Pumping Lemma for Context-Free Languages:

Goal for today: Show

$$L = \{a^n b^n c^n \mid n \geq 1\}$$

is not context-free

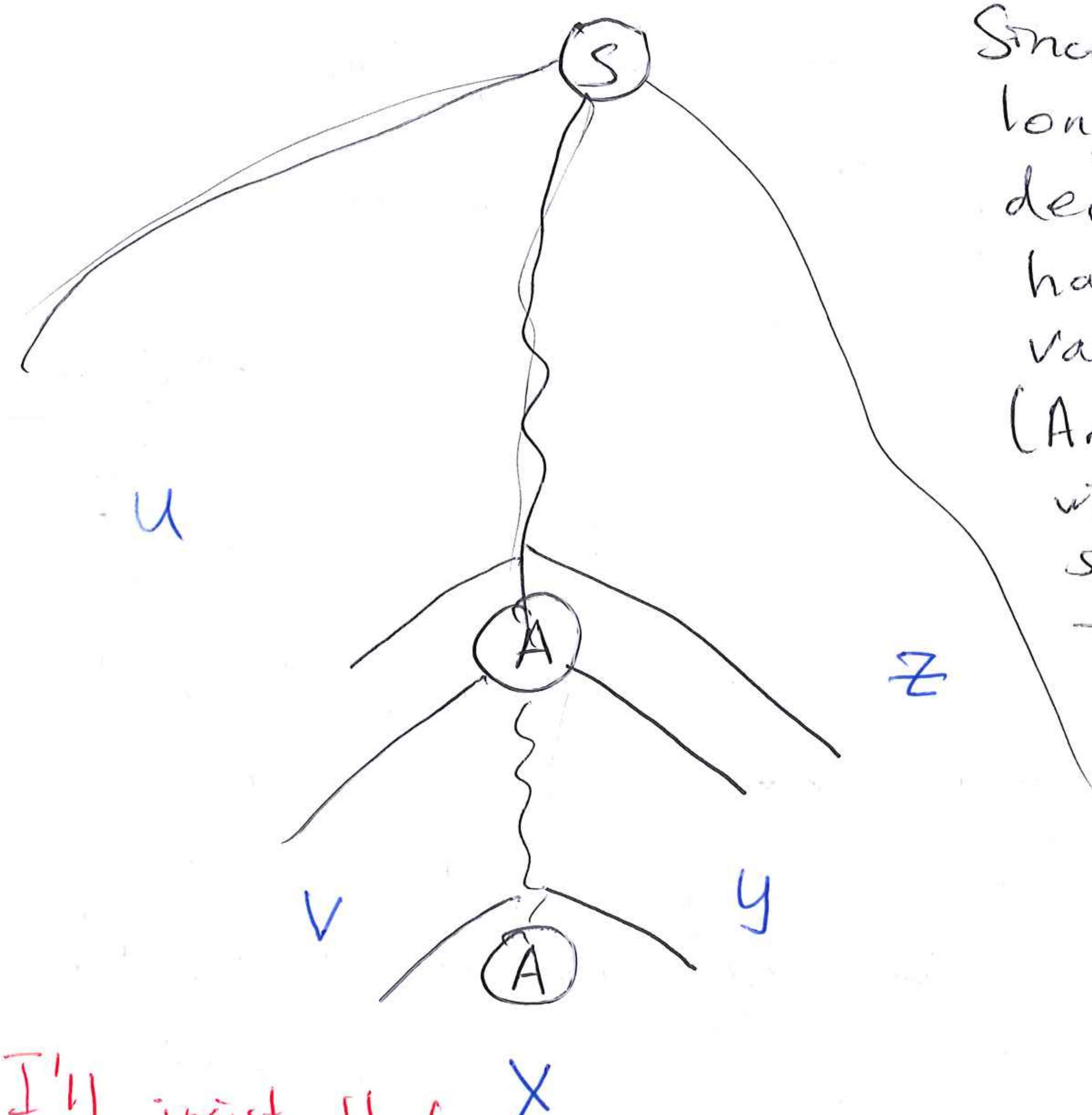
(i.e. it is not possible to make a CFG or PDA for L)

Pumping Lemma for context-free languages:

Let L be a context free language. Then there exists a ~~m~~ positive integer m so that for every string $w \in L$, $|w| \geq m$, ~~we~~ there exist u, v, x, y, z, with ~~uvxy~~ $w = uvxyz$, $|vxy| \leq m$, $vy \neq \lambda$ (i.e. $v \neq \lambda$ or $y \neq \lambda$) so that $uv^i xy^i z \in L$ for all i.

