |\mathbf{r} - nd\hat{\mathbf{x}}|$. Provided that$r >> nd$, we can expand this ad say that $r_n \simeq r - nd(x/r) = r - nd\cos\phi$. So if we say that $1/r_n \simeq 1/r$ and expand the exponentials to first order, we get:

$$\mathbf{A} = \sum \mathbf{A}_n = \frac{\mu_0 p_0}{4\pi r}\hat{\mathbf{z}}e^{i\omega(t-r/c)}\sum_{n=1}^{N} e^{-i\omega\cos\phi nd/c}$$

This prefactor is just the potential of a single dipole, so we have:

$$\mathbf{A} = \sum \mathbf{A}_n = \mathbf{A}_1\sum_{n=1}^{N} e^{-i\omega\cos\phi nd/c}$$

and then we can use the identity $\sum r^n = (1 - r^N)/(1 - r)$, to re-write this as:

$$\mathbf{A} = \sum \mathbf{A}_n = \mathbf{A}_1\frac{1 - e^{-iN\omega\cos\phi d/c}}{1 - e^{-i\omega\cos\phi d/c}}$$

which can be rewritten as:

$$\mathbf{A} = \sum \mathbf{A}_n = \mathbf{A}_1 e^i\Phi\frac{\sin(N\omega(\cos\phi)d/2c)}{\sin(\omega(\cos\phi)d/2c)}$$

where Φ is just a phase shift to the wave.

The electric fields of the array are therefore just the electric field of the single dipole multiplied by the factor:

$$\frac{\sin(N\omega(\cos\phi)d/2c)}{\sin(\omega(\cos\phi)d/2c)}$$

Note that along the y-axis, where $\cos\phi \to 0$, this factor goes to N. but it rapidly drops to around 1 along the x-axis. This gives a narrower beam perpendicular to the array.