

Stat/Math 452 (EO), Fall 2023 Assignment 1  
Due (via email to instructor): target date approx Oct 23

Suppose  $n$  independent, identically distributed observations are drawn from an exponential( $\lambda$ ) distribution, with pdf given by

$$f(x, \lambda) = \lambda e^{-\lambda x}, \quad 0 < x < \infty.$$

The data are  $X_1, X_2, \dots, X_n$ .

- (1) Show that  $\hat{\lambda}_1 = 1/\bar{X}$  is the ML estimate of  $\lambda$ .
- (2) Show that the sum of the observations has a gamma distribution, and then use the result to show that  $\bar{X}$  has a gamma distribution.
- (3) Use the result from (2) to derive the expected value of  $1/\bar{X}$ , and show that  $\hat{\lambda}_2 = \left(\frac{n-1}{n}\right)\hat{\lambda}_1$  is an unbiased estimate of  $\lambda$ .
- (4) Use the result from (2) to derive the expected value of  $1/\bar{X}^2$ , and thereby obtain the variance of  $\hat{\lambda}_1$ .
- (5) Derive the asymptotic distribution of  $\hat{\lambda}_1$  using large-sample ML theory.
- (6) Suppose a sample of size  $n = 7$  is drawn from an exponential distribution with true value of  $\lambda = 0.1$ .

Simulate the sampling distributions of  $\hat{\lambda}_1$ ,  $\hat{\lambda}_2$ , and a third estimate of  $\lambda$  given by

$$\hat{\lambda}_3 = -\frac{\log(.5)}{m},$$

where  $m$  is the sample median (the fourth ordered observation out of 7 total observations). Produce: (a) a histogram of the distribution for each of the three estimates, and (b) a table of bias, variance, and mean squared error for each estimate. You should simulate at least 1000 values for each type of estimate. You can use any computational software, but your instructor can offer the most help/support for R.

Hand in via email (**pdf or MS Word file**):

- I. Cover sheet, with name and typed paragraph describing the simulation results.
- II. Derivations, **neatly** hand written (engineering paper works well) and scanned/photoed, or typed (LaTeX, MS Word, etc.). Either option should be **converted to pdf or MS Word format (pdf is best; import photos into MS Word document then print to pdf file)**.
- III. Three histogram graphs, as separate figures or as separate panels in one figure. They should incorporate the same horizontal axis scale so that they can be visually compared easily. Again, **pdf is best**.
- IV. Table ( $3 \times 3$ ) of biases, variances, and MSEs.