Why is this true?

If L is CF, we have a grammar for it, and our string w has a derivation in the grammar. We'll insist our grammar has no λ or unit prods. Think about deriv tree for w.

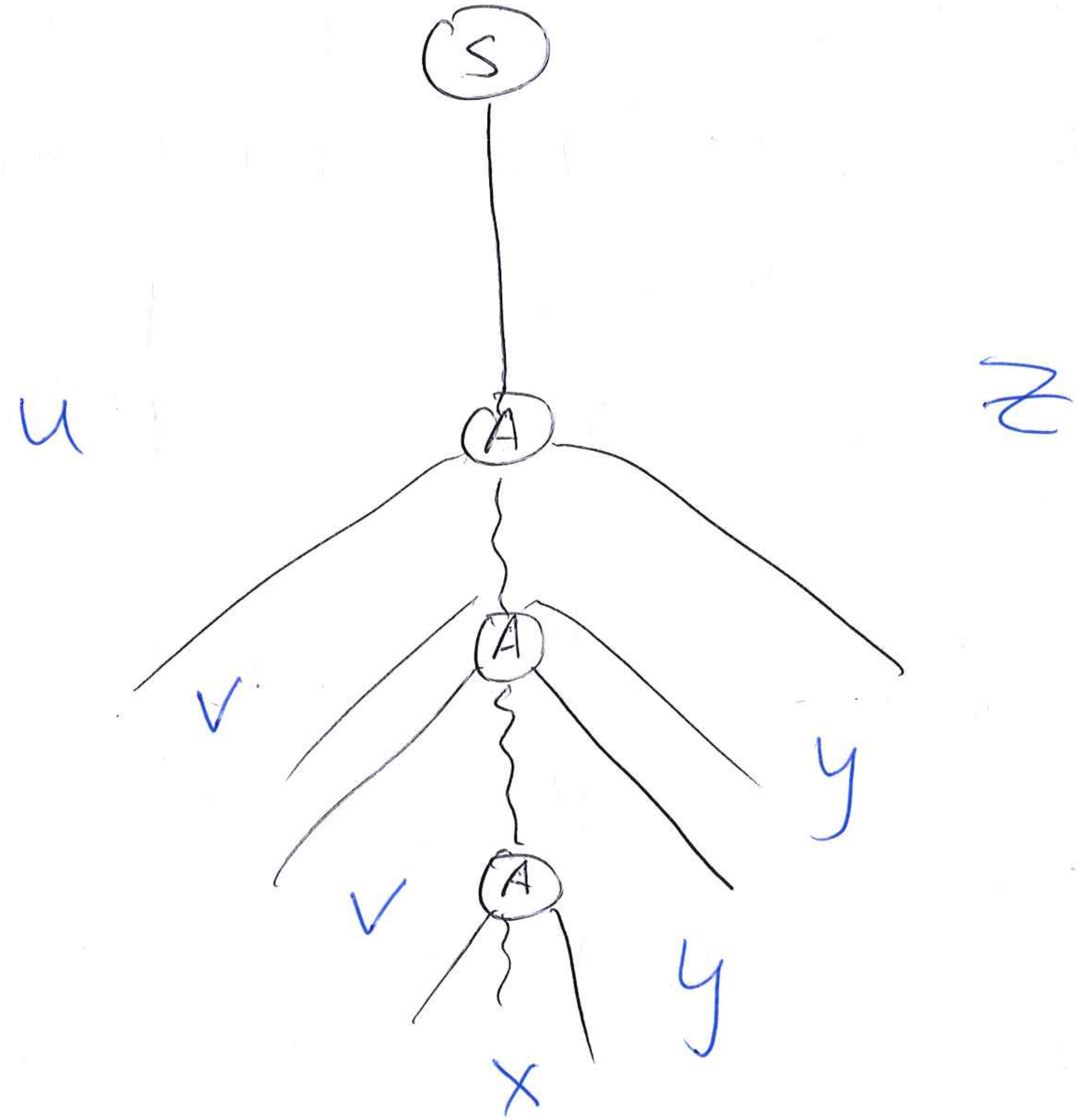


Since w is long, it's derivation has some var twice. (Any deriv with no vars showing up twice must be short.) Not only some var showing up twice, but twice in a path from S.

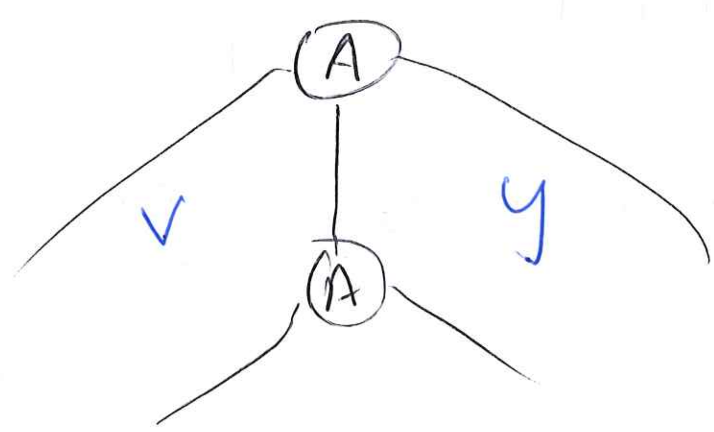
I'll insist that we don't have B — B in v, x, or y.

If we did, we could've picked B — B as our repeat.

If we have this deriv tree, we also have the deriv tree



We can repeat



as many times as we want.

Using the pumping lemma

$$L = \{a^n b^n c^n \mid n \geq 1\}$$

Given m, let

$$w = a^m b^m c^m \in L.$$

$$\underbrace{aa\cdots a}_{m}\underbrace{b\cdots\cdots b}_{m}\underbrace{c\cdots\cdots c}_{m}$$

Consider ways to write $w = uvxyz$,

~~$|xyz| \leq m$.~~ $|vxy| \leq m$.

What are the possibilities for v + y? Starting at left for vxy!

<u>Case 1:</u> $vxy = a^j$ for some j, $1 \leq j \leq m$.

