PHYS 542 Handout 8

Phase, Group and Energy velocities

The *phase velocity* indicates how fast the phase of a wave propagates through space. For a standard plane wave of the form:

$$\mathbf{E} = \mathbf{E_0}\hat{\mathbf{x}}e^{i(kz-\omega t)}$$

the phase factor has the form $kz - \omega t = k[z - (\omega/k)t]$. The phase speed is how fast a point of constant phase moves, since a point of constant phase is a point of fixed $z - (\omega/k)t$, the relevant phase speed is:

$$v_p = \frac{\omega}{k}$$

Consider the phase speed of a Drude conductor, for which $ck = \sqrt{\omega^2 - \omega_p^2}$. In this case

$$v_p = \frac{c}{\sqrt{1 - \omega_p^2/\omega^2}}$$

Note that this is always greater than the speed of light when $\omega > \omega_p$. This is okay, however, because this is not the speed at which information can travel.

Often, the speed at which information can travel is related to the *group velocity.* The group velocity emerges from a consideration of a quasi-monochromatic wave packet:

$$\mathbf{E} = \int E(k)\hat{\mathbf{x}}e^{i(kz-\omega t)}dk$$

Let us say that only a narrow range of frequencies around ω_0 contribute to this packer, so we may approximate ω as $\omega_0 + (k - k_0)\partial\omega/\partial k$. In this case the expression can be re-written as:

$$\mathbf{E} = e^{i(k_0 z - \omega_0 t)} \int E(k)\hat{\mathbf{x}}e^{i[(k-k_0)(z-(\partial\omega/\partial k)t)]}dk$$

The term in front represents the normal wave oscillations, while the integral gives the shape of the wave packet. Note that this part of the function only depends on $z - (\partial\omega/\partial k)t$, so the envelope that defines the wave packet must move at a speed given by the group velocity:

$$v_g = \frac{\partial\omega}{\partial k}$$

For a Drude conductor $\omega = \sqrt{\omega_p^2 - c^2k^2}$ and so the group velocity works out to be:

$$v_g = c\sqrt{1 - \omega_p^2/\omega^2}$$

which is sensibly less than the speed of light.

However, for Lorentz dielectrics, the group speed can be both infinite and negative, so in general the group speed is not the speed of information/energy transfer.

The actual speed of energy flow and information transfer (which really must be less than the group speed) is the **energy velocity**

$$\mathbf{v}_E = \frac{\langle \mathbf{S} \rangle}{\langle u_{EM} \rangle}$$

But to use this expression, we must be careful about how we evaluate the energy density in dispersive matter.

Energy in Dispersive Matter

In both Drude Conductors and Lorentz Dielectrics there is damping, so energy is being lost to heat. This requires us to be careful about how we define energy conservation:

Remember that in general

$$\frac{\partial u_{EM}}{\partial t} = \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{B} \cdot \frac{\partial \mathbf{H}}{\partial t}$$

For simplicity, let us consider the electric part of this energy and say that the electric field at a given location is time harmonic:

$$\mathbf{E} = \mathbf{E_0} \cos(kz - \omega t)$$

If we have a normal linear dielectric with real ϵ, then we can say:

$$\mathbf{D} = \epsilon \mathbf{E_0} \cos(kz - \omega t)$$

and so

$$\frac{\partial u_E}{\partial t} = \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} = \epsilon E_0^2 \cos \omega t \frac{\partial}{\partial t} \cos(kz - \omega t)$$

which can be re-written as a total time derivative:

$$\frac{\partial u_E}{\partial t} = \frac{\partial}{\partial t} \left[\frac{\epsilon}{2} E_0^2 \cos^2(kz - \omega t) \right]$$

and so we can identify the term in the square brackets as the energy density, as normal.

However, if ϵ is complex, so that $\epsilon = \epsilon_R + i\epsilon_I$ then the real part of the displacement field becomes:

$$\mathbf{D} = \epsilon_R \mathbf{E_0} \cos(kz - \omega t) + \epsilon_I \mathbf{E_0} \sin(kz - \omega t)$$

and so:

$$\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}=\frac{\partial}{\partial t}\left[\frac{\epsilon_R}{2}E_0^2\cos^2(kz-\omega t)\right]+\omega\epsilon_I E_0^2\cos^2(kz-\omega t)$$

This first term can be regarded as the energy in the while the latter can be regarded as a dissipation term:

$$\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}=\frac{\partial u_E}{\partial t}+Q$$

