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On the Spectral Entropy Behavior of Self-Organizing Processes

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Abstract

Using the spectral entropy techniques of Powell and Percival, we demonstrate a significant decrease in the spectral entropy of systems defined by both the logistic and Lorenz equations, as the control parameter increases past the initial bifurcation. This decrease in entropy may be related to self-organizing processes and the formation of coherent structures in the laminar to turbulence transition. Further increases in the control parameter forces the system through subsequent bifurcations, which in turn cause the flow to become less ordered. This reduction in order is reflected in the corresponding growth in the spectral entropy.

In addition, we show that the trajectories of a set of equations modeling free shear flows in internal cavities exhibit self-organizing behavior. The phase portraits of the solution trajectory reveal increasing complexity as the flow progresses downstream. The decrease in spectral entropy may imply the existence of self-organizing processes within the flow. This correlates with previous experimental observations describing vortex structures in free shear flows. The subsequent entropy increase indicates a collapse of the coherent structure into less-ordered turbulence.

Introduction

The work of Ruelle and Takens [1], which stated that a flow undergoes a series of bifurcations during its transition to turbulence, serves as a landmark study in both the dynamical systems and theoretical turbulence communities. The first bifurcation, or discrete transition, that the flow experiences marks the beginning of the transition region. The physical structure of the flow in the transition regime is fundamentally different than in the laminar regime. Succeeding bifurcations cause the fluid to become turbulent. By understanding the global effect of the bifurcation on the flow field and on the corresponding entropy, one may understand the relationship between transition and self-organization.

As some scientists incorporate dynamical systems techniques in understanding turbulence, Haken and others [2–4] study self-organizing behavior in nonlinear, nonequilibrium systems. Nonlinearities associated with fluid flows cause spontaneous interactions between fluid particles, which can result in the organization of coherent structures. The S-theorem as proposed by Klimontovich [4, 5] states that if a system recedes from its equilibrium state by increasing the control parameter, the entropy must decrease for self-organization to take place. This manifests itself in the form of coherent structures in the flow field. By monitoring the entropy through bifurcations and other changes in the flow, one may observe the location where structures form and collapse.

A technique to calculate the entropy using Fourier methods has been proposed by Powell and Percival [6]. This technique, called the spectral entropy method, incorporates the power spectral density of Hamiltonian systems to distinguish regular and irregular motions. In Part 1 of this work, we use the spectral entropy method to show a significant decrease in the entropy of two different systems: the logistic equation and the Lorenz equations. This decrease directly correlates with the initial symmetry breaking bifurcation of the system, and leads to the organization of distinct structures. Hence, self-organization arises due to a transition through the initial bifurcation of the flow.

In Part 2, we study an analytical model describing free shear flows in internal cavities. The results indicate that as the flow progresses downstream, the corresponding spectral entropy decreases then substantially increases, implying the formation and subsequent dissipation of coherent structures in the flow regime. Experimental evidence for this behavior has been presented by Isaacson [7].

The objectives of this presentation are two-fold. First, a viable link can be forged, connecting the dynamical systems theory of turbulence with self-organizational processes. Two simple examples show that the entropy decreases across the initial bifurcation, and increases through additional bifurcations *en route* to chaos. Second, a dynamical systems/spectral entropy approach to a realistic flow environment indicates self-organizational processes occur within the system. These structures have been observed experimentally.

1. Spectral entropy behavior in the logistic and Lorenz equations

Many theories [2–4] have been proposed which hypothesize the state of a system after undergoing a bifurcation. In this section, we apply the spectral entropy technique reported by Powell and Percival [6] to the simple difference equation (called the logistic [8] or quadratic equation),

$$x_{n+1} = rx_n(1 - x_n) \tag{1}$$

and the Lorenz equations,

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = -xz + rx - y \tag{2}$$
$$\frac{dz}{dt} = xy - bz.$$

For the Lorenz equations, $\sigma = 10$ and $b = 8/3$. The variable r serves as the control parameter for both equations (1) and (2). The spectral entropy denoted here as S, is defined

$$S = -\sum_k P_k \log P_k \tag{3}$$

where k represents the wavenumber and P_k is given by,

$$P_k = \frac{|f_k|^2}{\sum_{k'} |f_{k'}|^2} \tag{4}$$

The term $|f_k|^2$ represents the power spectral density of some trajectory at wavenumber k.

