RESEARCH STATEMENT

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My research interests lie broadly on objects in algebraic geometry and commutative algebra whose explicit study is amenable to combinatorial methods, and on the resulting combinatorics. Sometimes these objects are defined combinatorially but usually they have purely geometric or algebraic origins. Frequently the combinatorics arise from a group action on the objects involved, so representation theory becomes an important ingredient. While the results are usually interesting for purely geometric or algebraic reasons, they can also yield interesting new combinatorial formulas, or interesting new explanations of known ones.

Within this broad area I have two ongoing lines of research. One is to combinatorially construct free resolutions of modules which have or might be expected to have combinatorial descriptions, while the other is to closely study Schubert and related varieties, with an emphasis on their singularities. I will describe each of these separately, giving the necessary background and describing results and further directions in each section.

1. Combinatorial Free Resolutions

1.1. **Background.** Let S be the ring $k[x_1, x_2, \dots, x_n]$, where k is a field, with the usual grading by degree. Now let M be a finitely generated module over S. A **free resolution** of M is a sequence

$$0 \xrightarrow{\delta_{n+1}} F_n \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_{i+1}} F_i \xrightarrow{\delta_i} \cdots \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} 0$$

of free S-modules F_i and S-linear maps δ_i such that $M = \text{coker } \delta_1$ and, for each $i > 0$, $\text{im } \delta_{i+1} = \ker \delta_i$. Hilbert's Syzygy Theorem states that a finite free resolution always exists. Usually for me, M will be a graded module; in this case, M has a minimal graded free resolution which is unique up to homotopy. Therefore numerical data extracted from any minimal free resolution are invariants giving information about M. The finest invariants are the **graded Betti numbers** β_{ij} of M, where β_{ij} is the number of elements of degree j in a minimal generating set for F_i. From this information it is possible to calculate other useful invariants. For example, the **Hilbert series** is the generating series $H_M = \sum_{i\in\mathbb{Z}} \dim_k(M_d)t^d$, where M_d denotes the degree d part of M. We can use a free resolution to calculate this invariant since $H_M = \sum_i (-1)^i H_{F_i} = \sum_{i,j}(-1)^i \beta_{ij} t^j/(1-t)^n$. Some properties of M can also be determined from a free resolution. For example, let c be the codimension of the support of M; then M is **Cohen-Macaulay** if there is a free resolution with $F_{c+1} = 0$. If, in addition, F_c has rank 1 and $M = S/I$ for some ideal I of S, S/I is a **Gorenstein ring**. See [13] or [8] for more details on these notions and their significance.

Given a S-module M having some combinatorial structure, one can hope to construct a free resolution of M having a related combinatorial structure. For example, a free generating set for the F_i might be naturally indexed by certain combinatorial objects, or, at least, by terms in a combinatorial formula. Two cases gave been extensively studied in the literature. One case is the theory of resolutions of monomial ideals, where M is an ideal of S generated by monomials, and the related theory of resolutions of semigroup rings $k[Q]$ (for a given surjection of semigroups $\mathbb{N}^n \to Q$ making $k[Q]$ into an $S = k[\mathbb{N}^n]$-module). Much of this theory is explained in [28]. A second case consists of determinantal ideals, which are the ideals generated by all minors of a given fixed size inside a fixed matrix of variables, and, more broadly, Schur and Weyl modules which arise from the modular representation theory of GL_n. Here we have a compact but purely theoretical construction in characteristic 0 [34] using Lie theory, and a body of work giving more or less explicit descriptions, including [9, 1, 29, 2, 19, 12]; a few cases are still open in positive characteristic. While the more theoretical constructions are transparent and have obvious combinatorial structure, they give no method for writing down the actual maps δ_i. On the other hand, more explicit constructions give a method for writing down the maps, but these constructions are much more complicated, and their full combinatorial structure is less readily apparent.

The symmetric group S_n acts on our ring S by permuting the variables. If our S-module M is graded, and S_n acts on M in a way compatible with the S-module structure (making M into a left module over the skew group ring $S * S_n$), we can associate a **Hilbert-Frobenius series** to M; this is a symmetric function analog of the Hilbert series. To every S_n representation V we can associate a symmetric function $\mathrm{ch}(V)$ called the **Frobenius characteristic** of V [31, Sect. 7.18]. This has the property that, for each partition λ of n, $\mathrm{ch}(V^\lambda) = s_\lambda$ for the irreducible representation V^λ; furthermore, ch is additive on direct sums. The Hilbert-Frobenius series of M is then $\mathrm{ch}(M) = \sum_d \mathrm{ch}(M_d)t^d$. If there is also an S_n action on each F_i commuting with the maps δ_i, then $\mathrm{ch}(M) = \sum_{i=0}^{n}(-1)^n \mathrm{ch}(F_i)$.

