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Short Communication
Comparison of Spectral Entropy with Statistical Entropy in Selected Physical Systems

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Abstract

In an attempt to find a connection between the statistical entropy and the spectral entropy proposed by Powell and Percival, these properties are explicitly computed and compared for three different systems: Brownian motion of a particle in a fluid, Brownian motion in a harmonic potential, and the photon gas. It is shown that in these systems the statistical and the spectral entropy compare weakly since they quantify different statistical aspects.

1. Introduction

The fields of self-organization and nonlinear thermodynamics have grown rapidly in the last decade [1, 2]. Investigations have shown [3] that systems far from equilibrium can undergo transitions such that new, coherent structures are created. Familiar examples are Rayleigh-Bénard cells, and the spiral waves of the Belousov-Zhabotinsky reaction. As these so-called coherent structures become more ordered, their corresponding entropies decrease in many cases [4].

As the number of identified coherent structures increases, there has been a need for methods of numerically characterizing those structures. The spectral entropy as proposed by Powell and Percival [5] provides a convenient algorithm for assigning an entropy-like value to time-series data. It has been shown that this quantity can describe structural changes across bifurcations [6] and relative order of coherent structures in transitional flows [7]. The spectral entropy gives a measure of the broadness of the power spectrum.

2. Description of the spectral entropy

The spectral entropy provides a method of assigning a numerical value describing relative order to time series data. Using Fourier transform algorithms, a power

spectrum, $|f_k|^2$ where k is the wavenumber, of a time series can be obtained. The spectrum is then normalized by the relation,

$$P_k = \frac{|f_k|^2}{\sum_{k'} |f_{k'}|^2}. \tag{1}$$

This normalization procedure enforces the condition that the energy remains constant during the analysis. The constant energy condition is required when comparing the degree of self-organization [8] between different states of the system.

From the normalized value of the power spectra the spectral entropy is computed by

$$S^{\omega} = -\sum_k P_k \ln P_k. \tag{2}$$

This quantity describes the broadness of the power spectrum. If the energy is concentrated in only a few wavenumbers, the spectral entropy is low. A peak splitting increases the spectral entropy by one bit. When the spectrum is broadbanded, the spectral entropy becomes larger.

A natural question arises. Is there any similarity between the spectral entropy and the statistical thermodynamic entropy? For this purpose, we shall compute the two types of entropy for various models and compare their behavior.

3. Brownian particle in a fluid

We first examine the motion of a Brownian particle in a fluid, which has been studied extensively (see e.g., [9]). A velocity component of the particle is governed by the equation,

$$\dot{v} + \gamma v = \xi(t) \tag{3}$$

where γ is the damping coefficient, and $\xi(t)$ is Gaussian white noise with the properties,

$$\langle \xi(t) \rangle = 0, \qquad \langle \xi(t) \rangle \xi(t+\tau) \rangle = 2\varepsilon\delta(t).$$

In order to compute the thermodynamic entropy, one must obtain the stationary probability distribution $P(v)$ which in this case can be computed from the Fokker-Planck equation. In one dimension, it reads,

$$\frac{\partial P}{\partial t} - \frac{\partial}{\partial v}(\gamma v P) = \varepsilon \frac{\partial^2 P}{\partial v^2}. \tag{4}$$

The time independent solution of (4) yields the stationary probability distribution, given by

$$P^{\circ}(v) = \frac{1}{\sqrt{2\pi\varepsilon/\gamma}} \exp\left(-\frac{v^2}{2\varepsilon/\gamma} \right). \tag{5}$$

By the fluctuation-dissipation relation, $\varepsilon/\gamma = k_B T$, this is just a Maxwell distribution.

The statistical entropy can then be calculated as follows:

$$S = -\int_{-\infty}^{\infty} P^{v}(v) \log P^{v}(v)\, dv$$

$$= \frac{1}{2} \ln\left(\frac{\varepsilon}{\gamma}\right) + \text{constant.} \tag{6}$$

We will now compute the spectral entropy. The first step is to calculate the power spectrum. For the system described by (3), the power spectrum is [8],

$$F_{vv}(\omega) = \frac{2\varepsilon}{\gamma^2 + \omega^2}. \tag{7}$$

To compute the spectral entropy, one must normalize:

$$P^{\omega} = \frac{F_{vv}(\omega)}{\int_0^{\infty} F_{vv}(\omega)\, d\omega}. \tag{8}$$

The denominator can be computed to give

$$\int_0^{\infty} F_{vv}(\omega)\, d\omega = \frac{\pi\varepsilon}{\gamma}, \tag{9}$$

so that

$$P^{\omega} = \frac{2}{\pi} \frac{\gamma}{\gamma^2 + \omega^2}. \tag{10}$$

By normalizing the power spectrum in this fashion, we force the energy to remain constant during the calculation of the entropy. This is important when investigating the degree of self-organization within different states of the same system [10].