For example, we calculate the spectral entropy of the trajectory produced by equation (1) where $r = 3.6$ (see Fig. 1). The power spectral density is shown in Figure 2 and its spectral entropy curve is shown in Figure 3. The spectral entropy increases with every spike in the power spectral density. The spectral entropy is then the final value of the summation of equation (3).

The logistic equation provides a simple model for population dynamics, where growth and decay are represented by the terms x and $1 - x$. It is an extremely simple model which displays quite complex behavior, as seen in its bifurcation diagram, Figure 4. For values of the control parameter, r, lying between 0.0 and 3.0, there exists only a single steady-state solution. However the solution bifurcates at $r = 1$. This bifurcation marks the difference between extinction and a steady-state limit. Increasing the control parameter past $r = 3$ forces the system to undergo a period-doubling bifurcation. These period-doublings pile up on each other until the system becomes highly chaotic.

A curve of the spectral entropy versus the control parameter, r is shown in Figure 5. At the first bifurcation ($r = 1.0$), the spectral entropy takes a sharp drop, and stays very small until $r = 3.5$. In the region $1.0 < r < 3.5$, the power spectra are dominated by only one or two frequencies. By increasing the control parameter past $r = 3.5$ the number of frequencies required to describe the trajectory grows as the bifurcations come closer together. The rise in the entropy seen in Figure 5 is directly related to the number of frequencies present in the power spectral density.

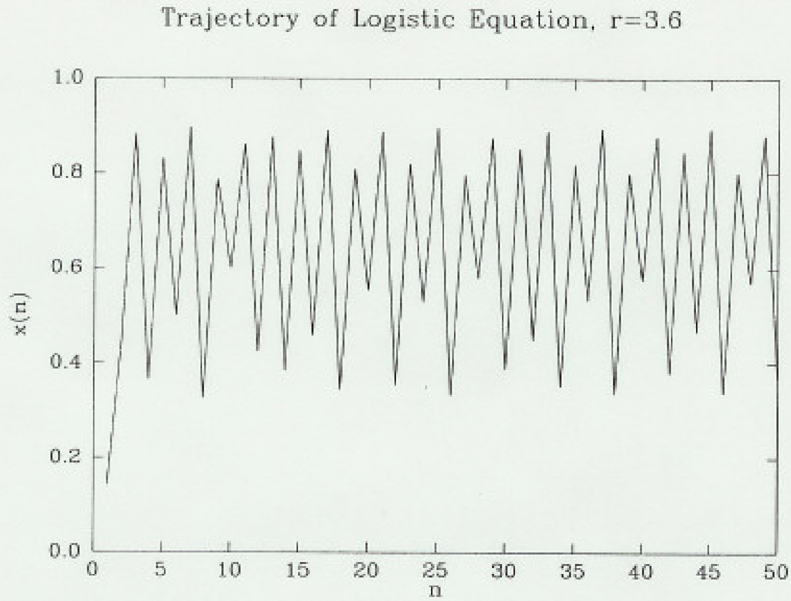

Fig. 1: Trajectory of x_n of the logistic equation, $r = 3.6$.

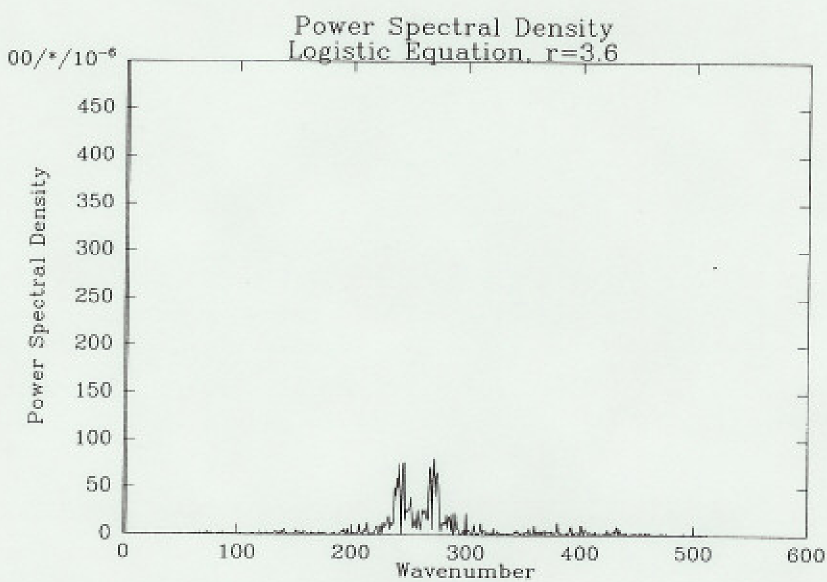

Fig. 2: Power Spectral Density of logistic equation, $r = 3.6$.

