**PHYS 542 Handout 2**

**Another example of a non-quasistatic system: An oscillating charge**

If instead of moving at a constant speed, a charge oscillates back and forth along the $x$ axis over a distance $\pm R$ with frequency $\omega$, then the charge density and current density are:

$$\rho = q\delta[\mathbf{r} - R\hat{\mathbf{x}}\sin(\omega t])$$

and

$$\mathbf{j} = q\omega R\cos(\omega t)\hat{\mathbf{x}}\delta[(\mathbf{r} - R\hat{\mathbf{x}}\sin(\omega t)]$$

If we assume this is a quasi-electrostatic system, then the electric and magnetic fields are:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r} - R\hat{\mathbf{x}}\sin(\omega t)}{|\mathbf{r} - R\hat{\mathbf{x}}\sin(\omega t)|^3}$$

$$\mathbf{B} = \frac{q\mu_0}{4\pi}\frac{(R\omega\hat{\mathbf{x}} \times \mathbf{r})\cos\omega t}{|\mathbf{r} - R\hat{\mathbf{x}}\sin(\omega t)|^3}$$

If we are far from the charge $r >> R$ these equations simplify to:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3}$$

$$\mathbf{B} = \frac{q\mu_0}{4\pi}\frac{(R\omega\hat{\mathbf{x}} \times \mathbf{r})\cos\omega t}{r^3}$$

but now note that

$$-\frac{\partial\mathbf{B}}{\partial t} = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{x}} \times \mathbf{r}}{r^3}\frac{R\omega^2}{c^2}\sin\omega t$$

This means that the electric field induced by this changing magnetic field will be of order $rR\omega^2/c^2$ times the quasi-static field. Thus the quasi-static approximation will only hold if $r << c^2/(R\omega^2)$. At larger distances, we cannot use this approximation.

**Quasi-static situations in conductors**

Recall that for DC conductors $\mathbf{j} = \sigma\mathbf{E}$. So in this case we have, from Gauss' law and the continuity equation:

$$-\frac{\partial\rho}{\partial t} = \nabla\cdot\mathbf{j} = \sigma\nabla\cdot\mathbf{E} = \sigma\frac{\rho}{\epsilon}$$

This implies that:

$$\frac{\rho}{\epsilon_0/\sigma} = -\frac{\partial\rho}{\partial t}$$

This implies a time constant of $\tau = \epsilon_0/\sigma$.

In order to have a quasi-electrostatic system, we need $\rho$ to change slowly, so $\tau$ needs to be large. For most conductors this is not the case, so quasi-electrostatic is a poor approximation for most conductors.

However, good conductors are often quasi-magnetostatic:

Note for quasi magnetostatic

$$\nabla\times\mathbf{B} = \mu_0\mathbf{j} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{dt} \simeq \mu_0\mathbf{j}$$

If we again use a linear conductor $\mathbf{j} = \sigma\mathbf{E}$, then

$$\nabla\times\mathbf{B} = \mu_0\left[\frac{1}{\sigma}\mathbf{E} + \epsilon_0\frac{\partial\mathbf{E}}{dt}\right]$$

Say $E$ varies on a timescale of $\tau$, then we need $\tau >> \sigma/\epsilon$, which can often be achieved with typical metals.

**Eddy currents**

Eddy currents are closed loops of current driven by changing electromagnetic field. These can influence the dynamics of objects and are one of the more straightforward applications of quasi-magnetostatics in conductors.

For example, consider a wire loop falling through a magnetic field at a speed $v$:

Field is in the $z$ direction, vertical is $y$, and the loop is in the $x-y$ plane. The magnitude of the field increases downwards, so the field in the lab frame is:

$\mathbf{B} = -\frac{y}{\ell}B_0\hat{\mathbf{z}}$

If we switch to the reference frame of the loop, $x' = x$, $y' = y - vt$, $z' = z$ (assume speed constant) the magnetic field passing through the loop is: $\mathbf{B} = \frac{-y'+vt}{\ell}B_0\hat{\mathbf{z}}$

Hence there is a circulating electric field along the ring is given by Farady's law:

$$\int \mathbf{E} \cdot d\ell = \frac{\partial}{\partial t} \int \mathbf{B} \cdot d\mathbf{S}$$

which means that

$$\mathbf{E} = -B_0 \frac{Rv}{2\ell} \hat{\phi}$$

where the $\hat{\phi}$ points around the loop.

If this wire is a conductor, then the current driven around the loop:

$$I = -\sigma B_0 A \frac{Rv}{2\ell}$$

where $\sigma$ is the conductivity or the material, and $A$ is the cross section of the loop.

Furthermore, we can compute the force on this wire:

$$\mathbf{F} = \int \mathbf{j} \times \mathbf{B} d^3 r$$

$$\mathbf{F} = \int \left( -\sigma B_0 \frac{Rv}{2\ell} \hat{\phi} \right) \times \left( -\frac{y}{\ell} B_0 \hat{\mathbf{z}} \right) d^3 r$$

$$\mathbf{F} = \sigma B_0^2 A \frac{R^2 v}{2\ell^2} \int \hat{\phi} \times \hat{\mathbf{z}} y d\phi$$

$$\mathbf{F} = \sigma B_0^2 A \frac{R^2 v}{2\ell^2} \int (-\sin\phi \hat{\mathbf{x}} + \cos\phi \hat{\mathbf{y}}) \times \hat{\mathbf{z}} R \sin\phi d\phi$$

$$\mathbf{F} = \frac{\pi}{2} \sigma B_0^2 A \frac{R^3 v}{\ell^2} \hat{\mathbf{y}}$$

Thus there is an upwards force on the wire loop.

