Closure properties of context-free languages

Facts: If L_1 + L_2 are context free languages,

1) $L_1 \cup L_2$ is a context-free lang.

   (Pf: Given grammars for L_1 + L_2 w/ start vars S_1 + S_2, create a new grammar with start var S and prod. $S \to S_1 \mid S_2$ along w/ all old prods. This new grammar generates $L_1 \cup L_2$)

2) $L_1 L_2$ is a context-free lang.

   (Pf: $S \to S_1 S_2$)

3) L_1^* is context-free

   (Pf: $S \to S_1 S \mid \lambda$)

4) $L_1 \cap L_2$ might not be a context-free language.

E.g. $L_1 = \{a^n b^n c^k \mid n \geq 0, k \geq 0\}$

$L_2 = \{a^k b^n c^n \mid n \geq 0, k \geq 0\}$

are both context-free, but

$L_1 \cap L_2 = \{a^n b^n c^n \mid n \geq 0\}$

is not.

Intuition - the reg. lang. proof (where you have a DFA that simulates running 2 DFAs simultaneously) doesn't work b/c you need 2 stacks to run 2 PDAs simultaneously.

But: ~~it~~

4') If L_1 is context-free and L_2 is regular, $L_1 \cap L_2$ is context-free. (running a PDA + a DFA simultaneously only requires one stack)

5) L_1 might not be context-free either.

Intuition: nondeterminism adds power to PDAs. (Our reg. lang. pf only worked on DFAs and relied on converting NFAs to DFAs)

E.g. We now easily know that:

① $\{a^n b^n w \mid n \geq 0, |w| = 5\}$ is context-free (concatenation of a CF lang w/reg lang)

② $\{a^{3n} b^{3n} \mid n \geq 0\}$ is context-free (intersection of CF lang w/reg lang)

③ $\{a^k b^k \mid 3 \text{ does not divide } k\}$ is context free.

More HW 4 problems:

Algorithm to decide if $L = \Sigma^*$ (for L regular)

the language of all strings (using the alphabet Σ)

Given a DFA M

....

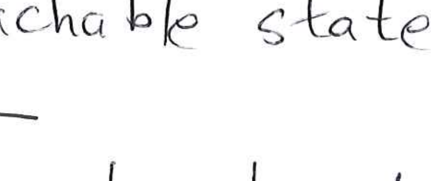
(You need something more clever than ~~one~~ testing one string at a time, b/c that will take an infinite amount of time.)

For every string to be accepted, every reachable state has to be final.

Algorithm to decide if L has infinitely many even length strings.

"Cheap sol'n": $K = \{ v \in \Sigma^* \mid |v| \text{ is even} \}$

K is regular b/c



Given a machine for L, we can build a machine for $L \cap K$ - check if that has reachable loops

~~Expensive~~ sol'n that you have to think about: Check if machine for L has any loops, but even length loops starting an odd distance from initial state don't count.

A pumping lemma problem:

$$L = \{a^n b^\ell \mid n/\ell \text{ is an integer}\}$$

Given m, let $w = a^m b^m$. Then let ~~we~~ $w = xyz$, $|xy| \le m$, $y \ne \lambda$. We know from the restrictions $y = a^j$ for some j, $1 \le j \le m$. Then $xy^2z = a^{m+j} b^m$ bad if $j = m$ (which is possible)

$xy^0z = a^{m-j} b^m$ also bad if $j = m$

$xy^3z = a^{m+2j} b^m$ also bad if $j = m$

We ~~didn't~~ don't want to allow adding/subtracting as many a's as there are b's. So if we have $(m+1)$ b's ...

Given m, let $w = a^{m+1} b^{m+1}$. Let $w = xyz$, $|xy| \leq m$, $y \neq \lambda$. ~~We~~ From the restrictions $y = a^j$ for some j, $1 \leq j \leq m$.

Then $xy^0z = a^{m+1-j} b^{m+1}$.

Since $1 \leq j \leq m$, $m+1-j < m+1$, and $m+1-j > 0$. Hence $\frac{m+1-j}{m+1}$ is not an integer, so

$xy^0z \notin L$.

Hence, L violates the pumping lemma and is not regular.