The Lorenz equations (2), on the other hand, model Rayleigh-Bénard convection. The control parameter, r represents the Rayleigh number, a measure of heat flux into the system. When the control parameter is less than one, the equations

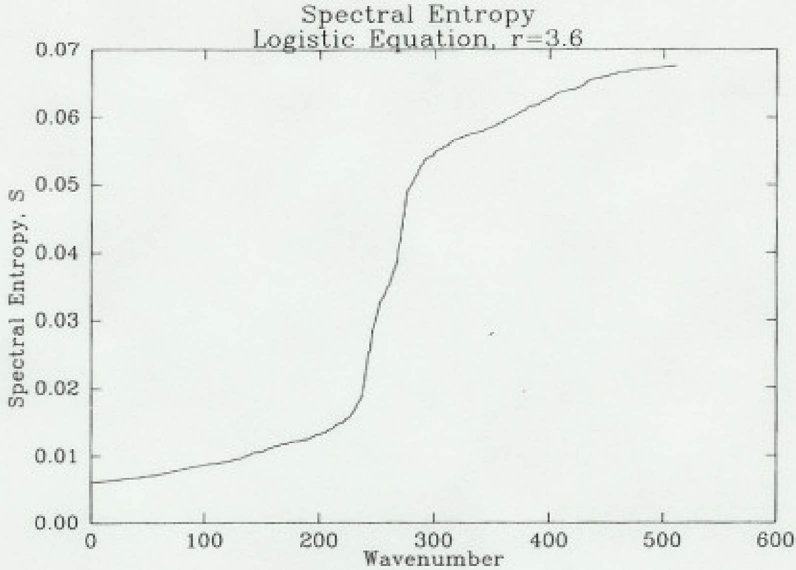

Fig. 3: Spectral Entropy of logistic equation, $r = 3.6$.

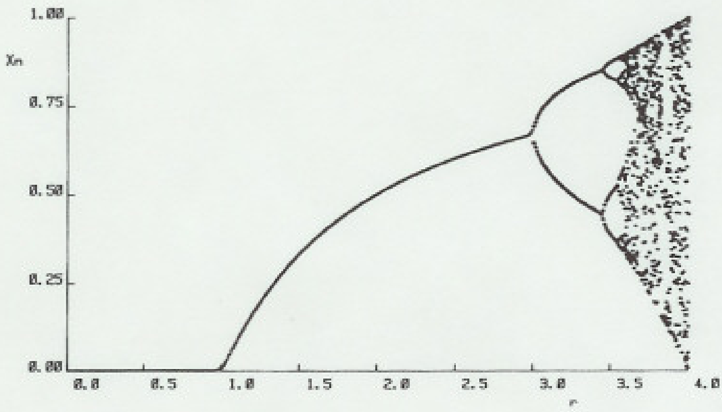

Fig. 4: Bifurcation diagram of logistic equation.

represent an unstable though stationary system where the fluid on the lower surface is warmer than fluid on the upper surface. Increasing the control parameter past $r = 1$ forces the system to bifurcate (see Fig. 6) and form macroscopic convection cells. The internal ordering process should result in a lower entropy. A turbulent region is located, past $r = 26$, where the large convection cells become unstable and collapse. The entropy should increase due to this loss in order.