In general, for quasi-monochromatic waves the energy density in the wave is:

$$u_{EM}=\frac{1}{2}\left[\frac{\partial}{\partial\omega}(\omega\epsilon_R)|\mathbf{E}|^2+\frac{\partial}{\partial\omega}(\omega\mu_R)|\mathbf{H}|^2\right]$$

and the dissipation constant is:

$$Q=\omega\left[\epsilon_I|\mathbf{E}|^2+\mu_I|\mathbf{H}|^2\right]$$

These expressions can be derived by considering a quasi-monochromatic wave. At any given location, this wave can be written in the following form (suppression the position dependence for the sake of simplcity):

$$\mathbf{E}(\mathbf{r})=\int\mathbf{E}(\omega)\cos(\omega t)d\omega$$

In this case the displacement field is:

$$\mathbf{D}(\mathbf{r})=\int\left[\epsilon_R\mathbf{E}(\omega)\cos(\omega t)+\epsilon_I\mathbf{E}(\omega)\sin(\omega t)\right]d\omega$$

In this case, we can compute the relevant product, but must take care to distinguish the integration constants:

$$\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}=\int\int d\omega_1 d\omega_2\mathbf{E}(\omega_1)\cdot\mathbf{E}(\omega_2)\left[\epsilon_R(\omega_2)\cos\omega_1 t\frac{\partial}{\partial t}\cos\omega_2 t+\epsilon_I(\omega_2))\cos\omega_1 t\frac{\partial}{\partial t}\sin\omega_2 t\right]$$

The second of these terms corresponds to the dissipation factor derived above, so we can say

$$Q=\int\int d\omega_1 d\omega_2\mathbf{E}(\omega_1)\cdot\mathbf{E}(\omega_2)\epsilon_I(\omega_2)\cos\omega_1 t\frac{\partial}{\partial t}\sin\omega_2 t$$

taking the derivative.

$$Q=\int\int d\omega_1 d\omega_2\mathbf{E}(\omega_1)\cdot\mathbf{E}(\omega_2)\omega_2\epsilon_I(\omega_2)\cos\omega_1 t\cos\omega_2 t$$

Provided the wave in quasi-monochromatic, we can approximate $\omega_2\epsilon_I(\omega_2)$ as the values they have at the average frequncy ω and so:

$$Q = \omega\epsilon_I(\omega)\int\int d\omega_1 d\omega_2 \mathbf{E}(\omega_1)\cdot\mathbf{E}\cos\omega_1 t\cos\omega_2 t$$

or

$$Q = \omega\epsilon_I(\omega)E^2$$

The first term is a bit trickier, but corresponds to $\partial u_E/\partial t$, so:

$$\frac{\partial u_E}{\partial t} = \int\int d\omega_1 d\omega_2 \mathbf{E}(\omega_1)\cdot\mathbf{E}(\omega_2)\epsilon_R(\omega_2)\cos\omega_1 t\frac{\partial}{\partial t}\cos\omega_2 t$$

To evaluate this, we recognize that ω_1 and ω_2 are dummy variables and so we can swap them around, so we can re-write this as:

$$\frac{\partial u_E}{\partial t} = \int\int d\omega_1 d\omega_2 \mathbf{E}(\omega_1)\cdot\mathbf{E}(\omega_2)\left[\frac{1}{2}\epsilon_R(\omega_2)\cos\omega_1 t\frac{\partial}{\partial t}\cos\omega_2 t + \frac{1}{2}\epsilon_R(\omega_1)\cos\omega_2 t\frac{\partial}{\partial t}\cos\omega_1 t\right]$$

Now we may approximate $\epsilon_R(\omega_2)$ as $\epsilon_R(\omega_1) + (\omega_2 - \omega_1)\partial\epsilon_R/\partial\omega$, then we get:

$$\frac{\partial u_E}{\partial t} = \int\int d\omega_1 d\omega_2 \mathbf{E}(\omega_1)\cdot\mathbf{E}(\omega_2)\left[\frac{1}{2}\epsilon_R(\omega_1)(\cos\omega_1 t\frac{\partial}{\partial t}\cos\omega_2 t + \cos\omega_2 t\frac{\partial}{\partial t}\cos\omega_1 t) + \frac{1}{2}\frac{\partial\epsilon_R}{\partial\omega}(\omega_2-\omega_1)\cos\omega_1 t\frac{\partial}{\partial t}\cos\omega_2 t\right]$$

The first term is now a total time derivative, and assuming ϵ_R does not vary much with frequency, this becomes:

$$\frac{\partial u_E}{\partial t} = \frac{\partial}{\partial t}\frac{\epsilon_R}{2}E^2 + \int\int d\omega_1 d\omega_2 \mathbf{E}(\omega_1)\cdot\mathbf{E}(\omega_2)\frac{1}{2}\frac{\partial\epsilon_R}{\partial\omega}(\omega_2-\omega_1)\cos\omega_1 t\frac{\partial}{\partial t}\cos\omega_2 t$$

