(Nondeterministic)
Pushdown Automata - (PDAs)

<u>Def'n</u>: A PDA is a tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, F, z)$ where

Q is a finite set (of states)

Σ is a finite (input) alphabet

Γ is a finite stack alphabet

$\delta \subseteq (Q \times (\Sigma \cup \{\lambda\}) \times \Gamma) \times (Q, \Gamma^*)$

a transition relation

$q_0 \in Q$ is the start state

$F \subseteq Q$ is the set of final states

$z \in \Gamma$ is the bottom-of-stack symbol.

Let's make a PDA for

$\{a^n b^n \mid n \geq 0\}$

Idea: put a symbol on the stack for every ~~A~~ a; pop it off for each b.

$\Sigma = \{a, b\}$ $\Gamma = \{z, A\}$ $Q = \{q_0, q_1, q_2, q_3\}$

$$
\begin{aligned}
\delta = \{ \; & (q_0, a, z) \longrightarrow (q_1, Az) \\
& (q_1, a, A) \longrightarrow (q_1, AA) \\
& (q_1, b, A) \longrightarrow (q_2, \lambda) \\
& (q_2, b, A) \longrightarrow (q_2, \lambda) \\
& (q_2, \lambda, z) \longrightarrow (q_3, z) \\
& (q_0, \lambda, z) \longrightarrow (q_3, z) \; \}
\end{aligned}
$$

convention - we put the end of the string on the stack first.

$F = \{q_3\}$.

Formal description of "how a PDA works" / acceptance.

__Def:__ An __instantaneous description__ of a PDA is an element of the set

$$Q \times \Sigma^* \times \Gamma^*$$

state it is in

input that's left

stack, top first

Given ~~an inst~~ two instantaneous descr.
(q, w, u), (q', w', u'), we say
(q, w, u) <u>yields in one step</u> (q', w', u')
and write

$$(q, w, u) \vdash (q', w', u')$$

if there exist, $\ell \in \Sigma$, $y \in \Gamma$, $v \in \Sigma^*$, $x \in \Gamma^*$
$t \in \Gamma^*$

so that

$$(q, \ell, y) \longrightarrow (q', t) \in \delta$$

and

$$w = \ell v, \quad u = yx, \quad w' = v, \quad u' = tx.$$

Given (q, w, u), $(q', w', u') \in Q \times \Sigma^* \times \Gamma^*$,
we say (q, w, u) <u>yields</u> (q', w', u') (<u>eventually</u>)
if there exist some integer $k \geq 0$,
(q_i, w_i, u_i) ~~(q_k, w_k, u_k)~~ $\in Q \times \Sigma^* \times \Gamma^*$ for each i,
$0 \leq i \leq k$ ~~$0 \leq k \leq i$~~, where $(q_0, w_0, u_0) = (q, w, u)$,
$(q_k, w_k, u_k) = (q', w', u')$ and

$(q_i, w_i, u_i) \vdash (q_{i+1}, w_{i+1}, u_{i+1})$ for each i, $0 \le i < k$. In this case we write

$$(q, w, u) \vdash^* (q', w', u').$$

A PDA M accepts a string w if there exists $q' \in F$, $u' \in \Gamma^*$ where

$$(q_0, w, z) \vdash^* (q', \lambda, u').$$

Another example:

$$L = \{ ww^R \mid \cancel{\{a,b\}} \; w \in \{a,b\}^* \}$$

Idea: push first half onto the stack, then pop it off, making sure we match the string.

$$(q_0, a, z) \longrightarrow (q_0, Az)$$
$$(q_0, b, z) \longrightarrow (q_0, Bz)$$
$$(q_0, a, A) \longrightarrow (q_0, AA)$$
$$(q_0, b, A) \longrightarrow (q_0, BA)$$

$(q_0, a, B) \longrightarrow (q_0, AB)$

$(q_0, b, B) \longrightarrow (q_0, BB)$

$(q_0, \lambda, z) \longrightarrow (q_1, z)$ alternative:

$\cancel{(q_0, \lambda, A) \longrightarrow (q_1, A)}$ $(q_0, a, A) \to (q_1, \lambda)$

$\cancel{(q_0, \lambda, B) \longrightarrow (q_1, B)}$ $(q_0, b, B) \to (q_1, \lambda)$

$(q_1, a, A) \longrightarrow (q_1, \lambda)$

$(q_1, b, B) \longrightarrow (q_1, \lambda)$

$(q_1, \lambda, z) \longrightarrow (q_2, z)$

$F = \{q_2\}$