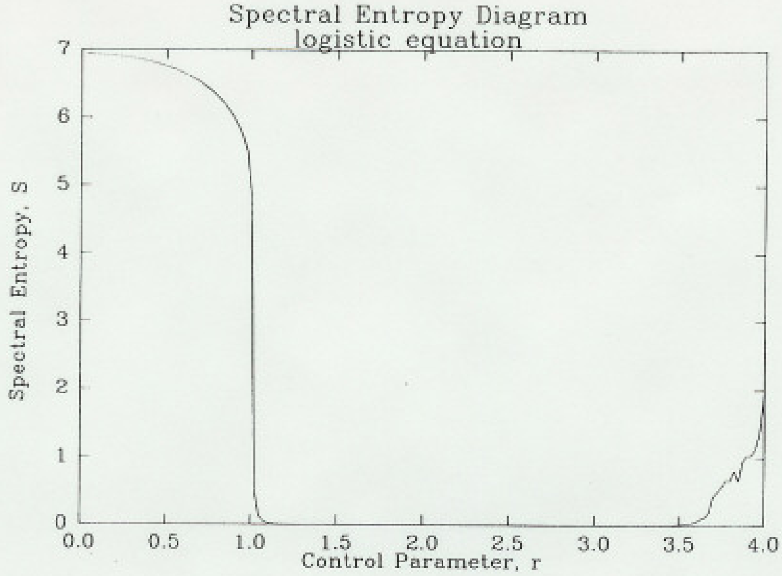

Fig. 5: Spectral entropy, S versus control parameter, r for logistic equation.

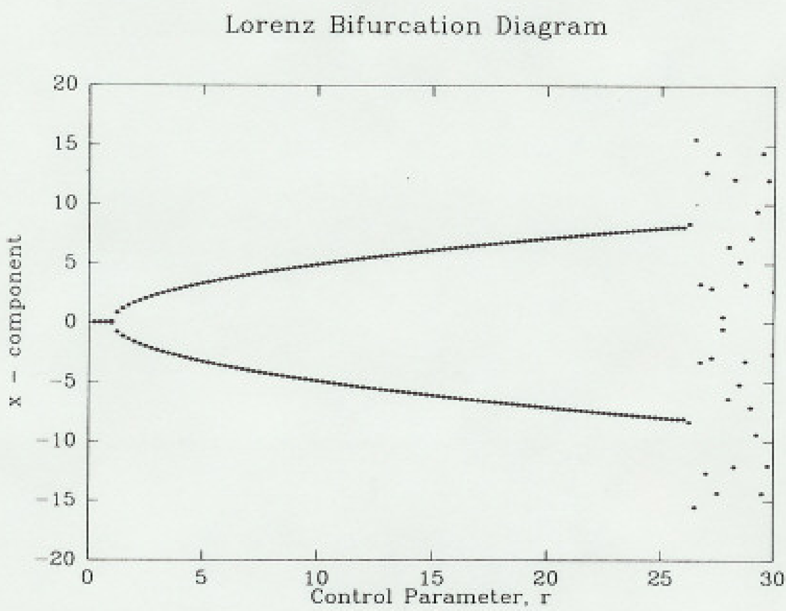

Fig. 6: Bifurcation diagram, Lorenz equations.

These patterns can be observed in the spectral entropies calculated for the x, y and z components of the Lorenz equations. Figure 6 is the Lorenz bifurcation diagram, and Figure 7 shows the corresponding spectral entropy of the x-component versus the control parameter, r. There is a sharp decrease in the entropy across the initial supercritical Hopf bifurcation at about $r = 1.0$. At about $r = 14$, there exists another bifurcation [9] which causes a slight increase in the entropy. The entropy stays low until the control parameter reaches a critical value and the system undergoes a subcritical Hopf bifurcation into a turbulent state. Figure 7 shows a corresponding rise in the entropy. The y and z components show similar characteristics to the x-component shown in Figure 7.

