PHYS 542 Handout 14

Basic Scattering Theory

If you have a small object that scatters an incident electromagnetic wave, this is often quantified using a differential cross section, which is the ratio of the power radiated in one direction divided by the incident flux, which has units of area.

$$\frac{d\sigma}{d\Omega} = \frac{r^2 \hat{\mathbf{r}} \cdot \langle \mathbf{S}_{rad} \rangle}{|\langle \mathbf{S}_{inc} \rangle|} = \frac{\langle dP/d\Omega \rangle}{0.5\epsilon_0 c E_{inc}^2} = r^2 \frac{E_{rad}^2}{E_{inc}^2}$$

Physically, the incident Electric field drives currents in object, which then radiates away. If the incident electric field is time-harmonic, then the currents should also be time-harmonic, so:

$$\mathbf{j}(\mathbf{r_1}, t) = \mathbf{j}(\mathbf{r}_1, \omega) e^{-i\omega t}$$

In this case, the vector potential is:

$$\mathbf{A}(\mathbf{r_2}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r_1}, t - |\mathbf{r_1} - \mathbf{r_2}|/c)}{|\mathbf{r_1} - \mathbf{r_2}|} d^3 r_1$$

which in the appropriate radiation limit is:

$$\mathbf{A}(\mathbf{r_2}, t) = \frac{\mu_0}{4\pi r_2} \int \mathbf{j}(\mathbf{r_1}, t - r_2/c + \hat{\mathbf{r}}_2 \cdot \mathbf{r_1}/c) d^3 r_1$$

$$\mathbf{A}(\mathbf{r_2}, t) = \frac{\mu_0}{4\pi r_2} e^{-i\omega(t - r_2/c)} \int \mathbf{j}(\mathbf{r_1}, \omega) e^{-i\omega \hat{\mathbf{r}}_2 \cdot \mathbf{r_1}/c} d^3 r_1$$

and so the Electric field is:

$$\mathbf{E} = \frac{i\omega\mu_0}{4\pi} \frac{e^{-i\omega(t - r_2/c)}}{r_2} \left[\hat{\mathbf{r}}_2 \times \left(\hat{\mathbf{r}}_2 \times \int \mathbf{j}(\mathbf{r_1}, \omega) e^{-i\omega \hat{\mathbf{r}}_2 \cdot \mathbf{r_1}/c} d^3 r_1 \right) \right]$$

Here is useful to consider the scattered radiation's wavevector, for which $\hat{\mathbf{r}}_2 = \hat{\mathbf{k}}$ and $\hat{\mathbf{r}}_2(\omega/c) = \mathbf{k}$

$$\mathbf{E} = \frac{ik}{4\pi\epsilon_0 c} \frac{e^{-i\omega(t - r_2/c)}}{r_2} \left[\hat{\mathbf{k}} \times \left(\hat{\mathbf{k}} \times \int \mathbf{j}(\mathbf{r_1}, \omega) e^{-i\mathbf{k} \cdot \mathbf{r_1}} d^3 r_1 \right) \right]$$

and so the square of the electric field is:

$$E^2 = \frac{1}{r^2} \frac{k^2}{(4\pi\epsilon_0 c)^2} \left| \hat{\mathbf{k}} \times \int \mathbf{j}(\mathbf{r_1}, \omega) e^{-i\mathbf{k} \cdot \mathbf{r_1}} d^3 r_1 \right|^2$$

and so the differential cross section is:

$$\frac{d\sigma}{d\Omega} = \frac{k^2}{(4\pi\epsilon_0 c E_{inc})^2}\left|\hat{\mathbf{k}} \times \int \mathbf{j}(\mathbf{r_1},\omega)e^{-i\mathbf{k}\cdot\mathbf{r_1}}d^3r_1\right|^2$$

