Pattern avoidance

Given two permutations w (of m objects) and v (of n objects),

11/16/18

E.g. $w = 5\,3\,6\,2\,7\,1\,4$
(a rearrangement of $1\ 2 \cdots m=7$)

$v = 3\ 4\ 1\ 2$ $(n=4)$

Def: We say w contains v if we can choose m places in w so that the objects there are in the same order as v.

E.g. $5\quad 3\quad 6\quad 2\quad 7\quad 1\quad 4$

Def: w *avoids* v if it does not contain v.

Q: Let $(v_1, \dots, v_k)$ be a set of permutations ("patterns") we want to avoid.

Let c_m = # permutations of m objects that avoid all k of $v_1, \dots, v_k$.

Understand the sequence $c_m(v_1, \dots, v_k)$.

Standard mathematical trick: understand instead

$$C(t) = \sum_{m=0}^{\infty} c_m t^m$$

E.g. Avoiding 21

The permutations avoiding 21

1, 12, 123, 1234, 12345

$c_m(21) = 1$ for all m

$$C(t) = 1 + 1t + 1t^2 + 1t^3 + 1t^4 + \cdots = \frac{1}{1-t}$$

Let L be a language on Σ.

We can define

$$c_m(L) = \# \text{ strings in } L \text{ of length } m.$$

and define

$$C_L(t) = \sum_{m=0}^{\infty} c_m(L)\, t^m$$

<u>Fact</u>: If L is a regular language, then

$C_L(t)$ is a rational function;

i.e. $C_L(t) = \dfrac{\text{polynomial in } t}{\text{polynomial in } t}$.

L = reg lang w/ reg exp a^*,

$c_m(L) = 1$ for all m.

$$C_L(t) = 1 + t + t^2 + t^3 + \cdots$$

$$= \frac{1}{1-t}$$

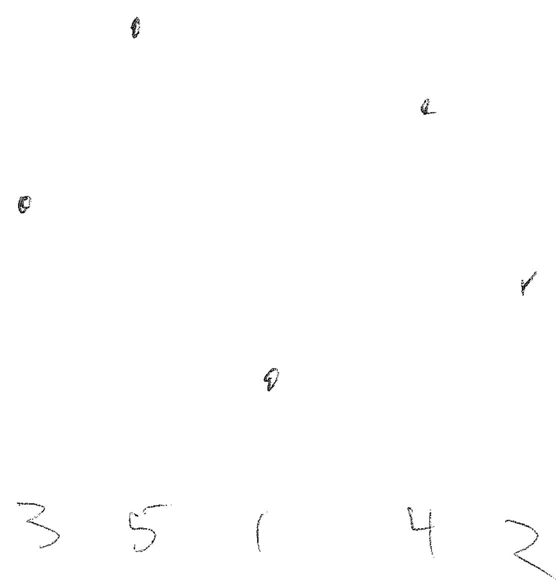
<u>Fact</u>: If L is a context-free language, then $C_L(t)$ is an algebraic function;

there is some polynomial in x & t, so that, when you solve for x, one of the roots is $C_L(t)$.