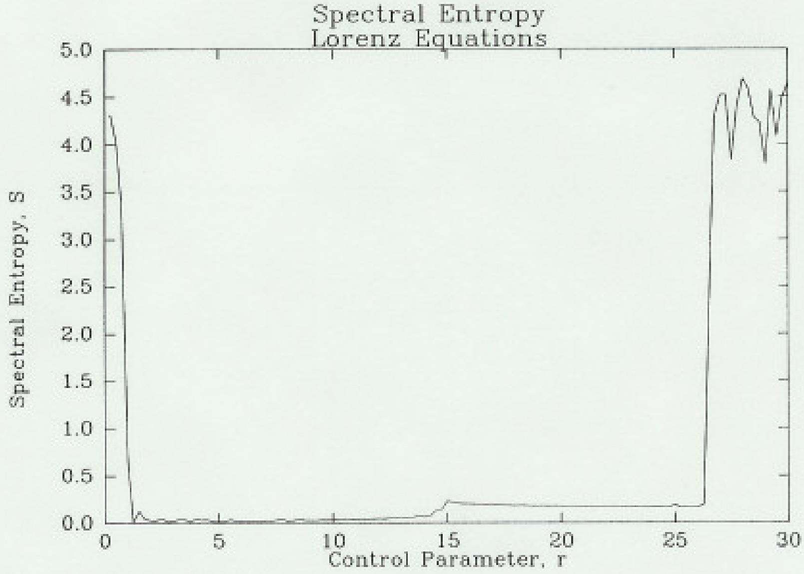

Fig. 7: Spectral entropy S versus control parameter, x-component, Lorenz equations.

The behavior of the spectral entropies of both the logistic and Lorenz equations is strikingly similar. The entropies remain relatively high until the first bifurcation is crossed. After passing the first bifurcation, only a few frequencies are present, hence the entropy remains low. As more frequencies emerge, the system becomes less coherent and increasingly chaotic, affecting the gain in the entropy.

In connecting the Ruelle-Takens theory with self-organization, one might hypothesize that the flow regime begins with some initial entropy in a laminar state. By increasing the control parameter of the system (say, the Reynolds or Rayleigh number), the flow undergoes its initial bifurcation whereupon the entropy decreases. The first bifurcation marks the beginning of the transition to turbulence and the initiation of self-organizing processes. In this region, coherent structures are formed. These structures exist until the next bifurcation is passed. At this point, additional frequencies emerge in the flow and the organized structures begin to lose their coherence. At this point, the entropy begins to climb. When the flow reaches the chaotic or fully turbulent regime, a broadband of frequencies arise in the flow, because the structures are extremely fine-scale and the entropy increases.

2. Deterministic model of a free shear layer

In an effort to describe the transition process in a deterministic manner, we employ the method used by Townsend [10] to reduce the Navier-Stokes equations into a set of first-order, ordinary differential equations. After decomposing the velocity term in the Navier-Stokes equations into its mean and fluctuating components, the fluctuating term is represented by,

$$u(x) = \sum_{k} a(\boldsymbol{k}) \exp(-i\boldsymbol{k} \cdot \boldsymbol{x}) \tag{5}$$

The final form of the equation for the Fourier amplitudes of the fluctuating velocities in homogeneous turbulence becomes,

$$\begin{aligned} \frac{da_i(\boldsymbol{k})}{dt} = &- \nu k^2 a_i(\boldsymbol{k}) - \frac{\partial U_i}{\partial x_l} a_l(\boldsymbol{k}) + 2 \frac{k_i k_l}{k^2} \frac{\partial U_l}{\partial x_m} a_m(\boldsymbol{k}) \\ &+ i \sum_{\boldsymbol{k}' + \boldsymbol{k}'' = \boldsymbol{k}} \left(k_i \frac{k_l k_m}{k^2} - \delta_{im} k_l\right) a_l(\boldsymbol{k}') a_m(\boldsymbol{k}'') \end{aligned} \tag{6}$$

where,

$$\frac{dk_i}{dt} = - \frac{\partial U_l}{\partial x_i} k_l \,. \tag{7}$$

Equation (7) accounts for the distortion of the wavenumbers due to shear.

For free shear layers in internal cavities, Isaacson, Denison and Crepeau [11] have shown that the Stuart [12] streamfunction,

$$\psi = \frac{1}{2} \ln(\cosh(2y) - \varrho \cos(2x)) \tag{8}$$

accurately models the mean velocity profile in the developing region of the shear layer. From equation (8), we see that ψ is a function of the location (x, y) in the flow, and from its derivatives, one may calculate values for the gradients, $\frac{\partial U_l}{\partial x_i}$.

The trajectory of the Fourier fluctuations may be found by integrating numerically a simplified version of equation (6) where only the first nonlinear term in the summation is included. By choosing some location in the flow field, equation (7) permits us to calculate the time-dependent values for each component of the wavenumber k. This in turn allows the integration of equation (6), so that the phase space trajectories of the components a_i may be observed.