1.2. **Thesis Work.** In my thesis work I have been exploring the free resolutions of a class of S-modules defined, in one way among others, in terms of the representation theory of S_n acting on S itself. Each irreducible representation V^λ of S_n occurs in S. Furthermore, there exists a positive integer $n(\lambda)$ such that V^λ does not occur in S in degree less than $n(\lambda)$, and V^λ has multiplicity 1 in the degree $n(\lambda)$ part of S. Let I_λ denote the ideal of S generated by this occurrence of V^λ. Now let $J_\lambda = \sum_{I_\mu \subsetneq I_\lambda} I_\mu$, and let M_λ be the module defined by $M_\lambda = I_\lambda / J_\lambda$. The symmetric group S_n acts on I_λ and M_λ via its action on S. I have conjectured that $\mathrm{ch}(M_\lambda) = H_\lambda(\mathbf{x};t)/\varphi_\lambda(t)$, where $H_\lambda(\mathbf{x};t)$ is a Hall-Littlewood polynomial, and $\varphi_\lambda(t)$ is a particular polynomial in t depending on λ. Furthermore, the generators of a particular (nonminimal) free resolution of M_λ appear to correspond to terms in a particular formula for $H_\lambda(\mathbf{x};t)/\varphi_\lambda(t)$, giving an algebraic explanation of this formula; I have constructed this resolution in some special cases. These resolutions also seem to be related to those of determinantal ideals mentioned above, and may give new insight into their structure.

1.3. **Proposed research.** Several other classes of S-modules may also have combinatorial free resolutions. One class consists of the cohomology ring for Springer fibers, which were given an explicit elementary description in [11, 15]. These S-modules are also S_n representations, and their Frobenius series are $\widetilde{H}_\lambda(\mathbf{x};t)$, a variant of the Hall-Littlewood polynomials. Finding free resolutions for them should lead to new explanations of formulas, probably known but possibly new, for the Hall-Littlewood polynomials. Another class of S-modules with interesting free resolutions are the ideals defining matrix Schubert varieties, originally introduced in [14] and further studied in [23]. These include as a special case the determinantal ideals mentioned above. While the complications in earlier explicit results on free resolutions of determinantal ideals appear at first to be quite discouraging, we now have both new information about matrix Schubert varieties and computational algebra systems such as Macaulay 2, which can compute small but non-trivial examples and help in computing larger ones. These advantages may make the problem of constructing these resolutions tractable. A solution for this problem should also lead to simpler descriptions of the known resolutions in the classical determinantal case. Finally, the ideals defining graph varieties [27] are also potentially interesting and understanding their free resolutions may lead to better understanding of certain notions in graph or matroid theory.

A **hyperplane arrangement** is simply a collection of m hyperplanes $H_1, \ldots, H_m$ in some vector space k^d. A **subspace arrangement** contained in a hyperplane arrangement is a collection of subspaces $V_1, \ldots, V_n$ such that each V_l is the intersection $\bigcap_{t \in T} H_t$ for some subset T of the hyperplanes. Let $\mathcal{B}_d$ be the **braid arrangement**, which consists of the hyperplanes defined by $x_i - x_j = 0$ whenever $1 \leq i \leq j \leq d$; then I_λ is the ideal of a particular subspace arrangement X_λ contained in $\mathcal{B}_d$. I intend to study the relationship of free resolutions of M_λ to the geometry and combinatorics of X_λ relative to $\mathcal{B}_d$. This geometric viewpoint may shed further light on the free resolutions of M_λ and I_λ, giving a description of the resolutions using the combinatorics of the arrangement. Such a result would also be a useful example which, combined with the known aforementioned case of monomial ideals, could lead to a general theory relating the combinatorics of an arrangement to free resolutions of modules describing parts of the arrangement.