For the last term, we use the same trick of swapping dummy variables is:

$$\frac{\partial u_E}{\partial t} = \frac{\partial}{\partial t}\frac{\epsilon_R}{2}E^2 +$$

$$\int\int d\omega_1 d\omega_2 \mathbf{E}(\omega_1)\cdot\mathbf{E}(\omega_2)\frac{1}{4}\frac{\partial\epsilon_R}{\partial\omega}\left[\omega_2\cos\omega_1 t\frac{\partial}{\partial t}\cos\omega_2 t - \omega_1\cos\omega_1 t\frac{\partial}{\partial t}\cos\omega_2 t + \omega_1\cos\omega_2 t\frac{\partial}{\partial t}\cos\omega_1 t - \omega_2\cos\omega_2 t\frac{\partial}{\partial t}\cos\omega_1 t\right]$$

Now note that you can move the minus signs on two of the terms as follows (since $\sin(-x) = -\sin(x)$

$$\frac{\partial u_E}{\partial t} = \frac{\partial}{\partial t}\frac{\epsilon_R}{2}E^2 + \int\int d\omega_1 d\omega_2 \mathbf{E}(\omega_1)\cdot\mathbf{E}(\omega_2)\times$$

$$\frac{1}{4}\frac{\partial\epsilon_R}{\partial\omega}\left[\omega_2\cos\omega_1 t\frac{\partial}{\partial t}\cos\omega_2 t + \omega_1\cos\omega_1 t\frac{\partial}{\partial t}\cos(-\omega_2 t) + \omega_1\cos\omega_2 t\frac{\partial}{\partial t}\cos\omega_1 t + \omega_2\cos\omega_2 t\frac{\partial}{\partial t}\cos(-\omega_1 t)\right]$$

and you can also swap the signs on corresponding frequencies in the other two terms since $\cos(x) = \cos(-x)$

$$\frac{\partial u_E}{\partial t} = \frac{\partial}{\partial t}\frac{\epsilon_R}{2}E^2 + \int\int d\omega_1 d\omega_2 \mathbf{E}(\omega_1)\cdot\mathbf{E}(\omega_2)\times$$

$$\frac{1}{4}\frac{\partial\epsilon_R}{\partial\omega}\left[\omega_2\cos(-\omega_1 t)\frac{\partial}{\partial t}\cos\omega_2 t + \omega_1\cos\omega_1 t\frac{\partial}{\partial t}\cos(-\omega_2 t) + \omega_1\cos(-\omega_2 t)\frac{\partial}{\partial t}\cos\omega_1 t + \omega_2\cos\omega_2 t\frac{\partial}{\partial t}\cos(-\omega_1 t)\right]$$

And so now we can combine the term pairs:

$$\frac{\partial u_E}{\partial t} = \frac{\partial}{\partial t}\frac{\epsilon_R}{2}E^2 + \int\int d\omega_1 d\omega_2 \mathbf{E}(\omega_1)\cdot\mathbf{E}(\omega_2)\times$$
$$\frac{1}{4}\frac{\partial \epsilon_R}{\partial \omega}\left[\omega_2\frac{\partial}{\partial t}\cos(-\omega_1 t)\cos\omega_2 t + \omega_1\frac{\partial}{\partial t}\cos(-\omega_2 t)\cos\omega_1 t\right]$$

And now we pull the time derivative completely out, and not that the sign on the ω inside the cosines is arbitrarty, so:

$$\frac{\partial u_E}{\partial t} = \frac{\partial}{\partial t}\frac{\epsilon_R}{2}E^2 + \frac{\partial}{\partial t}\int\int d\omega_1 d\omega_2 \mathbf{E}(\omega_1)\cdot\mathbf{E}(\omega_2)\frac{1}{4}\frac{\partial \epsilon_R}{\partial \omega}(\omega_2+\omega_1)\cos\omega_1 t\cos\omega_2 t$$

and assuming monochromatic waves, we can assume $\omega_1 \simeq \omega_2 \simeq \omega$, pull that outside the integrals and get:

$$\frac{\partial u_E}{\partial t} = \frac{\partial}{\partial t}\frac{\epsilon_R}{2}E^2 + \frac{\partial}{\partial t}\frac{\omega}{2}\frac{\partial \epsilon_R}{\partial \omega}E^2$$

And so we can identify the energy of the field as:

$$u_E = \frac{1}{2}\left(\epsilon_R + \omega\frac{\partial \epsilon_R}{\partial \omega}\right)E^2 = \frac{1}{2}\frac{\partial}{\partial \omega}(\omega\epsilon_R)E^2$$

As desired.