The trajectories shown in Figure 8 represent the evolution in time of the Fourier components of the velocity fluctuations in the downstream and spanwise directions at various locations within the flow. As x increases, the nature of the trajectory changes from limit cycle to quasi-periodic behavior. This might characterize the process that the flow undergoes during the transition process.

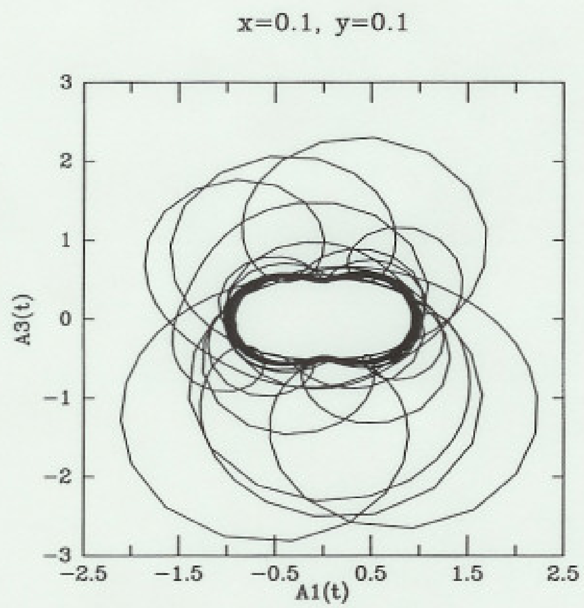

Fig. 8a: $a_1(t)$ versus $a_3(t)$ from equation (6), $x = 0.1$, $y = 0.1$.

Using the methods outlined in Part 1, we can compute the value of the spectral entropy as the flow progresses downstream. A decrease in the entropy represents self-organization, while an increase means additional turbulence. Figure 9 shows how the entropy changes at different locations at a vertical distance of 0.1. There exists a decreasing trend between the initial point and the distance downstream of 0.2. After this point the entropy begins to increase.

In terms of the physical processes within the flow field, the decrease in entropy may correlate with the spontaneous formation of flow structures. Isaacson [7] has reported the existence of tightly wound spiral vortices in the near region of a free shear flow in an internal cavity. After a short period, these small scale vortices decay very rapidly. This would be reflected by the sharp rise in entropy. As the structure decays and loses its coherency, the entropy must increase.

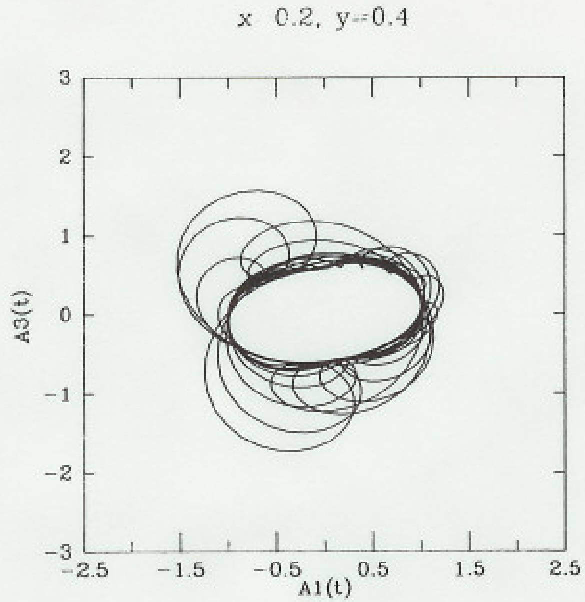

Fig. 8b: $a_1(t)$ versus $a_3(t)$ from equation (6), $x = 0.2$, $y = 0.4$.

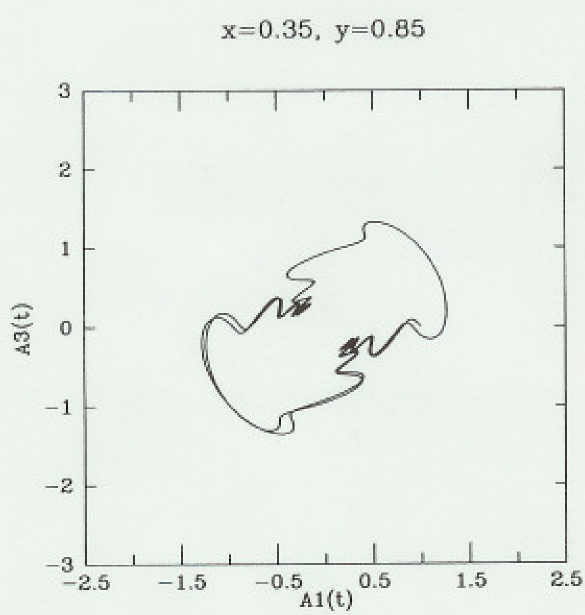

Fig. 8c: $a_1(t)$ versus $a_3(t)$ from equation (6), $x = 0.3$, $y = 0.75$.