The spectral entropy is then found to be

$$S^{\omega} = -\int_0^{\infty} P^{\omega} \ln P^{\omega}\, d\omega$$

$$= \ln \gamma + \text{constant.} \tag{11}$$

This dependence of the spectral entropy on the damping coefficient is to be expected since a large γ broadens the spectrum, which in turn increases the spectral entropy. Comparing equations (6) and (11) shows an inverse dependence on γ, and that there is no ε dependence of S^{ω}.

4. Brownian motion in a potential

Another important stochastic process is given by,

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \xi(t) \tag{12}$$

where the random term $\xi(t)$ is again Gaussian white noise with the same properties as mentioned previously. This Langevin equation describes a damped particle in a harmonic potential kicked by a random force.

The entropy for the process can be computed as above to be,

$$S = \ln\left\{\frac{\varepsilon}{\gamma\omega_0}\right\} + \text{constant}, \tag{13}$$

which is similar to the entropy of the free moving Brownian particle in a fluid. The power spectrum of the motion is computed [11] to be,

$$F_{xx}(\omega) = \frac{2\varepsilon}{\gamma^2\omega^2 + (\omega^2 - \omega_0^2)^2} \tag{14}$$

and the spectral entropy can be computed with the residual method to be

$$S^{\omega} = \ln(2\pi\gamma) + 2\left[\frac{\gamma}{2\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}}\arctan\left\{\frac{\gamma}{2\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}}\right\} - \frac{\gamma\pi}{4\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}}\right]. \tag{15}$$

For $\omega \gg \gamma$ we get

$$S^{\omega} = \ln(2\pi\gamma). \tag{16}$$

As in the case of the Brownian particle, the normalization of the spectrum causes the ε dependence to drop out, enforcing the constant energy condition. In (16), the ω_0 dependence is also eliminated. This is to be expected since the value of ω_0 simply changes the location of the peak in the energy spectrum. As in the previous case, the dependence on γ in equations (13) and (16) is different.

5. Photon gas

In addition, the thermodynamic entropy and the spectral entropy may be compared for the photon gas. Planck's radiation equation,

$$u_{\nu}(T, \nu) = \frac{8\pi}{c^3}\frac{h\nu^3}{e^{h\nu/kT} - 1} \tag{17}$$

gives the energy distribution as a function of both the temperature T and the frequency.

The thermodynamic entropy [12] as a function of the temperature and volume is

$$S = \frac{4}{3}bT^3V. \tag{18}$$

The spectral entropy can be computed after normalizing the spectral energy distribution above to give,

$$S^{\nu} = \ln T + \text{constant}. \tag{19}$$

The comparison between equations (18) and (19) does not exhibit any similarity, even when applying the constant energy condition as shown in the discussion.

6. Discussion

The advantage of the spectral entropy is that it can be easily computed from experimental data, and that it monitors changes in the power spectrum when calculated for different parameters. However, it is not equivalent to the Boltzmann entropy, as our cases show. In two cases, there was an inverse dependence on the damping coefficient γ, and for the photon gas, there was a significant difference in the functional dependencies.

The spectral entropy implies a fixed energy. This restriction has not been required thus far in the calculation of the Boltzmann entropy.

However, we can easily impose this condition and discuss, e.g.,

$$\left(\frac{\partial S}{\partial \gamma}\right)_{E=\varepsilon/2\gamma} \quad \text{or} \quad \left(\frac{\partial S}{\partial T}\right)_{E=bT^4V}. \tag{20}$$

Then we get:

$$\left(\frac{\partial S}{\partial \gamma}\right)_E = 0, \qquad \text{Brownian motion} \tag{21}$$

$$\left(\frac{\partial S}{\partial T}\right)_E = -\frac{4}{3} bT^2 V, \qquad \text{photon gas} \tag{22}$$

Again, the Boltzmann entropy and the spectral entropy depend differently on γ and T, respectively.

In conclusion, the statistical and spectral entropies are measures which quantify different statistical aspects (effective phase space volume and effective number of spectral components, respectively). The disagreement is related to the fact that the Boltzmann entropy can be obtained from stationary probability densities, i.e. it measures static properties. On the other hand, the spectral entropy quantifies dynamical properties – the power spectrum. Both entropies may complement one another in applications [6, 7].

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