PHYS 542 Handout 15

The Optical Theorem and Cross Sections

Whenever radiation hits an obstruction, some radiation is scattered in different directions and some is absorbed into the obstruction. By conservation of energy, both scattering and absorption must decrease the intensity of the incident radiation. In general, the electromagnetic fields in these situations can be written in the following form:

$$\mathbf{E}_{tot} = \mathbf{E}_{inc} + \mathbf{E}_{scat} = E_{inc}\hat{\mathbf{x}}e^{i(kz-\omega t)} + E_{inc}f(\theta,\phi)\hat{\mathbf{e}}(\theta,\phi)\frac{e^{i(kr-\omega t)}}{r}$$

where f is called the scattering amplitude. Note that f and $\hat{\mathbf{e}}$ can be combined into a vector scattering amplitude, if that is useful.

In this language, the only way for the incident radiation to be attenuated is via interference with the scattered radiation, since only radiation scattered at $\theta = 0$ can interfere with the incident radiation, the total amount of energy scattered or absorbed by the obstruction must be related to the forward scattering amplitude $f(\theta = 0)$. Quantitatively, this relationship is known as the optical theorem:

$$\sigma_{tot} = \frac{4\pi}{k}\Im[\hat{\mathbf{x}} \cdot \hat{\mathbf{e}}f(\theta = 0)]$$

To derive this relationship, we first must recognize that the total cross section has two parts, one due to absorption and one due to scattering:

$$\sigma_{tot} = \sigma_{abs} + \sigma_{scat}$$

The total scattering cross section is just:

$$\sigma_{scat} = \frac{r^2}{\langle|S_{inc}|\rangle}\int \langle\hat{\mathbf{r}} \cdot \mathbf{S}_{scat}\rangle d\Omega$$

However, the absorption cross section is determined by how quickly energy is dissipated by ohmic losses, which is given by $\mathbf{j} \cdot \mathbf{E}$. Hence the absorption cross section can be defined as:

$$\sigma_{abs} = \frac{1}{\langle|S_{inc}|\rangle}\int \langle\mathbf{j} \cdot \mathbf{E_{tot}}\rangle d^3r$$

Recall that energy conservation requires that the following relation holds at every point:

$$\frac{\partial u_{EM}}{\partial t} = -\mathbf{j} \cdot \mathbf{E} - \nabla \cdot \mathbf{S}$$

Since we have a steady-state system, this means that $\langle\mathbf{j} \cdot \mathbf{E_{tot}}\rangle = -\langle\nabla \cdot \mathbf{S}_{tot}\rangle$, and so we can re-write the absorption cross section as:

$$\sigma_{abs} = -\frac{r^2}{\langle|S_{inc}|\rangle}\int \langle\hat{\mathbf{r}} \cdot \mathbf{S}_{tot}\rangle d\Omega$$

and so the total cross section is:

$$\sigma_{tot} = -\frac{r^2}{\langle |S_{inc}| \rangle} \int \langle \hat{\mathbf{r}} \cdot (\mathbf{S}_{tot} - \mathbf{S}_{scat}) \rangle d\Omega$$

Now recall that $\mathbf{S} = (2\mu_0)^{-1} \Re(\mathbf{E} \times \mathbf{B}^*)$ and so we can rewrite the above as:

$$\sigma_{tot} = -\frac{r^2}{E_{inc}^2} \Re \int [\mathbf{E}_{tot} \times c\mathbf{B}_{tot}^* - \mathbf{E}_{scat} \times c\mathbf{B}_{scat}^*] \cdot \hat{\mathbf{r}} d\Omega$$

Recalling that $\mathbf{E}_tot = \mathbf{E}_{inc} + \mathbf{E}_{scat}$, we get the following terms:

$$\sigma_{tot} = -\frac{r^2}{E_{inc}^2} \Re \int [\mathbf{E}_{inc} \times c\mathbf{B}_{inc}^* + \mathbf{E}_{inc} \times c\mathbf{B}_{scat}^* + \mathbf{E}_{scat} \times c\mathbf{B}_{inc}^*] \cdot \hat{\mathbf{r}} d\Omega$$