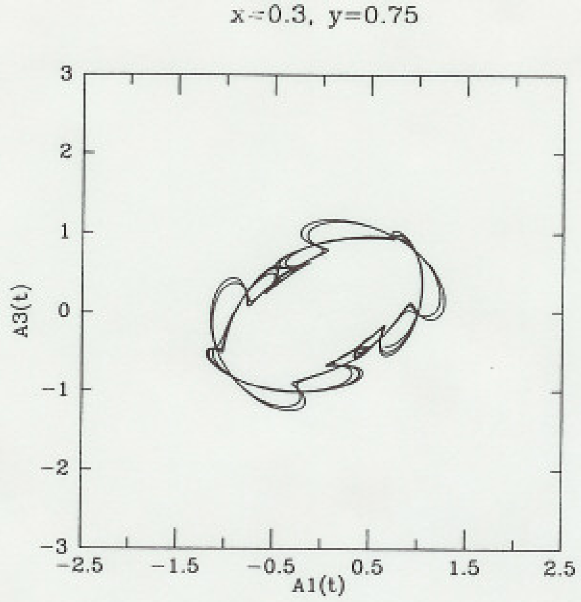

Fig. 8d: $a_1(t)$ versus $a_3(t)$ from equation (6), $x = 0.35$, $y = 0.85$.

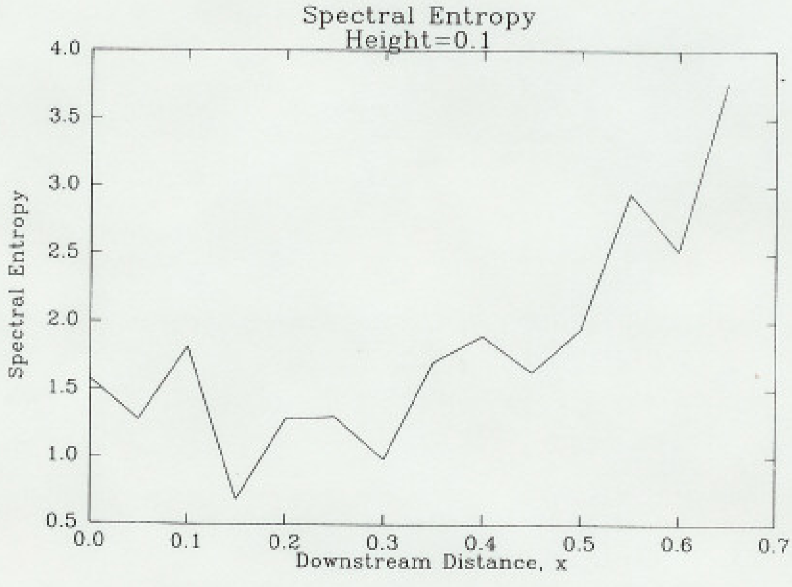

Fig. 9: Spectral entropy of equation (6) at $y = 0.1$.

3. Conclusions

We have shown, in the case of the logistic and Lorenz equations, that the spectral entropy decreases across the initial bifurcation and increases as the system becomes chaotic. According to the *S*-theorem of Klimontovich, the decrease in entropy of a system implies the existence of self-organizing processes. It is possible that these processes lead to the formation of coherent structures in fluid flows.

The deterministic model of free shear flows shows the evolution of attractors as the fluid progresses downstream. Spectral entropy calculations from these attractors reveal a decrease in the entropy from beginning to intermediate states. This indication of self-organization coincides with experimental data, which notes the presence of spiral vortices in this near region. The fact that the spectral entropy increases, which implies a reduction of order, explains the rapid dissipation of the spiral vortices into the flow field.

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