4/14/2014

Residues

$z = z_0$: an isolated singularity

Laurent series about z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z-z_0)^2} + \cdots$$

$$\operatorname*{Res}_{z=z_0} f(z) = b_1$$

Calculating residues (at poles):

Theorem If $z = a$ is a <u>simple</u> pole of f then

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z \to a} (z-a) f(z)$$

<u>Proof</u>: $z = a$ is a simple pole $\Rightarrow$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{(z-a)}$$

$$\Rightarrow (z-a) f(z) = \sum_{n=0}^{\infty} a_n (z-a)^{n+1} + b_1$$

$$\Rightarrow \lim_{z \to a} (z-a) f(z) = b_1 = \operatorname*{Res}_{z=a} f$$

Example $f(z) = \dfrac{3z-1}{z^2-1} = \dfrac{3z-1}{(z-1)(z+1)}$

simple poles @ $z = 1, -1$

$$\operatorname*{Res}_{z=-1} f(z) = \lim_{z \to -1} (z+1) f(z)$$

$$= \lim_{z \to -1} (z+1) \frac{3z-1}{(z-1)(z+1)}$$

$$= \lim_{z \to -1} \frac{3z-1}{z-1} = \frac{-4}{-2} = 2$$

Theorem [Residue at a pole of order m] :

If $z = a$ is a pole of order m, then

$$b_1 = \operatorname*{Res}_{z=a} f(z) = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m f(z)\right]$$

Proof : $\cancel{f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}}$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m}$$

$b_m \neq 0$

$$(z-a)^m f(z) = \sum_{n=0}^{\infty} a_n (z-a)^{m+n} + b_1 (z-a)^{m-1} + \cdots + b_{m-1}(z-a) + b_m$$

which is the Taylor series of $(z-a)^m f(z)$. about $z = a$;

$$b_1 = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[f(z)(z-a)^m\right]$$

by Taylor's Theorem .

Example

$$f(z) = \frac{\cos z}{z^6}$$

has a pole of order 6 at $z=0$.

$$\operatorname*{Res}_{z=0} \frac{\cos z}{z^6} = \lim_{z\to 0} \frac{1}{5!} \frac{d^5}{dz^5} \cos z$$

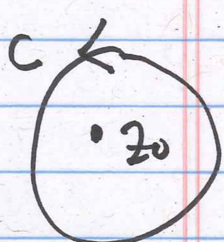
$$= \lim_{z\to 0} \frac{1}{5!} (-\sin z) = 0$$

z_0 : an isolated singularity

About z_0, the series expansion of f

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots$$

$$b_1 = \operatorname*{Res}_{z=z_0} f(z) \quad ; \quad b_1 = \frac{1}{2\pi i} \int_C f(z)\, dz$$



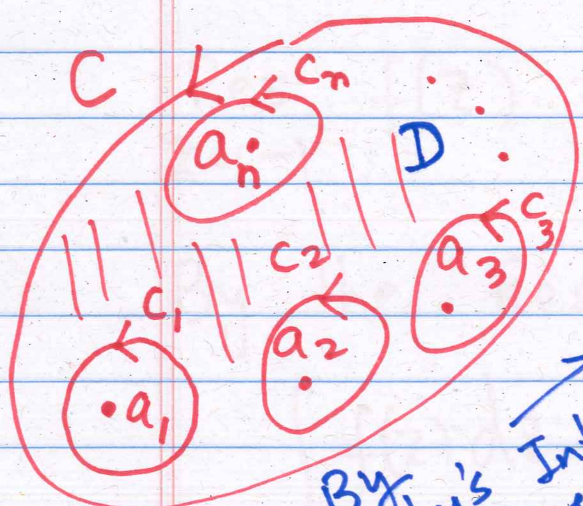
$$\Rightarrow \frac{1}{2\pi i} \int_C f(z)\, dz = \operatorname*{Res}_{z=z_0} f(z)$$

$$\Rightarrow \int_C f(z)\, dz = 2\pi i \operatorname*{Res}_{z=z_0} f(z)$$

THE RESIDUE THEOREM

Let f be analytic on and inside a simple closed curve C except for a finite no. of isolated singularities $a_1, a_2, \ldots, a_n$ inside C. Then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} f(z)$$



Proof: f is analytic in D; D is bounded by $C, C_1, \ldots, C_n$.