2. Singularities of Schubert Varieties

2.1. **Background.** Let G be a semisimple reductive algebraic group, $B \subseteq G$ a Borel subgroup, and $T \subseteq B$ the maximal torus. For example, G could be $\mathrm{SL}_n(\mathbb{C})$ and B the subgroup of upper triangular matrices, in which case T consists of all diagonal matrices with determinant 1. Let $W = N(T)/T$ denote the **Weyl group**, where $N(T)$ is the normalizer of T in G; in the case where $G = \mathrm{SL}_n$ it is the symmetric group S_n. The quotient G/B is called the **flag variety**, and it has a cell decomposition

$$G = \bigcup_{w \in W} BwB/B$$

known as the **Bruhat decomposition**. The closure of BwB/B in G/B is known as a **Schubert variety** and will be denoted X_w.

It is known which Schubert varieties are singular [25, 4, 6]. In the case where $G = \mathrm{SL}_n$, it is known precisely where the singularities of X_w are [10, 26, 7, 21, 16] and what X_w looks like near a generic singular point [10]. The results referenced are stated in terms of a combinatorial condition on elements of W known as pattern avoidance [5, 6].

A possibly singular point p on a variety X is **Gorenstein** if its local ring $\mathcal{O}_{X,p}$ is Gorenstein; all smooth points are Gorenstein. A variety is considered to be Gorenstein if all of its points are Gorenstein. Roughly speaking, being Gorenstein means that a variety is as good as smooth for some purposes, for example Serre duality. A **dualizing sheaf** for a variety is an analog for the purposes of Serre duality of the sheaf of top exterior forms on a smooth variety. Precise definitions can be found in [13] and [18].

Another measure of how singular a point p is on a variety X is its **multiplicity**. This is defined as the degree of the projective tangent cone $\mathrm{Proj}(\mathrm{gr}_{\mathfrak{m}_p}\mathcal{O}_{X,p})$, which is the projective variety whose homogeneous coordinate ring is the associated graded ring of $\mathcal{O}_{X,p}$, using the filtration induced by the maximal ideal $\mathfrak{m}_p$. In particular, the multiplicity is 1 at p if and only if p is a smooth point on X. In this case there are known combinatorial formulas for multiplicities when the Borel subgroup B is replaced by a maximal parabolic subgroup P, so that the known results describe multiplicities of singular points on Schubert varieties of Grassmannians [22, 24, 30, 20] or symplectic Grassmannians [17] rather than of points on Schubert varieties of flag varieties.

2.2. **Completed Work.** In joint work with Alexander Yong [33], we have combinatorially characterized which Schubert varieties are Gorenstein for $G = \mathrm{SL}_n$ using a new variant of pattern avoidance. This was accomplished not by a local analysis of singularities but by global computations involving the class group and a previously known computation of the dualizing sheaf. In addition, we have a new formula for the dualizing sheaf in the case where X_w is Gorenstein. This formula gives new information about the geometry of Gorenstein Schubert varieties and in particular may provide a useful clue to those working on calculating the sheaf cohomology of non-dominant line bundles on Schubert varieties [3].

I have also calculated the multiplicity of the most singular point on a Schubert variety of G/B for a single isolated case when $G = \mathrm{SL}_n$ [32]. These multiplicities turn out to be Catalan numbers, and in this paper I also explore some combinatorics which relate the multiplicities to some classical sets also enumerated by the Catalan numbers and one of their q-analogs.

2.3. **Proposed Research.** We hope to extend this work to all of the other reductive algebraic groups. This should be possible using essentially only the methods used when $G = \mathrm{SL}_n$, along with an extension of our variant of pattern avoidance along the lines of [5, 6]. In addition, in parallel with the study of which points on X_w are singular, we would like to determine for non-Gorenstein X_w exactly which points on them are not Gorenstein. If global techniques do not suffice for this problem, it might be solved by analyzing the singularities locally. A free resolution of the local rings would give more than enough information; these local rings are similar to the coordinate rings of matrix Schubert varieties mentioned above, and should therefore have related free resolutions.

I also hope to make more calculations towards a general combinatorial formula for multiplicities of points on Schubert varieties. Multiplicity is a rather coarse invariant, and progress might be more easily made by attempting to compute finer invariants such as a multi-graded multiplicity or even an entire free resolution of the local ring $\mathcal{O}_{X,p}$ or its associated graded ring $\mathrm{gr}_{\mathfrak{m}_p}\mathcal{O}_{X,p}$.

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