RESEARCH STATEMENT

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This statement outlines some of the research questions I have been recently considering. I have outlined far more questions than I can pursue in depth. My plan is to pursue only the most promising of these lines of research as time and interest allow. All of these problems are amenable to investigation by calculation using computers in concrete examples, although I have given descriptions of the problems in more theoretical terms. Undergraduates can participate in solving small parts of these problems; indeed, I advised three students this past summer working on a problem mentioned at the end of Section 2.

My research is on specific spaces and rings from algebraic geometry and commutative algebra which can be explicitly studied by combinatorial methods, and on the resulting combinatorics. Frequently the combinatorics come from a group action on the space or ring, so representation theory becomes an important ingredient. Such study hopefully gives us interesting information about specific geometric or algebraic information about the objects involved. It can however also lead to either new formulas and concepts which are entirely combinatorial, or to interesting new explanations or interpretations of known combinatorial facts.

I take the opportunity here to give in some detail the general and historical background to Schubert varieties, which is the class of objects I have done the most work on. Other geometric objects I am studying include Springer fibers and Hilbert schemes of points in the plane. On the algebraic side, I am interested in understanding the action of S_n on the polynomial ring, and in homological questions concerning rings coming from some combinatorial questions. I will describe these other problems more briefly and in a more technical manner.

1. Schubert varieties and their singularities

1.1. **General and historical background.** Consider the following problem first studied in the 19th century. Fix four randomly chosen lines in 3-space, and to distinguish them later color all of these lines red. (To be precise, for technical reasons, we should use complex projective lines in complex projective 3-space $\mathbb{P}^3$.) Now we want to know the number of lines which touch all four red lines. Enumerative geometers in the 19th century developed methods to get answers to this and similar problems, culminating in the work of Schubert [26], Pieri, and others. For this particular problem, it turns out the answer is two. Their methods, however, were not entirely rigorous, and, indeed, the 19th century geometers knew that, while they seemed to always get correct answers when working with linear spaces, a naive application of their methods could give incorrect answers for more complicated problems, such as that of counting the conics tangent to five other conics in the plane. Hilbert in his list of problems in 1900 asked, as his Fifteenth Problem, for a rigorous justification for these methods.

The solution to Hilbert's problem, completed in the 1960s, is as follows. The **Grassmannian** (of lines in 3-space) is a geometric object whose points correspond one-to-one with lines in $\mathbb{P}^3$, with nearby points in the Grassmannian corresponding to nearby lines in $\mathbb{P}^3$. For each red line, we define a **Schubert variety** as the set of points corresponding to lines which touch our red line; this is a closed subset of the Grassmannian. The class of the Schubert variety in the homology of the Grassmannian is independent of which line is our red line, and intersection corresponds to cup product in cohomology, so we take the fourth power of the cohomology class Poincaré dual to the homology class of this Schubert variety, and find that it is twice the cohomology class Poincaré dual to the class of a point. This tells us, assuming the red lines were not in some particularly special position relative to each other, that the four Schubert varieties intersect in two points, which correspond to the two lines in the original answer.

This problem can be generalized, not only to higher dimensional planes in higher dimensional spaces, but also to intersection conditions on **flags**. A flag is a chain of linear subspaces $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_\ell \subseteq \mathbb{C}^n$, and we can ask how many flags (of vector spaces of certain specified dimensions) intersect some fixed number of other flags so as to satisfy various dimension conditions on the vector spaces in the flags. Finding ways to count solutions to such intersection problems is the subject of **Schubert calculus**, which aims to find formulas giving the number of solutions from the intersection conditions. It is possible to solve these problems by multiplying certain polynomials, known as **Schubert polynomials**, expanding the result as a linear combination of Schubert polynomials, and looking at a particular coefficient. However, although from the geometry it is obvious that the answer must be a positive integer, this algebraic method gives no hint as to why the answer must be positive. Current research in Schubert calculus seeks combinatorial formulas from which one can tell, just by looking at the formula, that one will get a nonnegative number. In the original case of single vector spaces (rather than general flags), such rules, of which there are many, are known as Littlewood-Richardson rules.

My research focuses not on Schubert calculus, but on the Schubert varieties themselves, and particularly their singularities. There is a **flag variety**, whose points correspond to flags just as points in the Grassmannian correspond to lines (or, in general, planes of various dimensions). In general, if we fix a base flag which we color red, we can define a **Schubert variety** to be the set of all points in the flag variety corresponding to flags which satisfy some prescribed set of intersection conditions with the red base flag. For the case of complete flags (with a subspace of each dimension), the reasonable intersection conditions correspond to permutations in the symmetric group S_n. In general, a Schubert variety can be **singular**, meaning that they can have points with neighborhoods which do not look like an open subset of $\mathbb{C}^n$. For example, in the Schubert variety of lines in $\mathbb{P}^3$ meeting a fixed red line mentioned above, neighborhoods of the point corresponding to the red line itself look like the product of the cone defined in $\mathbb{C}^3$ by $x^2 + y^2 = z^2$ with a (complex) line.