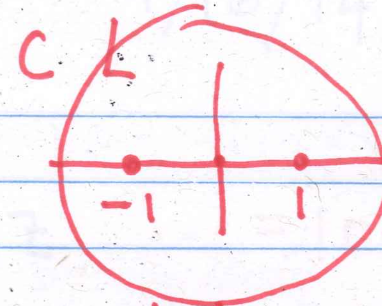
$$\int_C f = \int_{C_1} f + \int_{C_2} f + \cdots + \int_{C_n} f$$

By Cauchy's Integral Theorem

$$= 2\pi i \operatorname*{Res}_{z=a_1} f + 2\pi i \operatorname*{Res}_{z=a_2} f + \cdots + 2\pi i \operatorname*{Res}_{z=a_n} f$$

$$\boxed{\int_C f = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} f}$$

Example $\quad f(z) = \dfrac{3z-1}{z^2-1}$



Integrate f about $C: |z| < 3$

f has <u>two</u> <u>simple</u> poles at $z = \pm 1$

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z \to 1} (z-1) \frac{(3z-1)}{(z-1)(z+1)} = 1$$

$$\operatorname*{Res}_{z=-1} f(z) = \lim_{z \to -1} (z+1) \frac{(3z-1)}{(z-1)(z+1)} = 2$$

By the Residue Theorem

$$\int_C f(z)\,dz = \int_C \frac{3z-1}{z^2-1} = 2\pi i\,(1+2)$$

$$= 6\pi i$$

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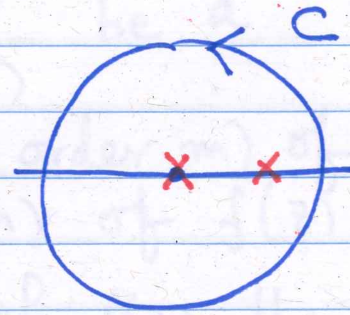
Residue Thm

Evaluate $\int_C \frac{5z-2}{z(z-1)}\,dz$ $\cdots = 10\pi i$

C: circle $|z| = 2$.



$f(z) = \frac{5z-2}{z(z-1)}$ has singularities inside C @ $z = 0, 1$.

Residue @ $z = 0$: (simple pole)

$$\lim_{z\to 0} z \frac{5\cancel{z}-2}{z(z-1)} = \lim_{z\to 0} \frac{5z-2}{z-1} = 2$$

or,

$$\frac{5z-2}{z(z-1)} = \frac{2-5z}{z(1-z)} = \frac{2-5z}{z}\left[1+z+z^2+\cdots\right]$$

$$= \frac{2}{z} + 2 + 2z^2 + \cdots \quad (-5-5z\cdots)$$

$$b_1 = \operatorname*{Res}_{z=0} f = 2$$

$$\operatorname*{Res}_{z=1} f = \lim_{z\to 1} \cancel{(z-1)} \frac{5z-2}{z\cancel{(z-1)}} = 3$$

By the Residue Thm $\int_C \frac{5z-2}{z(z-1)}$

$$= 2\pi i\,(2+3) = 10\pi i$$

Another result at calculating residues at simple poles :

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Thm. Let $f(z) = \dfrac{p(z)}{q(z)}$ be a rational function. A zero (of order m) of $q(z)$ is a pole (of order m) of $f(z)$.

