# Continuous Distributions

Module 15

*Statistics 251: Statistical Methods*

*Updated 2019*

## Continuous distributions (from chapter 5 in the textbook)

Continuous Random Variable

- A random variable $X$ is called **continuous** if it can take any value within a finite or infinite interval of the real number line $(-\infty, \infty)$
- Some examples would be measurements of length, strength, lifetime, pH, etc.

## Density Function of a Continuous Random Variable

A continuous random variable $X$ cannot have a PMF (probability mass function). The reason for this is that:

$P(X = x) = 0$

That is, there is no area under a curve at a single point

## Probability Density Function (pdf)

Probability Density Function (pdf):

The probability density function (pdf) of a continuous random variable $X$ is a nonnegative function $f_X$ with the property that $P(a < X < b)$ equals the area under it and above the interval $[a, b]$. Thus,

$$P(a < X < b) = \text{area under } f_X$$

$$a \leq X \leq b$$

The area under a curve is found by integration, such as:

$$P(a < X < b) = \int_a^b f(x)\ dx$$

But no calculus will be necessary to compute with the distributions

## Cumulative Distribution Function

The cumulative distribution function, or **CDF**, of a random variable $X$ gives the probability of events of the form $[X \leq x]$, for all numbers $x$

*Notation for the cumulative distribution function is*:

CDF or $F_X(x) = P(X \leq x)$

## Rules of Probibility

1. $0 \leq p(x) \leq 1$
2. $\int_{-\infty}^{\infty} f(x)\ dx = 1$ (ie the total probability must sum to 1)
3. Complement Rule
4. Addition Rule (for disjoint and non-disjoint events)
5. Multiplication Rule (for independent events)
6. Conditional Probability Rule

All the previously learned rules apply to continuous distributions.

## Uniform Distribution

In the context of probability distributions, a uniform distribution refers to a probability distribution for which all of the values that a random variable can take on occur with equal probability. This probability distribution is defined as follows.

A random variable $X$ is said to be uniform if all values of $X$ are equally likely

$$X \sim U(a,b) \quad for \;\; a < X < b$$

$$P(X \leq x) = \frac{x-a}{b-a}$$

## Uniform $EX$, $VX$, $SDX$

$$EX = \frac{a+b}{2}$$

$$VX = \frac{(b-a)^2}{12}$$

$$SDX = \sqrt{VX}$$

## Uniform Example I

Say that $Y$ has a uniform distribution on the interval $[2,5]$. Find the following:

- $f(y)$
- $F_Y(y)$
- $P(2 < Y < 3)$
- $EX$, $VX$, $SDX$

## Uniform Example II

$$f(y) = \begin{cases} \frac{1}{5-2} & 2 \leq y \leq 5 \\ 0 & otherwise \end{cases}$$