There are a number of topological and algebro-geometric invariants measuring how unlike open subsets of $\mathbb{C}^n$ the neighborhoods of a point are. I am particularly interested in combinatorial formulas for the values they take at various points of a Schubert variety. The singular points of Schubert varieties were independently given a combinatorial characterization, in terms of the permutation the Schubert variety corresponds to, by four different groups [9, 24, 7, 19, 15]. For Schubert varieties on Grassmannians, Krattenthaler [21] showed that the multiplicity (one measure of how unlike $\mathbb{C}^n$ the neighborhood is) at any particular point is the number of systems of nonintersecting lattice paths with certain specified starting and ending points. Another topological measure is given by Kazhdan-Luzstig polynomials, which are of independent interest in representation theory.

1.2. **Past Work.** I gave jointly with Alexander Yong [32] a combinatorial characterization of which Schubert varieties are Gorenstein. A Gorenstein variety is one that behaves like a nonsingular one for certain algebro-geometric purposes. Some features of this characterization led us to develop in [33] the purely combinatorial notion of **interval pattern avoidance**, which generalizes classical pattern avoidance. In the classical definition, a permutation $w \in S_n$ avoids a permutation $v \in S_m$ if none of the length m (not necessarily consecutive) subsequences of w have elements in the same order as v. Our new definition generalizes the notion of pattern avoidance to intervals in Bruhat order, a partial ordering on S_n, rather than individual permutations. We prove our notion is general enough to provide in principle a combinatorial answer to any local geometric question about Schubert varieties.

The notion of Schubert varieties can be generalized to any simple Lie group; the definition above is for the case of the group SL_n. Even less is known about the singularities for these generalized Schubert varieties; it is only known which ones have a singular point [22, 4, 6], but not which points are singular. I have further generalized in [31] the notion of pattern interval avoidance so that it can provide combinatorial answers to questions about these Schubert varieties, using in part a notion of pattern avoidance due to Billey and Postnikov [6] and given a geometric interpretation by Billey and Braden [5].

1.3. **Current Projects.** In collaboration with Alexander Yong, I am working to use this definition to give a combinatorial characterization of the singular points in these more general Schubert varieties, and to characterize which of them are Gorenstein. We are also interested in seeing where various other properties of singularities hold and fail to hold.

I am also working with Isaiah Lankham on the purely combinatorial problem of enumerating the permutations which avoid particular intervals. The Stanley-Wilf conjecture, now proven by Marcus and Tardos [25], states that the number of permutations in S_n which avoid a certain pattern is bounded by c^n for some constant c depending only on the pattern. Our aim is to find out if this property also holds for interval pattern avoidance.

Furthermore, I have recently started a project with Victor Reiner and Alexander Yong to find short presentations for the cohomology rings of Schubert varieties, and another project with Brant Jones to understand how Deodhar's algorithm for calculating Kazhdan-Luzstig polynomials [12] relates to partial resolutions of singularities for Schubert varieties.

2. Hilbert schemes of points in the plane and S_n representations of the polynomial ring

The **Hilbert scheme** of n points in $\mathbb{C}^2$ is a geometric object whose points correspond to ideals I of $\mathbb{C}[x, y]$ such that $\mathbb{C}[x, y]/I$ has dimension n as a vector space. Generically, the ideal I vanishes at n distinct points of $\mathbb{C}^2$, but otherwise some of these points can coincide, and, vaguely speaking, the ideal in such a case keeps the information not only of how many points lie on top of each other, but also of a set of directions from which the points came together.

There is a correspondence under a general framework of Bridgeland, King, and Reid [8] and established for the Hilbert scheme by Haiman [17] between sheaves of modules on the Hilbert scheme and S_n-equivariant modules on $\mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$. (Technically, this is a correspondence only on the derived categories.) Given an ideal I vanishing at n distinct points, we can associate the point on the Hilbert scheme corresponding to I with the $n!$ ways to list in some order the n points at which I vanishes, thereby giving $n!$ points in $(\mathbb{C}^2)^n$. Roughly speaking, the correspondence extends this natural association to the remainder of the Hilbert scheme.