The first term is just the Poynting flux through the sphere of the incident wave, which is zero because there is no source for this wave inside the sphere, so:

$$\sigma_{tot} = -\frac{r^2}{E_{inc}^2} \Re \int [\mathbf{E}_{scat} \times c\mathbf{B}_{inc}^* + \mathbf{E}_{inc} \times c\mathbf{B}_{scat}^*] \cdot \hat{\mathbf{r}} d\Omega$$

Now we can note that:

$$\mathbf{E}_{inc} = E_{inc} \hat{\mathbf{x}} e^{i(kz - \omega t)}$$

$$\mathbf{E}_{scat} = E_{inc} f(\theta, \phi) \hat{\mathbf{e}}(\theta, \phi) \frac{e^{i(kr - \omega t)}}{r}$$

$$c\mathbf{B}_{inc} = E_{inc} \hat{\mathbf{y}} e^{i(kz - \omega t)}$$

$$c\mathbf{B}_{scat} = E_{inc} f(\theta, \phi) [\hat{\mathbf{r}} \times \hat{\mathbf{e}}(\theta, \phi)] \frac{e^{i(kr - \omega t)}}{r}$$

subbing these in gives:

$$\sigma_{tot} = -r \Re \int \left[f e^{-ik(z-r)} \hat{\mathbf{e}} \times \hat{\mathbf{y}} + f^* e^{ik(z-r)} \hat{\mathbf{x}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{e}}) \right] \cdot \hat{\mathbf{r}} d\Omega$$

Now things get tricky because f could in principle be a very complex function of θ and ϕ.

To make further progress, one needs to exploit the following identity that in the limit where $kr >> 1$

$$e^{ikz} = e^{ikr\cos\theta} = 2\pi i \left[\frac{e^{-ikr}}{kr} \delta(\hat{\mathbf{z}} + \hat{\mathbf{r}}) - \frac{e^{ikr}}{kr} \delta(\hat{\mathbf{z}} - \hat{\mathbf{r}}) \right]$$

Where does this identity come from? Well, it involves a bunch of bessel functions:

First, in general, one can always re-write a plane wave in a spherical basis:

$$e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty} (2\ell + 1) i^\ell j_\ell(kr) P_\ell(\cos\theta)$$

and furthermore, the polynomials $P_\ell(\cos\theta)$ can be written in terms of spherical harmonics:

$$e^{ikr\cos\theta} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} i^\ell j_\ell(kr) Y_{\ell m}(\hat{\mathbf{z}}) Y^*_{\ell m}(\hat{\mathbf{r}})$$

Note $Y_{\ell m}(\hat{\mathbf{e}})$ is just the standard harmonic evaluated at the angles where the give unit vector points. Next, it turns out that if kr is large, then the spherical bessel functions approach: $\sin(kr - \ell\pi/2)/kr$, and so we can say:

$$e^{ikr\cos\theta} = \frac{4\pi}{kr} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} i^\ell \sin(kr - \ell\pi/2) Y_{\ell m}(\hat{\mathbf{z}}) Y^*_{\ell m}(\hat{\mathbf{r}})$$

Next, we re-write the sine function in terms of complex exponentials:

$$e^{ikr\cos\theta} = -\frac{2\pi i}{kr} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} i^\ell (e^{i(kr-\ell\pi/2)} - e^{-i(kr-\ell\pi/2)}) Y_{\ell m}(\hat{\mathbf{z}}) Y^*_{\ell m}(\hat{\mathbf{r}})$$

Now note $e^{\pm i\ell\pi/2} = (\pm i)^\ell$, so

$$e^{ikr\cos\theta} = -\frac{2\pi i}{kr} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} (e^{ikr} - e^{-ikr}(-1)^{+\ell}) Y_{\ell m}(\hat{\mathbf{z}}) Y^*_{\ell m}(\hat{\mathbf{r}})$$

or

$$e^{ikr\cos\theta} = -\frac{2\pi i}{kr} \left[e^{ikr} \sum_{\ell m} Y_{\ell m}(\hat{\mathbf{z}}) Y^*_{\ell m}(\hat{\mathbf{r}}) - e^{-ikr} \sum_{\ell m} (-1)^\ell Y_{\ell m}(\hat{\mathbf{z}}) Y^*_{\ell m}(\hat{\mathbf{r}}) \right]$$