If z_0 is a simple pole of $f(z)$, then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)}$$

$$f(z) = \frac{1 \nearrow p}{z^6+1 \searrow q}$$

zeros of z^6+1 are the 6th roots of -1 :

$$\underbrace{e^{i\pi/6},\ e^{i\pi/2},\ e^{i5\pi/6},\ e^{i7\pi/6},\ e^{i9\pi/6},\ e^{i11\pi/6}}_{\text{all simple poles of } \frac{1}{z^6+1}}$$

$$\operatorname*{Res}_{z=e^{i\pi/6}} \frac{1}{z^6+1} = \left.\frac{1}{6z^5}\right|_{z=e^{i\pi/6}} = \frac{1}{6} e^{-i\frac{5\pi}{6}}$$

Similarly $\displaystyle\operatorname*{Res}_{z=e^{i\pi/2}} f(z) = \frac{1}{6} e^{-5\pi/2\, i}$

$\vdots$

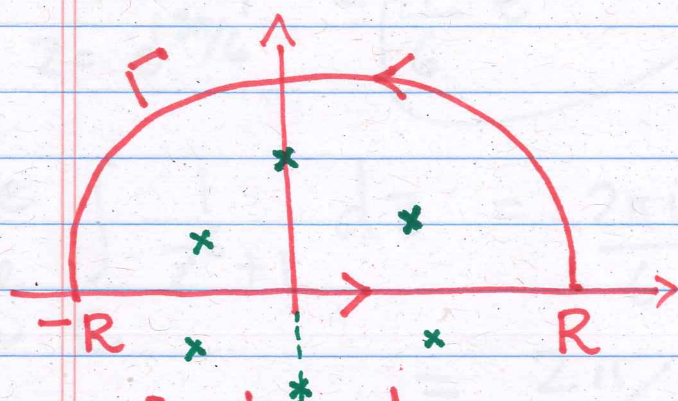
Applications of the Residue Thm in evaluating real integrals:

① Integrals of the form $\int_{-\infty}^{\infty} F(x)\,dx$

$F(x)$ is a rational function

$$\int_0^\infty \frac{dx}{x^6+1} \qquad \text{Let } f(z) = \frac{1}{z^6+1}$$

function of a complex variable



$C : [-R, R] \cup \Gamma$

Γ : upper boundary of the circle of radius R

$5\pi/6 = \pi - \pi/6$

$\frac{7\pi}{6} = \pi + \pi/6$

$$\int_C \frac{1}{z^6+1}\,dz$$

$z^6 + 1 = 0$ if

$z^6 = -1$

$= e^{i(\pi + 2n\pi)}$

$z = e^{i\pi\frac{(2n+1)}{6}}$, $n = 0, 1, \dots, 5$ | $z = e^{i\frac{(2n+1)\pi}{6}}$ $n = 0, 1, \dots, 5$

$\Rightarrow z = \underbrace{e^{i\pi/6}}_{z_1}, \underbrace{e^{i3\pi/6}}_{z_2}, \underbrace{e^{i5\pi/6}}_{z_3}, e^{i7\pi/6}, e^{i9\pi/6}, e^{i11\pi/6}$ outside C

are the singularities of $\frac{1}{z^6+1}$, all poles of order 1.

$$\int_C \frac{1}{z^6+1}\,dz = \int_{-R}^{R} \frac{dx}{x^6+1} + \int_\Gamma \frac{dz}{z^6+1}$$

$$\operatorname*{Res}_{z=e^{i\pi/6}} f = \lim_{z\to e^{i\pi/6}} \left(z - e^{i\pi/6}\right)\frac{1}{z^6+1} = \frac{1}{6}e^{-i5\pi/6}$$

$$\operatorname*{Res}_{z=e^{i3\pi/6}} f = \frac{1}{6}e^{-i5\pi/2} \qquad \operatorname*{Res}_{z=e^{i5\pi/6}} f = \frac{1}{6}e^{-\frac{25\pi i}{6}}$$

