More pumping lemma examples

$L = \{ w^3 w \mid w \in \{a,b\}^* \}$

is not context free:

Given $m$, let

$$w = a^m b^m a^m b^m \in L$$

We need to consider all the possible ways to write $w = uvxyz$, with $|vxy| \leq m$, $v \neq \lambda$. We consider cases from left to right in $w$.

Case 1: $vxy = a^k$ for some $k$, and $z = a^l b^m a^m b^m$. So $vy = a^l$ for some $l$, $1 \leq l \leq m$. Then $uxz = a^{m-l} b^m a^m b^m \notin L$.

(Halfway through is in the middle of the second bunch of $a$'s, so it were some string repeated twice, that string would end with an $a$, but the whole string doesn't end with an $a$.)
Case 2: \(xy = a^p b^q, z = b^m a^r b^m\) for some \(p, q, r\). Then \(yz = a^{p'} b^{q'}\) and \(uxz = a^{m - p'} b^{m - q'} a^m b^m \neq L\).

Case 3: \(xy = b^p, z = b^r a^m b^m\) as in Case 3. We can use the same argument to get \(uxz = a^m b^{m - p'} a^m b^m \neq L\).

Case 4: \(xy = b^p a^q\) for some \(p, q\). We have \(yz = b^{p'} a^{q'}\) for some \(p', q'\), and \(uxz = a^m b^{m - p'} a^{m - q'} b^m\). If \(p' \neq q'\), then the halfway point moved, and the two halves either end w/ different letters or begin w/ different letters. If \(p' = q'\), then the first half has \(m\) a's but the second half has \(m - q'\) a's. So \(uxz \neq L\).

We have 3 more cases, which are just the first 3 cases backwards. Hence \(L\) is not context-free.
What happens if you try to prove some CFL lang is not CFL using pumping lemma?

\[ L = \{ w w^R \mid w \in \{a,b\}^* \} \]

This is context free; a grammar is:

\[ S \rightarrow aSa | bSb | \lambda \]

Let's say we tried to prove this isn't CFL by pumping lemma:

take \( w = a^m b^m a^m \in L \)

We'd in the analogue of case 1+2, we'd still create \( u x z \notin L \).

But for case 3, or case 4,

\[ a \quad \overline{b} \quad a \]

When we deleted a bunch of b's (or repeated some of them many times) - if it's an even # b's, it's still in L.
So this doesn't work. (It shouldn't, since $L$ is context-free)

Another example:

$L = \{ a^j b^k \mid j^2 = k^3 \}$

$L$ is not context-free.

*Pf:* Given $m$, let $w = a^m b^{m^2}$.

We really only need to think about the case where $v = a^p$, $y = b^q$.

(If both were just $a$'s, or both were just $b$'s, deleting throws off equality. If one had both $a$'s & $b$'s, then repeating throws off order. Puts some $b$'s before $a$'s.) Note—all we know about $p + q$ is that $0 \leq p+q \leq m$. Actually, if $p+q$ were not both positive, we'd also be able to delete & throw off equality.
If \( p = 1 \), what would \( q \) have to be so that \( uxz = a^{m-p} b^{m^2-q} \in L \)?

We would need

\[
(m-1)^2 = m^2 - q
\]

\[
m^2 - 2m + 1 = m^2 - q
\]

\[
2m - 1 = q
\]

As long as \( m > 1 \) (which we can assume) this is impossible since \( q < m \).

If \( p \) were bigger? – actually, we would need

\[
(m-p)^2 = m^2 - q
\]

\[
m^2 - 2pm + p^2 = m^2 - q
\]

\[
2pm - p^2 = q
\]

\[
p(2m-p) = q
\]

If \( p \geq 1 \), and

\[
2m - p \geq m \quad \text{(since } p \leq m \text{), so}
\]

\[
p(2m-p) \geq m,
\]

and \( q < m \), so

this is impossible.
Summarize the last page of scratch work.

In this case
\[ uxz = a^{m-p} b^{m^2-q} \]

We need to show that \( uxz \not\in \mathcal{L} \),
or, equivalently,
\[ (m-p)^2 \neq m^2 - q. \]

But, we know that
\[ p \geq 1, \quad 2m-p \geq m, \quad \text{so} \]
\[ p(2m-p) \geq m, \quad \text{and} \quad q < m, \quad \text{so} \]
\[ p(2m-p) \neq q. \]

Hence
\[ 2mp - p^2 \neq q, \]
\[ m^2 - (2mp^2 - p^2) \neq m^2 - q, \]
\[ m^2 = 2mp + p^2 \neq m^2 - q, \]
\[ (m-p)^2 \neq m^2 - q, \]
which is what we wanted to show. \( \square \)