Pumping Lemma for Context-Free Languages:

Goal for today: Show

\[ L = \{ a^n b^n c^n \mid n \geq 1 \} \]

is not context-free

(i.e. it is not possible to make a CFG or PDA for \( L \))

Pumping Lemma for context-free languages:

Let \( L \) be a context-free language, then there exists a positive integer \( m \) so that for every string \( w \in L, \mid w \mid \geq m, \) there exist \( u, v, x, y, z, \) with \( w = uvxyz, \mid vxyl \mid \leq m, vy \neq \lambda \) (i.e. \( v \neq \lambda \) or \( y \neq \lambda \)) so that \( uv^i xy^i z \in L \) for all \( i \).
Why is this true?

If $L$ is CF, we have a grammar for it, and our string $w$ has a derivation in the grammar. We'll insist our grammar has no or unit for $w$.

Since $w$ is long, its derivation has some var twice. (Any deriv with no vars showing up twice must be short.) Not only some var showing up twice, but twice in a path from $S$.

I'll insist that we don't have $X$ in $V, X,$ or $Y$.

If we did, we could've picked it as our repeat.
If we have this deriv tree, we also have the deriv tree.

We can repeat as many times as we want.
Using the pumping lemma

\[ L = \{ a^n b^n c^n \mid n \geq 1 \} \]

Given \( m \), let

\[ w = a^m b^m c^m \in L. \]

\[ \underbrace{a a \ldots a}_{m} \underbrace{b b \ldots b}_{m} \underbrace{c c \ldots c}_{m} \]

Consider ways to write \( w = uvxyz \),

\[ |vxy| \leq m, \quad |vxy| \leq m. \]

What are the possibilities for \( v \) and \( y \)? Starting at left for \( vxy \):

**Case 1**: \( vxy = a^j \) for some \( j \), \( 1 \leq j \leq m \).

So \( vy = a^k \) for some \( k \leq j \). (\( 1 \leq k \))

So \( u^k x y z = a^{m-k} b^m c^m \notin L \).

**Case 2**: \( vxy = a^j b^k \) for some \( j, k \), \( 1 \leq j + k \leq m \).

(If \( j = 0 \) or \( k = 0 \), we'll be in case 1 or 2)

If \( v \) or \( y \) has both \( a \)'s and \( b \)'s, then

\[ uv^2 x y^2 z \]

will have \( a \)'s after \( b \)'s and hence \( uv^2 x y^2 z \notin L \).
Otherwise, \( v = a^p b^q \) for some \( p + q \). Then
\[
uv^2xy^2z = a^{m+p} b^{m+q} c^m \notin L
\]
since \( m+p \neq m \), or \( m+q \neq m \).
(since \( v \neq \lambda \) or \( y \neq \lambda \))

Case 3: \( vxy = b^j \) for some \( j \), \( 1 \leq j \leq m \).
Then \( vy = b^k \) for some \( k \leq j \) (\( 1 \leq k \)),
so \( uxz = a^m b^m k c^m \notin L \)

Case 4: \( vxy = b^j c^k \) for some \( j, k \). We can repeat the case 2 argument with \( b \)'s & \( c \)'s instead of \( a \)'s & \( b \)'s.

Case 5: \( vxy = c^k \) for some \( k \). We can repeat the case 1 or case 3 argument with \( c \)'s instead.

Hence, in all cases, we have \( uv^i x y^i z \notin L \) for some \( i \), so the pumping lemma fails and \( L \) is not context free.