Thomson Scattering

Consider a single free charge particle, and say that it is being illuminated by the incident flied: $\mathbf{E}_{inc} = E_0\hat{\mathbf{y}}e^{i(kz-\omega t)}$, this applies a time-harmonic force $q\mathbf{E}$ on the particle, and so the acceleration is $\mathbf{a} = (q/m)\mathbf{E}$, which can be written in the form of the time variation of the current:

$$\frac{\partial \mathbf{j}}{\partial t} = q\mathbf{a} = \frac{q^2}{m}\mathbf{E}_{inc}$$

which means that

$$\mathbf{j} = -i\frac{q^2 E_0}{m\omega}\hat{\mathbf{y}}e^{i(kz-\omega t)}\delta(\mathbf{r}-\mathbf{r_0})$$

if we place the charge at the origin, then this gives:

$$\int \mathbf{j}(\mathbf{r_1},\omega)e^{i\mathbf{k}\cdot\mathbf{r_1}}d^3r_1 = -i\frac{q^2 E_0}{m\omega}\hat{\mathbf{y}}$$

which means that

$$\frac{d\sigma}{d\Omega} = \frac{k^2}{(4\pi\epsilon_0 c E_{inc})^2}\left|\hat{\mathbf{k}} \times \frac{q^2 E_0}{m\omega}\hat{\mathbf{y}}\right|^2 = \left(\frac{q^2}{4\pi\epsilon_0 mc^2}\right)^2 |\hat{\mathbf{k}} \times \hat{\mathbf{y}}|^2$$

The prefactor corresponds to the classical electron radius r_e and so:

$$\frac{d\sigma}{d\Omega} = r_e^2[1-(\hat{\mathbf{k}}\cdot\hat{\mathbf{y}})^2]$$

integrating over all angles gives the cross-section

$$\sigma = \frac{8}{3}\pi r_e^2$$

Note that this frequency independent.

Rayleigh Scattering

Another simple example of scattering occurs with you have an object that can be polarized by an incident electric field, and so we can say the electric dipole moment $\mathbf{p} = \epsilon_0 \alpha \mathbf{E}_{inc}$. If the incident electric field is $\mathbf{E}_{inc} = E_0 \hat{\mathbf{y}} e^{i(kz-\omega t)}$ then:

$$\mathbf{p} = \epsilon_0 \alpha E_0 \hat{\mathbf{y}} e^{i(kz-\omega t)}$$

and the radiation from the dipole is:

$$\mathbf{E}_{rad} = \frac{\mu_0}{4\pi r} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \ddot{\mathbf{p}})$$

$$\mathbf{E}_{rad} = \frac{\alpha k^2}{4\pi} E_0 [\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \hat{\mathbf{y}})] \frac{e^{i(kz-\omega t)}}{r}$$

and so the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 k^4}{16\pi^2} |\hat{\mathbf{k}} \times \hat{\mathbf{y}}|^2$$

Note that the cross section scales like $1/\lambda^4$ in this case.

The Born Approximation

In the above calculation, we actually made use of something, called the Born approximation. Note that for a generic linear material we can say that $\mathbf{j} = \sigma \mathbf{E}$ or $\mathbf{P} = \epsilon_0 \chi \mathbf{E}$. Remember that for time-harmonic fields the susceptability χ and the conductivity σ are related to each other by the expression $\sigma = i\epsilon_0 \chi \omega$, so we can say $\mathbf{j} = i\epsilon_0 \chi \omega \mathbf{E}$. Note that the electric field in this relation is the **total** electric field, and so includes both the incident and radiated fields. However, in the **Born Approximation**, we assume that the incident field is always much stronger than the radiated field, and so we can say:

$$\mathbf{j} \simeq i\epsilon_0 \chi \omega \mathbf{E}_{inc} = i\epsilon_0 \chi \omega E_{inc} \hat{\mathbf{e}}_0 e^{i(kz-\omega t)}$$

(where $\hat{\mathbf{e}}_0$ is the generic polarization vector).

Substituting this into the expression for the cross section yields the following expression:

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2} \left| \hat{\mathbf{k}} \times \hat{\mathbf{e}}_0 \right|^2 \left| \int \chi(\mathbf{r_1}, \omega) e^{ikz} e^{-i\mathbf{k}\cdot\mathbf{r_1}} d^3 r_1 \right|^2$$

or, defining the incident wavevector as $\mathbf{k_0} = k\hat{\mathbf{z}}$:

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2} \left| \hat{\mathbf{k}} \times \hat{\mathbf{e}}_0 \right|^2 \left| \int \chi(\mathbf{r_1}, \omega) e^{i(\mathbf{k_0}-\mathbf{k})\cdot\mathbf{r_1}} d^3 r_1 \right|^2$$

Note that for a small scatterer, where $(\mathbf{k_0} - \mathbf{k}) \cdot \mathbf{r} << 1$ everywhere that $\chi \neq 0$, we get:

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2} \left| \hat{\mathbf{k}} \times \hat{\mathbf{e}}_0 \right|^2 \left| \int \chi(\mathbf{r_1}, \omega) d^3 r_1 \right|^2$$

And if χ is constant over a volume V, then:

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{16\pi^2}\left|\hat{\mathbf{k}}\times\hat{\mathbf{e}}_0\right|^2\chi^2V^2$$

which is the Rayleigh limit again!