Note that if you try to do this calculation in the lab frame, you must note that the contour you are integrating over is changing, and so:

$$\frac{\partial}{\partial t} \int \mathbf{B} \cdot d\mathbf{S} = \int \left[ \frac{\partial \mathbf{B}}{\partial t} + \mathbf{v}(\nabla \cdot \mathbf{B}) - \nabla \times (\mathbf{v} \times \mathbf{B}) \right] \cdot d\mathbf{S}$$

$$\frac{\partial}{\partial t} \int \mathbf{B} \cdot d\mathbf{S} = \int [\mathbf{E} + \mathbf{v} \times \mathbf{B}] \cdot d\ell$$

The RHS is the EMF on the electrons in the wire, which generates the same current as above.

**Skin Depth**

Even in the quasi-magnetostatic limit, time-varying currents in conductors do not flow like constant currents due to the skin effect.

In the quasi-magnetostatic limit for a linear conductor:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} = \mu_0 \sigma \mathbf{E}$$

combining this with Faraday's law $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ yields the following differential equation for the current:

$$\nabla^2 \mathbf{j} = \mu_0 \sigma \frac{\partial \mathbf{j}}{\partial t}$$

Say the current varies with time like $\cos(\omega t)$, or equivalently $e^{i\omega t}$, then

$$\nabla^2 \mathbf{j} = i\mu_0 \sigma \omega \mathbf{j}$$

if we say

$$\mathbf{j} = \mathbf{j}_0 e^{\kappa x}$$

then the above expression indicates that

$$\kappa^2 = i\mu_0 \sigma \omega$$

which means

$$\kappa = \pm \sqrt{\frac{\sigma \mu_0 \omega}{2}} (1 + i)$$

which defines a length called the skin depth

$$\delta = \sqrt{\frac{2}{\sigma \mu_0 \omega}}$$

In this case the current has a spatial form:

$$\mathbf{j} = \mathbf{j}_0 e^{-x/\delta} e^{i(x/\delta + \omega t)}$$

or, equivalently:

$$\mathbf{j} = \mathbf{j}_0 e^{-x/\delta} \cos\left(x/\delta + \omega t\right)$$

and the following form also works:

$$\mathbf{j} = \mathbf{j}_0 e^{-x/\delta} \cos\left(x/\delta - \omega t\right)$$

Thus the current flowing back and forth in a conductor is confined to a layer $\delta$ thick.

The electric field in the conductor is:

$$\mathbf{E} = \frac{1}{\sigma}\mathbf{j}_0 e^{-x/\delta}\cos\left(x/\delta - \omega t\right)$$

Hence the displacement current from the time-variable electric field is:

$$\mathbf{j}_D = -\frac{\omega\epsilon}{\sigma}\mathbf{j}_0 e^{-x/\delta}\sin\left(x/\delta - \omega t\right)$$

This will be small compared to the real current so long as $\omega << \sigma/\epsilon$, which is valid for quasi-magnetostatic situations.

Also, we can get the magnetic field from Farady's Law:

$$\mathbf{B} = -\frac{1}{\delta\sigma\omega}\mathbf{j}_0 e^{-x/\delta}\left[\cos\left(x/\delta + \omega t\right) - \sin(x/\delta - \omega t)\right]$$

Physically, the skin depth arises because the changing current produces time-varying magnetic fields in the wire, which in turn produce circulating electric fields that generate currents in the conductor.

**Situation illustrating the physics of the Skin Effect:**

Consider an infinite slab of material of thickness $T$ centered on the $x - y$ plane that carries a current $\mathbf{j}$ along the $y$ direction.

First, assume the current is constant in time and space $\mathbf{j} = j_0\hat{\mathbf{y}}$. By symmetry, the magnetic field here must point in the $x$-direction, and so Ampere's law:

$$\nabla \times \mathbf{B} = \mu_0\mathbf{j}$$

can be re-written using the definition of curl to be:

$$\frac{\partial B_x}{\partial z} = \mu_0 j_0$$

which means that

$$B_x = \mu_0 j_0 z$$

So the magnetic field varies linearly throughout the slab.

However, what if we now assume that the current varies sinusoidally

$$\mathbf{j} = j_0\sin(\omega t)\hat{\mathbf{y}}$$

If we assume quasi-magnetostatic limit applies, then we can use the same procedure and estimate that

$$\mathbf{B} = \mu_0 j_0 z\sin(\omega t)\hat{\mathbf{x}}$$

However, this changing magnetic field will produce an electric field given by Farady's law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Again, by symmetry the only net electric field is in the $y$ direction, so

$$-\frac{\partial E_y}{\partial z} = -\mu_0 \sigma_0 j_0 z \cos(\omega t)$$

which means

$$E_y = \frac{1}{2}\mu_0 \omega j_0 z^2 \cos(\omega t)$$

which will produce a correction to the current:

$$j' = \frac{1}{2}\mu_0 \sigma_0 \omega z^2 j_0 \cos(\omega t)$$

This correction increases the current flowing along the surfaces of the slab and is of order $z^2/\delta^2$. Thus if $T << \delta$, it is a small effect.