So $vy = a^k$ for some $k \leq j$. $(1 \leq k)$

So $u\cancel{v}x\cancel{y}z = a^{m-k} b^m c^m \notin L$.

<u>Case 2:</u> $vxy = a^j b^k$ for some j, k, $1 \leq j+k \leq m$

(if $j = 0$ or $k = 0$, we'll be in case 1 or 3)

If v or y has both a's + b's, then ~~the~~ uv^2xy^2z will have a's after b's and hence $uv^2xy^2z \notin L$.

~~If~~ Otherwise, ~~$v = a^j$~~ $v = a^p$ & $y = b^q$ for some p & q. Then

$$uv^2xy^2z = a^{m+p}b^{m+q}c^m \notin L$$

since $m+p \neq m$, or $m+q \neq m$. (since $v \neq \lambda$ or $y \neq \lambda$)

<u>Case 3:</u> $vxy = b^j$ for some j, $1 \leq j \leq m$. Then $vy = b^k$ for some $k \leq j$ $(1 \leq k)$, So $uxz = a^m b^{m-k} c^m \notin L$

<u>Case 4:</u> $vxy = b^j c^k$ for some j, k. We can repeat the case 2 argument w/ b's & c's instead of a's & b's.

<u>Case 5:</u> $vxy = c^k$ for some k. We can repeat the case 1 or case 3 argument with c's instead.

Hence, in all cases, we have $uv^ixy^iz \notin L$ for some i, so the pumping lemma fails and L is not context free.