The Hilbert scheme has a decomposition into cells indexed by monomial ideals such that a point corresponding to an ideal I is in the cell of the monomial ideal M if M is the ideal of leading terms (for a given term order) of elements of I. In joint work with Mark Haiman we show that closed unions of such cells correspond to particular unions of subspaces of $\mathbb{C}^{2n}$ which we call **partition varieties**

Algebraically, we show that the ideals of polynomials which vanish on partition varieties are particularly distinguished in the representation theory of S_n on the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ in one set of variables. In the decomposition of $\mathbb{C}[x_1, \ldots, x_n]$ into irreducible representations of S_n, there is a lowest degree in which any particular irreducible representation appears, and it appears with multiplicity one in that degree. These ideals defining these unions of subspaces are generated by these particular irreducible subrepresentations.

We also link these ideals to the cohomology rings of Springer fibers, which are particular closed subsets of the flag variety defined in the next section. Characters of S_n representations have a standard encoding as symmetric functions, and the characters of graded representations (such as the polynomial ring) can be encoded as symmetric functions with coefficients in $\mathbb{Q}[t]$. Under this encoding, the cohomology rings of Springer fibers are well known [18, 10] to have characters which are Hall-Littlewood polynomials, which are a particular special basis of symmetric functions over the field $\mathbb{Q}(t)$. Our work shows that these ideals also have graded characters which are specific sums of a slightly different version of the Hall-Littlewood polynomials.

I am hoping that this connection can also give insights into other ideals generated by irreducible representations, and to questions about bases for the polynomial ring which are compatible with the decomposition into irreducible S_n representations. One such basis was found by Allen [1], and a similar basis was independently discovered by Ariki, Terasoma, and Yamada [2], but conjectures about other bases remain unproven. Allen also proposed a conjectured basis for the cohomology ring of the Springer fiber which is related to the basis he found for the entire polynomial ring. Three undergraduate students worked with me this past summer on trying to use the connection with the ideals mentioned above to prove this conjecture.

3. Springer fibers

Let X be a nilpotent $n \times n$ matrix. The Springer fiber for X is the set of all complete flags $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n$ such that $XV_i \subseteq V_{i-1}$, considered as a closed subset of the flag variety. These were studied in depth, for example in [10, 29, 30, 20, 28] because they gave a natural geometric construction of the irreducible representations of S_n as well as a natural interpretation for Hall-Littlewood polynomials. (These constructions also extend, with somewhat weaker statements, for other Lie groups and their Weyl groups.) The Springer fiber has a cell decomposition due to Shimomura [27]. I am interested in finding formulas for the homology classes (in the homology of the flag variety) for the closures of these cells.

Beyond intrinsic interest, there is the hope that these classes may form part of a basis for S_n-modules defined by Garsia and Haiman [13] to help in proving the Macdonald positivity conjecture. Macdonald polynomials are a basis of symmetric functions over $\mathbb{Q}(q,t)$ with some particularly nice properties, and Macdonald postivity is the statement, conjectured by Macdonald [23] that the expansion of a Macdonald polynomial as a linear combination of Schur functions produces coefficients which are polynomials with positive integer coefficients. While Haiman has used the algebraic geometry of the Hilbert scheme to prove that the Garsia-Haiman modules have dimension $n!$ [17], and as a corollary the Macdonald positivity conjecture, explicit bases for the modules remain to be found. Haglund, Haiman, and Loehr [16] proved a combinatorial formula for Macdonald polynomials which in particular suggests a particular indexing set for such a basis, and parts of this formula are the same as the Shimomura's formula for the dimensions of his cells. This similarity suggests that formulas for these homology classes, written as polynomials, might help with finding such a basis.

4. Regularity of graph coloring varieties

In a new attempt to find new and hopefully more efficient methods for finding all colorings of a graph, de Loera, Lee, Margulies, and Onn [11] recently studied graph coloring varieties, which are finite sets of points, each of which corresponds to a coloring of the graph. (Graph coloring varieties were originally introduced by Bayer [3].) In addition, these varieties are defined by a system of polynomial equations, one for each edge, which are easy to write down. However, since graph coloring is an NP-complete problem, one expects that solving this system of equations, or even finding out if this system of equations has any solutions at all, will take an exponential amount of time as a function of the number of edges. Nevertheless, computations show that, for many families of graphs, finding a solution is quite easy.

The **Castelnuevo-Mumford regularity** of a system of polynomial equations is a number with a rather technical definition involving homological algebra, specifically Tor; among other applications, regularity gives an upper bound on the complexity of finding out if the system has a solution. Along with Jesus de Loera, Chris Hillar, Susan Margulies, and several other potential collaborators, I am trying to gain some theoretical understanding of why many computations are actually efficient in this case, in part by trying to understand the regularity of these systems of equations.

Similar systems of equations are defined for other combinatorial and graph-theoretic problems such as that of finding a Hamiltonian cycle, and we are interested in considering similar questions for those systems.

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