A. more complicated pattern avoidance example:

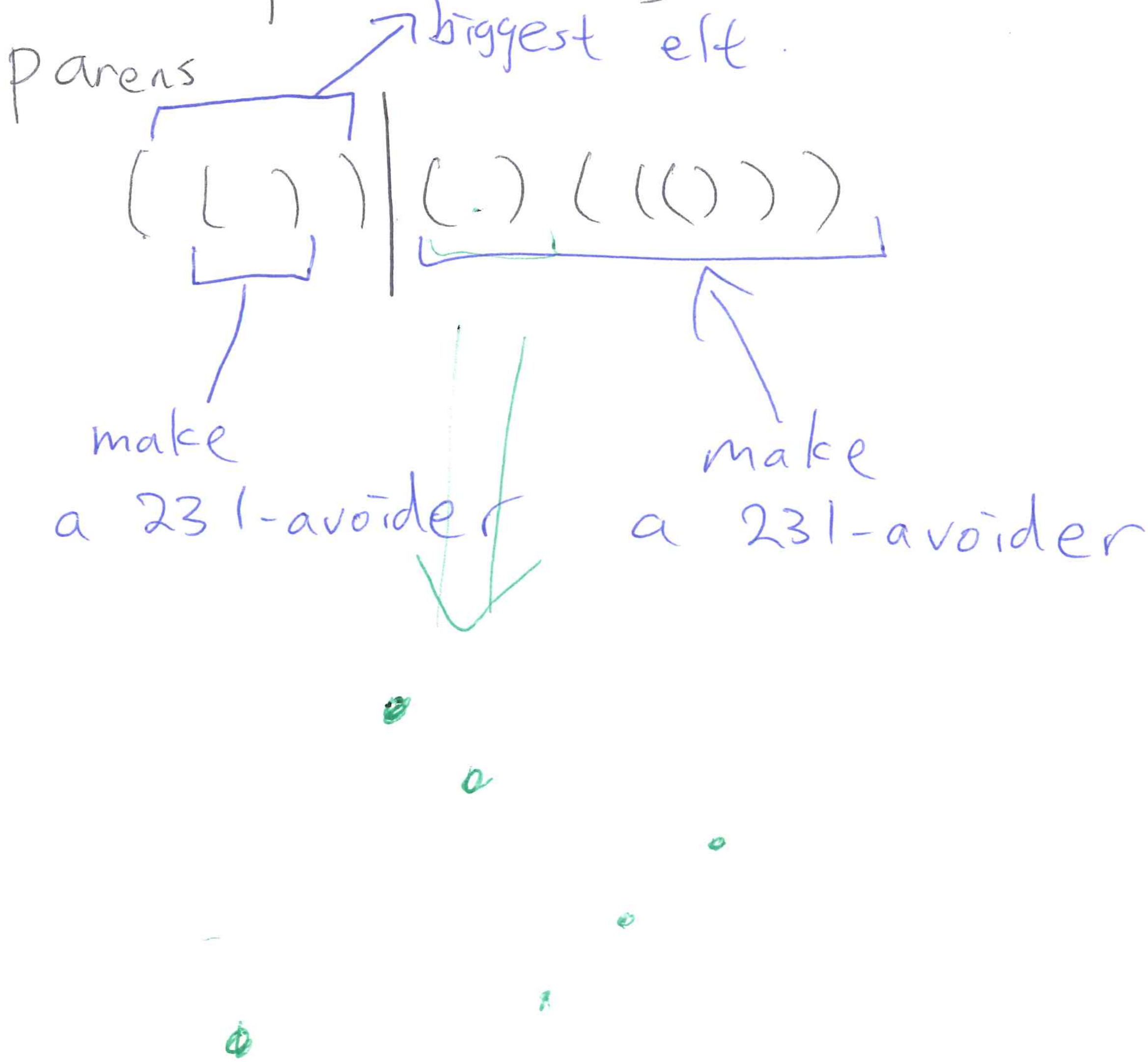
Avoiding $v=231$:

Graph of a perm: 35142



If I avoid 231

We have a bijection between perms avoiding 231 of n ~~letters~~ objects with proper parenthesizations using n ('s and n)'s



Counting # parenthesizations:

$$c_m = \sum_{i=0}^{m-1} c_i c_{m-1-i} \qquad c_0 = 1 \quad c_1 = 1$$

$$\sum_{m=0}^{\infty} c_m t^m = \underbrace{\sum_{m=0}^{\infty} \left(\sum_{i=0}^{m-1} c_i c_{m-1-i} \right) t^m}_{\substack{\shortparallel \\ ??}}$$

$$C(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots\cdots$$

$$C(t)^2 = (c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots)(c_0 + c_1 t + c_2 t^2 + \cdots)$$

$$= c_0^2 + c_0 c_1 t + c_1 c_0 t + (c_0 c_2 + c_1 c_1 + c_2 c_0) t^2$$

$$+ (c_0 c_3 + c_1 c_2 + c_2 c_1 + c_3 c_0) t^3 + \cdots\cdots$$

$$= \sum_{i=0}^{1-1} c_i c_{1-1-i} + \underbrace{\sum_{i=0}^{2-1} c_i c_{2-1-i}}_{\substack{\shortparallel \\ c_0 c_1 + c_1 c_0}} + \cdots$$

$$= \sum_{m=1}^{\infty} \left(\sum_{i=0}^{m-1} c_i c_{m-1-i} \right) t^{m-1}$$

This means

$$C(t) = t[C(t)]^2 + 1 \text{; i.e.}$$

$$t[C(t)]^2 - C(t) + 1 = 0, \text{ so}$$

$$\cancel{C(t) = C(t)^2 -}$$

$$C(t) = \frac{1 \pm \sqrt{1-4t}}{2t}$$

So $C(t) = \dfrac{1 + \sqrt{1-4t}}{2t}$

counts valid parens $\Longleftrightarrow$ 231 avoiders.

Conjecture from 1980s: (Noonan-Zeilberger)

~~If~~ The sequence

$c_m(v_1, \ldots, v_k)$ from any patterns $v_1, \ldots, v_k$ to be avoided is P-recursive: there are polynomials

$P_1, \ldots, P_\ell$ so that

$$P_1(n)c_n + P_2(n)c_{n-1} + P_3(n)c_{n-2} + \cdots + P_\ell(n)c_{n-\ell+1} = 0$$

for all n. (This is a recursive formula for c_n (w/ some omitted initial cond))

Disproved in 2015 by Garrabrant-Pak:

They show that, for any Turing machine M, there exist a list of patterns $v_1, \ldots, v_k$ so that parity of $\#$ strings of ~~accep~~ length n accepted by M

equals the parity of the number of permutations avoiding $v_1, \dots, v_k$.

It's known that the counting sequence for arbitrary TMs get arbitrarily bad - in particularly, it's not necessarily P-recursive.