Pushdown automata:

Trace through instantaneous descriptions for one run of a PDA. (The last one we talked about Wed)

How does that PDA accept abaaba?

A PDA accepts $w$ if

$$(q_0, w, z) \xrightarrow{*} (q'_1, \lambda, u')$$

where $q'_1 \in F$

So to say how our PDA accepts abaaba— we should trace some seq. of instant. descr. that leads to a final state:

\[ (q_0, abaaba, Z) \]
\[ \xrightarrow{A} (q_0, baaba, Az) \]
\[ \xrightarrow{A} (q_0, aaba, BAz) \]
\[ \xrightarrow{A} (q_0, aba, ABAz) \]

\[ (q_0, a, z) \rightarrow (q_0, Az) \]
\[ (q_0, b, A) \rightarrow (q_0, BA) \]
\[ (q_0, a, B) \rightarrow (q_0, AB) \]
(q₀, λ, A) → (q₁, A)
(q₁, a, A) → (q₁, λ)
(q₁, b, B) → (q₁, λ)
(q₁, a, A) → (q₂, λ)
(q₁, λ, z) → (q₂, z)

Since q₂ ∈ F, we accept abaaba.

Equivalence of PDAs and CFGs

Goal: Prove that a language is given a language L, there exists a PDA M that accepts L iff there exists a CFG G that generates L.

How to do this?

1. Give a method for taking a CFG G and constructing a PDA M from G so that a string is accepted by M if and only if it is generated by G.
Give a method for taking a PDA $M$ and constructing a grammar $G$, so that a string can be generated by $G$ if and only if it can be accepted by $M$.

Let's do (1):

We have a grammar $G$ - step (1) is to find an equivalent grammar $G'$ in CNF.

We'll have the PDA mimic a leftmost derivation - in CNF, at any time in a leftmost derivation, we have:

- the letters will be the part of the string already processed by the PDA, and the variables will be on the stack.
E.g. Grammar $G$ is given by

$$S \rightarrow AB \mid AA$$
$$A \rightarrow SB \mid a$$
$$B \rightarrow BA \mid b$$

(leftmost)

Derivation in the grammar:

$$S \rightarrow AA \rightarrow SBA \rightarrow ABBBA$$
$$\quad \rightarrow aBBA \rightarrow abBA \rightarrow abBAA \rightarrow abbAA$$
$$\quad \rightarrow abbaA \rightarrow abbaa.$$ 

I want my PDA to accept the string $abbaa$ through a process mimicking this derivation:

$$\begin{align*}
(q_0, \text{abbaa}, \varepsilon) & \xrightarrow{S \rightarrow AA} (q_1, \lambda, S) \\
& \xrightarrow{A \rightarrow SB} (q_1, \lambda, A) \\
& \xrightarrow{S \rightarrow AB} (q_1, \lambda, S) \\
& \xrightarrow{\text{always have this transition}} (q_1, \text{abbaa}, \varepsilon)
\end{align*}$$
Formally, our construction algorithm for getting our PDA from our CFG:

- $M$ always has 3 states: $\{q_0, q_1, q_2\}$
- $F = \{q_2\}$
- $\Sigma = \Sigma$ for $G$, $\Gamma = \$ \cup \Sigma$ for $G$

The transitions in $M$ are:
For each production
\[ A \rightarrow BC \]
in \( G \), we have
\[ (q_0, \lambda, A) \rightarrow (q_1, BC) \]

For each production
\[ A \rightarrow a \]
in \( G \), we have
\[ (q_1, a, A) \rightarrow (q_1, \lambda) \]