Now we can note that $(-1)^\ell Y_{\ell m}(\hat{\mathbf{z}}) = Y_{\ell m}(-\hat{z})$ so:

$$e^{ikr\cos\theta} = -\frac{2\pi i}{kr} \left[e^{ikr} \sum_{\ell m} Y_{\ell m}(\hat{\mathbf{z}}) Y^*_{\ell m}(\hat{\mathbf{r}}) - e^{-ikr} \sum_{\ell m} Y_{\ell m}(-\hat{\mathbf{z}}) Y^*_{\ell m}(\hat{\mathbf{r}}) \right]$$

Finally, since $Y_{\ell m}$ are a complete basis set, the sums above correspond to the required delta functions:

$$e^{ikr\cos\theta} = -\frac{2\pi i}{kr} \left[e^{ikr} \delta(\hat{\mathbf{z}} - \hat{\mathbf{r}}) - e^{-ikr} \delta(\hat{\mathbf{z}} + \hat{\mathbf{r}}) \right]$$

which is the above identity.

Subbing this in for $e^{\pm ikz}$, we get the following:

$$\sigma_{tot} = -\frac{2\pi}{k} \Re \int \left[\left[-if e^{2ikr} \delta(\hat{\mathbf{z}} + \hat{\mathbf{r}}) + if \delta(\hat{\mathbf{z}} - \hat{\mathbf{r}}) \right] \hat{\mathbf{e}} \times \hat{\mathbf{y}} + \left[-if^* \delta(\hat{\mathbf{z}} - \hat{\mathbf{r}}) + if^* e^{-2ikr} \delta(\hat{\mathbf{z}} + \hat{\mathbf{r}}) \right] \hat{\mathbf{x}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{e}}) \right] \cdot \hat{\mathbf{r}} d\Omega$$

Now integrating over the delta functions allows us to replace $\hat{\mathbf{r}}$ with $\pm\hat{\mathbf{z}}$ and get:

$$\sigma_{tot} = -\frac{2\pi}{k}\Re\left[-if_-e^{2ikr}(\hat{\mathbf{e}}_- \times \hat{\mathbf{y}})\cdot(-\hat{\mathbf{z}}) + if_+(\hat{\mathbf{e}}_+ \times \hat{\mathbf{y}})\cdot\hat{\mathbf{z}} + if_+^*[\hat{\mathbf{x}}\times(-\hat{\mathbf{z}}\times\hat{\mathbf{e}}_+)]\cdot(-\hat{\mathbf{z}}) - if_-^*e^{-2ikr}[\hat{\mathbf{x}}\times(\hat{\mathbf{z}}\times\hat{\mathbf{e}}_-)]\cdot\hat{\mathbf{z}}\right]$$

where $f_\pm\hat{\mathbf{e}}_\pm$ is the scattering amplitude evaluated along the $\pm z$ axis After moving out the sign factors, all the vector products are just $\hat{\mathbf{e}}\cdot\hat{\mathbf{z}}$, so this becomes:

$$\sigma_{tot} = -\frac{2\pi}{k}\Re\left[-i(f_+ - f_+^*)(\hat{\mathbf{x}}\cdot\hat{\mathbf{e}}_+) + i(f_-e^{-2ikr} + f_-^*e^{+2ikr}))(\hat{\mathbf{x}}\cdot\hat{\mathbf{e}}_-)\right]$$

Since we are asked to take the real part of something times i, this is equivalent to the negative of the imaginary part:

$$\sigma_{tot} = \frac{2\pi}{k}\Im\left[(f_+ - f_+^*)(\hat{\mathbf{x}}\cdot\hat{\mathbf{e}}_+) - (f_-e^{-2ikr} + f_-^*e^{+2ikr}))(\hat{\mathbf{x}}\cdot\hat{\mathbf{e}}_-)\right]$$

But now note that the second term is the sum of a factor and its complex conjugate, this is purely real, and so disappears:

$$\sigma_{tot} = \frac{2\pi}{k}(\hat{\mathbf{x}}\cdot\hat{\mathbf{e}}_+)\Im[f_+ - f_+^*]$$

and the remaining term is f minus its complex conjugate, which is purely imaginary and equal to twice the imaginary part. Recalling that this + subscripts indicate this is evaluated in the forward direction, we get the desired relationship:

$$\sigma_{tot} = \frac{4\pi}{k}(\hat{\mathbf{x}}\cdot\hat{\mathbf{e}})\Im[f(\theta = 0)]$$