$$F_Y(y) = P(X \leq x) = \frac{x-a}{b-a} = \frac{x-2}{5-2}$$

$$P(2 < Y < 3) = \frac{3-2}{5-2} - \frac{2-2}{5-2} = \frac{1}{3} \approx 0.33$$

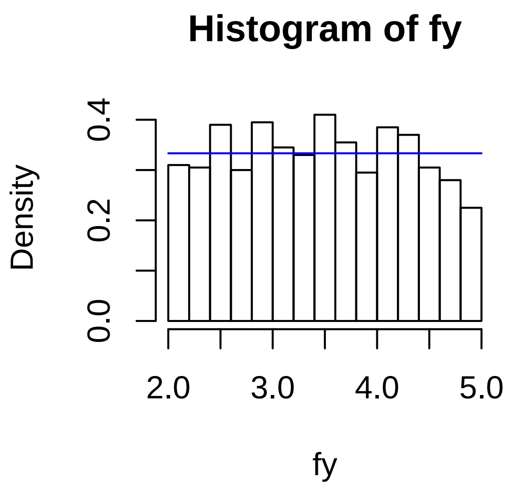
**Uniform Example III**

$$EX = \frac{b+a}{2} = \frac{5+2}{2} = 3.5$$

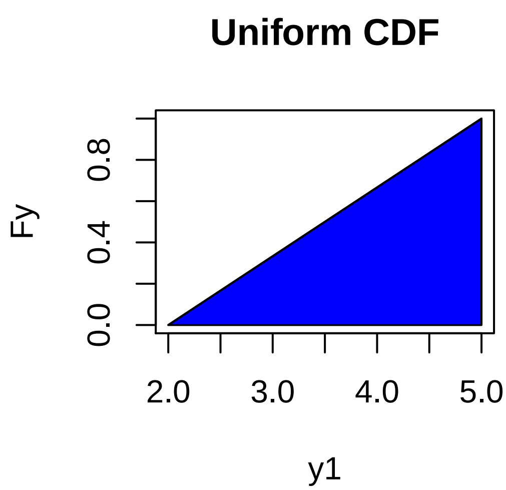
$$VX = \frac{(b-a)^2}{12} = \frac{(5-2)^2}{12} = 0.75$$

$$SDX = \sqrt{VX} = \sqrt{0.75} = 0.866025$$

**Graph of Uniform pdf**




**Graph of Uniform CDF**




**The Exponential Distribution**

$X$ is said to have an **exponential distribution** with parameter $m$ and pdf:

$$f(x;m) = \begin{cases} me^{-mx} & x \geq 0 \\ 0 & otherwise \end{cases}$$

and will always have CDF:

$$P(X \le x) = 1 - e^{-mx}$$

## $EX$, $VX$, $SDX$

$$EX = \frac{1}{m}$$

$$VX = \frac{1}{m^2}$$

$$SDX = \sqrt{VX}$$

### Exponential Example I

Suppose that the useful time (in years) of a PC is exponentially distributed with parameter $\lambda = 0.25$. A student entering a four-year undergraduate program inherits a two-year old PC from his sister who just graduated. Find the probability that the useful lifetime of the PC will last at least until he graduates (assume within 4 years). Let $X$ denote the useful lifetime of the PC.

$f(x; m) = f(x; 0.25) = \frac{1}{4} e^{-\frac{1}{4}x}$

$P(X > 4 + 2 | X > 2) = \frac{P(X>4+2 \cap X>2)}{P(X>2)}$

$= \frac{P(X>6)}{P(X>2)} = \frac{e^{-.25*6}}{e^{-.25*2}} = 0.367879$

### Exponential Example II

About how long would you expect the PC to last, on average? (this is the question to find the mean). Find $EX$, $VX$, $SDX$

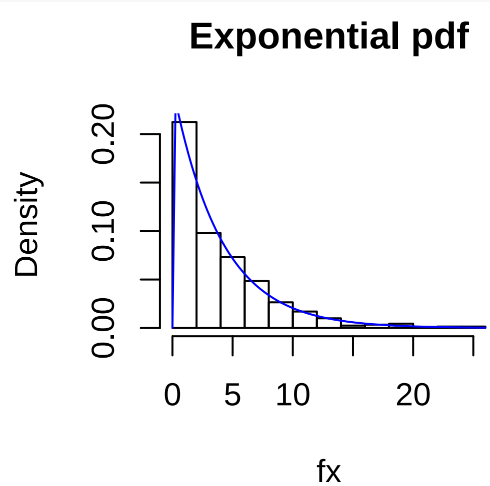
$EX = \frac{1}{m} = \frac{1}{0.25} = 4$

$VX = \frac{1}{m^2} = \frac{1}{(0.25)^2} = 16$

$SDX = \sqrt{VX} = 4$

### Graph of Exponential pdf

```
fx=rexp(1000,rate=.25)
hist(fx,prob=T,main="Exponential pdf")
curve(dexp(x,rate=.25),col='blue',add=T)
```



### Graph of Exponential CDF

```
x1=seq(0:1000); F=antiD((1/4)*exp(-.25*x)~x); Fx1=F(x1)-F(0)
plot(x1,Fx1,type='l',main="Exponential CDF",ylim=c(0.2,1.1))
polygon(c(x1,x1[length(x1)]),c(Fx1, Fx1[1]),col='blue')
```