By the residue Theorem $\displaystyle\int_C \frac{1}{z^6+1}\,dz = \frac{2\pi i}{6}\left[e^{-i5\pi/6} + e^{-i\frac{5\pi}{2}} + e^{-\frac{25}{6}i\pi}\right]$

$$= 2\pi/3$$

$$\Rightarrow \frac{2\pi}{3} = \int_{-R}^{R} \frac{dx}{x^6+1} + \int_\Gamma \frac{dz}{z^6+1}$$

$$\Gamma: z = Re^{i\theta} \qquad 0 \le \theta \le \pi$$

$$dz = Rie^{i\theta}\,d\theta$$

$$\lim_{R\to\infty}\left|\int_\Gamma \frac{dz}{z^6+1}\right| = \lim_{R\to\infty}\left|\int_0^\pi \frac{iRe^{i\theta}\,d\theta}{R^6e^{i6\theta}+1}\right| \le \lim_{R\to\infty}\int_0^\pi \frac{R\,d\theta}{|R^6e^{i6\theta}+1|}$$

$$= \lim_{R\to\infty}\int_0^\pi \frac{R\,d\theta}{R^6\left|e^{i6\theta}+\frac{1}{R^6}\right|} \longrightarrow 0$$

Taking the limit as $R \to \infty$

$$\frac{2\pi}{3} = \int_{-\infty}^{\infty} \frac{dx}{x^6+1} + \underbrace{\cancel{\lim_{R\to\infty} \int_{\Gamma} \frac{dz}{z^6+1}}}_{0}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}$$

$$\int_{0}^{\infty} \frac{dx}{x^6+1} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{\pi}{3}$$

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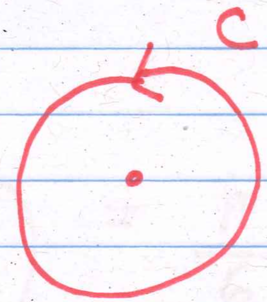
Application of the Residue Theorem in evaluating real integrals

(2) $\int_0^{2\pi} G(\sin\theta, \cos\theta)\, d\theta$

G: is a rational function of $\sin\theta$, $\cos\theta$

Example $\int_0^{2\pi} \frac{d\theta}{13-5\cos\theta}$



Set $z = e^{i\theta}$ $\Rightarrow dz = ie^{i\theta} d\theta$ $\Rightarrow d\theta = \frac{1}{iz} dz$, $0 \le \theta \le 2\pi$

on the unit circle

$z = \cos\theta + i\sin\theta$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\frac{1}{13-5\cos\theta} = \frac{1}{13 - \frac{5}{2}\left(z+\frac{1}{z}\right)} = \cdots = \frac{-2z}{(5z-1)(z-5)}$$

C: unit circle

$$\int_0^{2\pi} \frac{d\theta}{13-5\cos\theta} = \int_C \frac{-2\cancel{z}\, dz}{(5z-1)(z-5)i\cancel{z}} = \frac{-2}{i}\int_C \frac{dz}{(5z-1)(z-5)}$$

Let $f(z) = \frac{-2}{i(5z-1)(z-5)}$. Singularities at $z = 1/5, 5$

$1/5$ → inside C

$$\operatorname*{Res}_{z=\frac{1}{5}} f(z) = \lim_{z\to\frac{1}{5}} \left(z-\frac{1}{5}\right)\frac{-2}{i}\,\frac{1}{(5z-1)(z-5)}$$

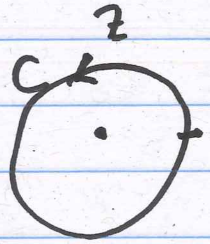
$$= \lim_{z\to\frac{1}{5}} \cancel{\left(z-\frac{1}{5}\right)}\left(-\frac{2}{i}\right)\frac{1}{5\cancel{\left(z-\frac{1}{5}\right)}(z-5)}$$

$$= -\frac{i}{12}$$

$$-\frac{2}{i}\int_C \frac{dz}{(5z-1)(z-5)} = 2\pi i\left(-\frac{i}{12}\right) = \frac{\pi}{6}$$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{13-5\cos\theta} = \frac{\pi}{6}.$$

General idea : $I = \int_0^{2\pi} G(\sin\theta, \cos\theta)\, d\theta.$



$$z = e^{i\theta} \Rightarrow dz = i e^{i\theta} d\theta$$

$$\Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$$

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) \qquad \sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$$

$$I = \int_C \tilde{G}(z)\,\frac{dz}{iz}$$

C : unit circle

Find singularities of $\tilde{G}$ inside C ; use Residue Thm.