PHYS 542 Handout 4 Supplement

Superpositions of waves and Free-space diffraction

In general, one rarely has pure monochromatic plane waves, but since these a complete orthogonal basis set, you can often write other waves as an integral over these types of waves:

$$\mathbf{E}(\mathbf{r},t) = \Re\left[\frac{1}{(2\pi)^3}\int d^3k \mathbf{E}_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]$$
$$c\mathbf{B}(\mathbf{r},t) = \Re\left[\frac{1}{(2\pi)^3}\int d^3k (\hat{\mathbf{k}}\times\mathbf{E}_{\mathbf{k}}) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right]$$

In this case, the average energy density and Poynting vectors are:

$$\langle u_{em}\rangle = \frac{\epsilon_0}{2}\frac{1}{(2\pi)^3}\int d^3k |\mathbf{E}_{\mathbf{k}}|^2$$
$$c\langle P_{em}\rangle = \frac{\epsilon_0}{2}\frac{1}{(2\pi)^3}\int d^3k \hat{\mathbf{k}} |\mathbf{E}_{\mathbf{k}}|^2$$

We want to create a EM wave packet with a single frequency ω, which means that all the components must have the same k. However, they can be oriented in different directions, so they have the components $[q_x, q_y, \sqrt{k^2-q_x^2-q_y^2}]$, which can be approximated as $[q_x, q_y, k-(q_x^2-q_y^2)/2k]$ provided $q_x, q_y << k$, so

$$\mathbf{E}(\mathbf{r},t) = \int dq_x dq_y \mathbf{E}_{\mathbf{q}} e^{i(q_x x + q_y y - (q_x^2+q_y^2)z/2k + kz - \omega t)}$$

or, equivalently:

$$\mathbf{E}(\mathbf{r},t) = e^{i(kz-\omega t)}\int dq_x dq_y \mathbf{E}_{\mathbf{q}} e^{i(q_x x + q_y y - (q_x^2+q_y^2)z/2k)}$$

So this does behave like a plane wave. We just need to choose $\mathbf{E}_{\mathbf{q}}$ so that we get something like $\mathbf{E_0}e^{-(x^2+y^2)/w_0^2}$ when $z=0$. A choice that does this is $\mathbf{E}_{\mathbf{q}} = w_0^2 E_0 \hat{\mathbf{x}} e^{-w_0^2(q_x^2+q_y^2)/4}$. Let's try it out at $z=0$ where

$$\mathbf{E}(\mathbf{r},t) = e^{i(kz-\omega t)} w_0^2 E_0 \hat{\mathbf{x}} \int dq_x dq_y e^{-w_0^2(q_x^2+q_y^2)/4 + i(q_x x + q_y y)}$$

$$\mathbf{E}(\mathbf{r},t) = e^{i(kz-\omega t)} w_0^2 E_0 \hat{\mathbf{x}} \left[\int dq_x e^{-w_0^2 q_x^2/4 + i q_x x}\right]\left[\int dq_y e^{-w_0^2 q_y^2/4 + i q_y y}\right]$$

Now to evaluate the integrals, we need to "complete the squares"

$$\int dq_x e^{-w_0^2 q_x^2/4 + i q_x x} = \int dq_x e^{-(w_0^2/4)(q_x^2 - 2(2ix/w_0^2)q_x)}$$

$$\int dq_x e^{-w_0^2 q_x^2/4+iq_x x} = \int dq_x e^{-(w_0^2/4)(q_x^2-2(2ix/w_0^2)q_x-4x^2/w_0^4+4x^2/w_0^4)}$$

$$\int dq_x e^{-w_0^2 q_x^2/4+iq_x x} = \left[\int dq_x e^{-(w_0^2/4)(q_x^2-2(2ix/w_0^2)q_x-4x^2/w_0^4)}\right] e^{-x^2/w_0^2)}$$

$$\int dq_x e^{-w_0^2 q_x^2/4+iq_x x} = \left[\int dq_x e^{-(w_0^2/4)(q_x-2ix/w_0^2)^2}\right] e^{-x^2/w_0^2}$$

$$\int dq_x e^{-w_0^2 q_x^2/4+iq_x x} = \left[\int dq_x e^{-[(w_0/2)q_x-ix/w_0]^2}\right] e^{-x^2/w_0^2}$$

$$\int dq_x e^{-w_0^2 q_x^2/4+iq_x x} = \frac{2}{w_0}\left[\int d\mathcal{X} e^{-\mathcal{X}^2}\right] e^{-x^2/w_0^2}$$

$$\int dq_x e^{-w_0^2 q_x^2/4+iq_x x} = \frac{2}{w_0}\sqrt{\pi} e^{-x^2/w_0^2}$$

and so we get the desired shape of the beam at $z = 0$

$$\mathbf{E}(\mathbf{r},t) = e^{i(kz-\omega t)} E_0 \hat{\mathbf{x}} \frac{4\pi w_0^2}{w_0^2} e^{-x^2/w_0^2 - y^2/w_0^2}$$

But what happens at other values of z?, well now we have to do the full integral:

$$\mathbf{E}(\mathbf{r},t) = e^{i(kz-\omega t)} w_0^2 E_0 \hat{\mathbf{x}} \left[\int dq_x e^{-(w_0^2/4+iz/2k)q_x^2+iq_x x}\right]\left[\int dq_y e^{-(w_0^2/4-iz/2k)q_y^2/+iq_y y}\right]$$

again, we can evaluate the integrals by completing the squares as above, and in this case, we get:

$$\mathbf{E}(\mathbf{r},t) = E_0 \hat{\mathbf{x}} e^{i(kz-\omega t)} \frac{4\pi w_0^2}{w_0^2 + i2z/k} e^{-(x^2+y^2)/(w_0^2+i2z/k)}$$

Now see that when $z/k >> w_0^2$ we get something like:

$$\mathbf{E}(\mathbf{r},t) = E_0 \hat{\mathbf{x}} e^{i(kz-\omega t)} \frac{-i2\pi w_0^2 k}{z} e^{-i(x^2+y^2)/(2z/k)} e^{-(x^2+y^2)/(4z^2/k^2 w_0^2)}$$

So the width is $2z/kw_0$, which means that the smaller the width w_0, the more the beam spreads out at larger distances. Also, there is a phase shift across the beam because it is beginning to become a spherical beam. Finally note the energy density is starting to go like $1/z^2$.