More, generally, let us consider a sphere of finite radius a with constant χ, then the cross section becomes:

$$\frac{d\sigma}{d\Omega} = \frac{k^4\chi^2}{16\pi^2}\left|\hat{\mathbf{k}}\times\hat{\mathbf{e}}_0\right|^2\left|\int_V e^{i(\mathbf{k_0}-\mathbf{k})\cdot\mathbf{r_1}}d^3r_1\right|^2$$

Let us define f_1 as the angle between the vectors $\mathbf{r_1}$ and $\mathbf{k_0}-\mathbf{k}$, then we can say $\mathbf{k_0}-\mathbf{k})\cdot\mathbf{r_1} = |\mathbf{k_0}-\mathbf{k}|r_1\cos f_1$, and we can do the volume integral in terms of the coordinates r_1, f_1 and ϕ_1

$$\frac{d\sigma}{d\Omega} = \frac{k^4\chi^2}{16\pi^2}\left|\hat{\mathbf{k}}\times\hat{\mathbf{e}}_0\right|^2\left|\int_0^a\int_{-1}^{-1}\int_0^{2\pi} e^{i|\mathbf{k_0}-\mathbf{k}|r_1\cos f_1}r_1^2dr_1d\cos f_1d\phi_1\right|^2$$

The ϕ_1 integral just gives 2π, which can be pulled out of the square:

$$\frac{d\sigma}{d\Omega} = \frac{k^4\chi^2}{4}\left|\hat{\mathbf{k}}\times\hat{\mathbf{e}}_0\right|^2\left|\int_0^a\int_{-1}^{-1} e^{i|\mathbf{k_0}-\mathbf{k}|r_1\cos f_1}r_1^2dr_1d\cos f_1\right|^2$$

The integral over $\cos f_1$ can then be done:

$$\frac{d\sigma}{d\Omega} = \frac{k^4\chi^2}{4}\left|\hat{\mathbf{k}}\times\hat{\mathbf{e}}_0\right|^2\left|\int_0^a\frac{1}{i|\mathbf{k_0}-\mathbf{k}|r_1}\left[e^{i|\mathbf{k_0}-\mathbf{k}|r_1}-e^{-i|\mathbf{k_0}-\mathbf{k}|r_1}\right]r_1^2dr_1\right|^2$$

which can be re-written as:

$$\frac{d\sigma}{d\Omega} = \frac{k^4\chi^2}{|\mathbf{k_0}-\mathbf{k}|^2}\left|\hat{\mathbf{k}}\times\hat{\mathbf{e}}_0\right|^2\left|\int_0^a\sin(|\mathbf{k_0}-\mathbf{k}|r_1)r_1dr_1\right|^2$$

The last integral turns out to give the first Bessel function:

$$\frac{d\sigma}{d\Omega} = \frac{k^4\chi^2}{|\mathbf{k_0}-\mathbf{k}|^2}\left|\hat{\mathbf{k}}\times\hat{\mathbf{e}}_0\right|^2|a^2J_1(|\mathbf{k_0}-\mathbf{k}|a)|^2$$

Finally, we can note that $|\mathbf{k_0}-\mathbf{k}|^2 = 2k^2-2k^2\cos\theta = 4k^2\sin^2\theta/2$, where θ is the angle between the incident and scattered wavevectors, so this expression can be written as:

$$\frac{d\sigma}{d\Omega} = \frac{k^2\chi^2a^4}{4\sin^2(\theta/2)}J_1^2[2ka\sin(\theta/2)]\left|\hat{\mathbf{k}}\times\hat{\mathbf{e}}_0\right|^2$$

This expression again gives $d\sigma/d\Omega \propto k^4$ where $ka << 1$, but more generally, $J_1(x)/x$ has its first null when $x = 3.83$, which means the forward-scattered light has a null where $2ka\sin(\theta/2) = 3.83$, which corresponds to $\theta \simeq 1.22\lambda/d$, the classic diffraction result.