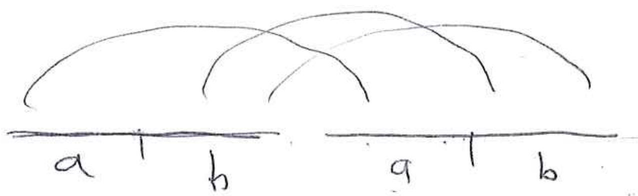
More pumping lemma examples

$L = \{ww \mid w \in \{a,b\}^*\}$

is not context free:



repeated

$w = uvxyz$

Given m, let

$$w = a^m b^m a^m b^m \in L$$

We need to consider all the possible ways to write $w = uvxyz$, with $|vxy| \le m$, ~~v≠~~ $vy \ne \lambda$. We consider cases from left to right in w.

Case 1: $vxy = a^k$ for some k, and $z = a^j b^m a^m b^m$. So $vy = a^\ell$ for some ℓ, $1 \le \ell \le m$. Then $uxz = a^{m-\ell} b^m a^m b^m \notin L$. (Half way through is in the middle of the second bunch of a's, so if it were some string repeated twice, that string would end with an a, but the whole string doesn't end w/ an a.)

Case 2: $vxy = a^p b^q$, $z = b^r a^m b^m$ (for some p, q, r). Then $vy = a^{p'} b^{q'}$, and $uxz = a^{m-p'} b^{m-q'} a^m b^m \notin L$.

Case 3: $vxy = b^p$, $z = b^r a^m b^m$. We can use the same argument as in case 1 to get $uxz = a^m b^{m-p'} a^m b^m \notin L$.

Case 4: $vxy = b^p a^q$ for some p, q. We have $vy = b^{p'} a^{q'}$ for some p', q', and $uxz = a^m b^{m-p'} a^{m-q'} b^m$. If $p' \neq q'$, then the half way point moved, and the two halves either end w/ different letters or begin w/ different letters. If $p' = q'$, then the first half has m a's but ~~$m-q$~~ the second half has $m - q'$ a's. So $uxz \notin L$.

We have 3 more cases, which are just the first 3 cases backwards. Hence L is not context-free.

What happens if you try to prove Some CF lang is not CF using pumping lemma?

$$L = \{ ww^R \mid w \in \{a,b\}^* \}.$$

This is context free; a grammar is

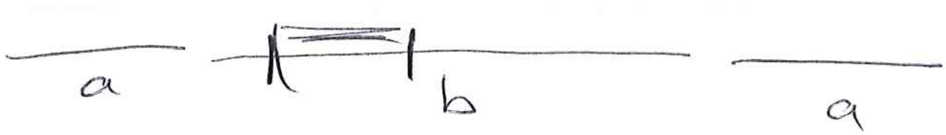
$$S \to aSa \mid bSb \mid \lambda.$$

Let's say we tried to prove this isn't CF by pumping lemma:

take $w = a^m b^{2m} a^m \in L$

~~We~~ In the analogues of case 1 & 2, we'd still create $uxz \notin L$.

But for case 3, or case 4.



When we deleted a bunch of b's (or repeated some of them many times) - if it's an even # b's, it's still in L.

So this doesn't work. (It shouldn't, since L is context-free)

Another example:

$$L = \{ a^j b^k \mid j^2 = k \}$$

L is not context-free.

Pf: Given m, let $w = a^m b^{m^2}$.

We ~~should~~ really only need to think about the case where $v = a^p$, $y = b^q$. (If both were just a's, or both were just b's, deleting throws off equality. If one had both a's & b's, then repeating throws off order. puts some b's before a's) Note - all we know about p & q is that $0 \le p+q \le m$. Actually, if p & q were not both positive, we'd also be able to delete & throw off equality.

If $p=1$, what would q have to be so that $uxz = a^{m-p} b^{m^2-q} \in L$?

We would need

$$(m-1)^2 = m^2 - q$$

$$m^2 - 2m + 1 = m^2 - q$$

$$2m - 1 = q$$

As long as $m > 1$ (which we can assume) this is impossible since $q < m$.

If p were bigger? -- actually, we would need

$$(m-p)^2 = m^2 - q$$

$$m^2 - 2pm + p^2 = m^2 - q$$

$$2pm - p^2 = q$$

$$p(2m-p) = q$$

$p \geq 1$, and $2m - p \geq m$ (since $p \leq m$), so $p(2m-p) \geq m$, and $q < m$, so this is impossible.

Summarize the last page of scratch work.

In this case

$$uxz = a^{m-p} b^{m^2-q}.$$

We need to show that $uxz \notin L$, or, equivalently,

$$(m-p)^2 \neq m^2 - q.$$

But, we know that

$$p \geq 1,\ 2m-p \geq m, \text{ so}$$

$$p(2m-p) \geq m, \text{ and } q < m, \text{ so}$$

$$p(2m-p) \neq q.$$

Hence

$$2mp - p^2 \neq q,$$

$$m^2 - (2mp - p^2) \neq m^2 - q,$$

$$m^2 - 2mp + p^2 \neq m^2 - q,$$

$$(m-p)^2 \neq m^2 - q,$$

which is what we wanted to show. $\square$