Back to Energy speed in Lorentz Dielectrics

The actual speed of energy flow and information transfer (which really must be less than the group speed) is the **energy velocity**

$$\mathbf{v}_E = \frac{\langle \mathbf{S} \rangle}{\langle u_{EM} \rangle}$$

Imagine a wave of the form:

$$\mathbf{E} = E_0 \hat{\mathbf{x}} \cos(kz - \omega t)$$

then the corresponding magnetic field is:

$$\mathbf{H} = H_0 \hat{\mathbf{y}} \epsilon_0 c E_0 (n_R \cos(kz - \omega t) + n_I \sin(kz - \omega t)$$

and so the average poynting vector is:

$$\langle \mathbf{E} \times \mathbf{H} \rangle = \frac{\epsilon_0}{2} n_R c E_0^2$$

If we use the above expressions for the energy density, we have:

$$\langle u_{EM} \rangle = \left\langle \frac{1}{2} \frac{\partial}{\partial \omega} (\epsilon_R \omega) E^2 + \frac{1}{2} \frac{\partial}{\partial \omega} (\mu_R \omega) H^2 \right\rangle$$

$$\langle u_{EM} \rangle = \frac{\epsilon_0}{4} E_0^2 \left(\frac{\epsilon_R}{\epsilon_0} + \frac{\omega}{\epsilon_0} \frac{\partial \epsilon_R}{\partial \omega} + n_R^2 + n_I^2 \right)$$

$$\langle u_{EM} \rangle = \frac{\epsilon_0}{2} E_0^2 \left(n_R^2 + \frac{\omega}{2\epsilon_0} \frac{\partial \epsilon_R}{\partial \omega} \right)$$

so the energy speed is:

$$\mathbf{v}_E = c \hat{\mathbf{z}} \frac{n_R}{n_R^2 + \frac{\omega}{2\epsilon_0} \frac{\partial \epsilon_R}{\partial \omega}}$$

For materials with no dissipation, like Drude conductors above the plasma frequency, this energy velocity is the same as the group velocity.

However, for a Lorentz dielectric, the above expression is not completely valid because the above expression for the average energy density is only an approximation, and cleanly separating Q from u_{EM} in this case requires expanding things to higher-order frequency derivatives of ϵ_R. An easier approach is provided by Loudon 1970 ("The propagation of electromagnetic energy through an absorbing dielectic" J. Phys A 3 233) and Oughstun 2006 (*Electromagnetic and Optical Pulse Propagation 1* Springer), and for this we start by writing the three parts of the non-dissipative electromagnetic energy density associated with the wave itself , the kinetic energy of the charges and the potential energy of the charges:

$$u_{EM} = \frac{1}{2}(\epsilon_0 E^2 + \mu_0 H^2) + \frac{1}{2}nmv^2 + \frac{1}{2}nm\omega_0^2 r^2$$

and for time-hamonic signals $v = r\omega$ so this becomes:

$$u_{EM} = \frac{1}{2}(\epsilon_0 E^2 + \mu_0 H^2) + \frac{1}{2}nm(\omega_0 + \omega)r^2$$

and for a Lorentz dielectric:

$$\mathbf{r} = \Re\left[\frac{q/m}{(\omega_0^2 - \omega^2) + i\omega\Gamma}\mathbf{E}\right]$$

this means

$$u_{EM} = \frac{1}{2}(\epsilon_0 E^2 + \mu_0 H^2) + \frac{1}{2}\frac{\omega_p^2(\omega_0^2 + \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2}E^2$$

and so

$$\langle u_{EM}\rangle = \frac{1}{4}\epsilon_0 E_0^2\left(1 + n_R^2 + n_I^2 + \frac{\omega_p^2(\omega_0^2 + \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2}\right)$$

Now if we recall that for a Lorentz Dielectric:

$$n_R^2 - n_I^2 = \epsilon_R = 1 + \frac{\omega_p^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2}$$

$$2n_R n_I = \epsilon_I = \frac{\omega_p^2\Gamma\omega}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2}$$

We can rewrite the last term as $\epsilon_R - 1 + 2\epsilon_I\omega/\gamma$, which becomes:

$$\langle u_{EM}\rangle = \frac{1}{2}\epsilon_0 E_0^2(n_R^2 + 2n_R n_I\omega/\Gamma)$$

and so the energy speed becomes (using the same Poynting vector as before:

$$v_E = \frac{c}{n_R + n_I\omega/\Gamma}$$

which is always less than the speed of light.