RESEARCH STATEMENT

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1. Introduction

My research lies at the intersection of combinatorics and algebraic geometry. In general, I study specific details of various particular geometric objects which have rich and interesting combinatorial structures. I am interested both in using combinatorics to describe the geometry of the objects involved and in using the geometric objects to prove purely combinatorial theorems. As tools for this study, I also use commutative algebra, as I often work with explicit equations defining these objects, and representation theory, as frequently both the geometric and combinatorial structure arises from group actions on the geometric objects. This work frequently inspires purely combinatorial questions which are interesting and accessible problems for undergraduate research.

My previous research has included work on the Hilbert schemes of points and their relationship to symmetric functions. I have also extensively studied toric varieties and their relationship to polytopes. However, my current research focuses on Schubert varieties and other subvarieties of flag varieties and their relationship to the combinatorics of Coxeter groups such as the symmetric group. I am particularly interested in local geometric properties of singularities and in combinatorial characterizations of where and on which Schubert varieties various properties hold.

Schubert varieties were first studied because of the role they play in partially solving Hilbert's Fifteenth Problem. Nineteenth century mathematicians developed various methods to give answers to questions in enumerative geometry, culminating in a classic treatise by Schubert [Sch79]. For example, one of the simplest nontrivial questions they answered was to count the number of lines intersecting four generically positioned given (complex projective) lines in (complex projective) 3-dimensional space. Hilbert asked for a rigorous justification for their methods. There is a geometric object, an example of a Grassmannian, which parameterizes lines in 3-space, and the subset parameterizing those lines intersecting a given line is one example of a Schubert variety. Therefore, the enumerative geometry question can be reinterpreted as counting the points in the intersection of four particular Schubert varieties inside the Grassmannian. This number can in turn be calculated using intersection theory. These calculations turn out to have interesting combinatorial underpinnings and interpretations. Research into the combinatorics of these calculations and the geometry underlying these combinatorial rules continues and is known as modern Schubert calculus.

My main interests are not in Schubert calculus but in the geometry of the Schubert varieties themselves, and especially their singularities. One reason for studying singularities of Schubert varieties is their connection with Kazhdan–Lusztig polynomials [KL79]. These polynomials have interpretations in the representation theory of Lie algebras [BK81] as well as the representation theory of the symmetric group and of finite groups of Lie type [KL79]. However, they are also measures of how far certain points on a Schubert variety are from being smooth [KL80], so they can be studied from this viewpoint. Another reason for studying singularities of Schubert varieties is that it leads to interesting combinatorics, in particular combinatorics related to the symmetric group and other Coxeter groups. Among the combinatorics that has arisen are new notions of pattern avoidance [BB03, BP05, 2, 3], which is a subject originally from theoretical computer science, and Bruhat graphs, which are graphs on certain sets of permutations. Furthermore, Schubert varieties provide a large class of examples of fairly but not completely well behaved higher dimensional varieties, so they can be an important testing ground for developing new ideas in symplectic and algebraic geometry.

The remainder of this statement is a more technical introduction to the subject, a description of my past contributions and ongiong work, and ideas for future work. I know many of the details will be difficult for non-specialists to fully understand, but I hope to convey the spirit and overall structure of my research. At the end I also include a section on ongoing and potential future work with undergraduate students related to some of these new combinatorial ideas.

2. Definitions

Let G be the general linear group $\mathrm{Gl}_n(\mathbb{C})$. We pick a basis of $\mathbb{C}^n$, identifying G with a group of matrices, and let B be the subgroup of upper triangular matrices, and T the subgroup of diagonal matrices. The quotient G/B forms a smooth complex algebraic variety known as the **flag variety**.

The fixed points of G/B under the action of T are precisely the cosets of the permutation matrices. For $w \in S_n$, let e_w be this point. The flag variety has a cell decomposition into the cells C_w where C_w is the B-orbit of e_w; these cells are known as **Schubert cells**. The closure of C_w is known as a **Schubert variety** and denoted X_w.

Some notions from the combinatorics of the symmetric group will also be helpful. The dimension of C_w and X_w are equal to the **length** of the permutation w, denoted $\ell(w)$. If we take the adjacent transpositions s_i switching i and $i+1$, for all i, $1 \leq i \leq n-1$, to be our generating set for S_n, then $\ell(w)$ is defined algebraically as the smallest number of generators needed to write w. Combinatorially, $\ell(w)$ can be equivalently defined as the number of pairs $i < j$ with $w(i) > w(j)$.

Each Schubert cell C_w is isomorphic (in the sense of algebraic geometry, which in turn implies holomorphic and diffeomorphic as well) to $\mathbb{C}^{\ell(w)}$. A Schubert variety X_w partitions into Schubert cells, and a partial order known as **Bruhat order** can be defined by declaring that $v \leq w$ if $C_v \in X_w$. Bruhat order also has a number of alternative combinatorial definitions. For example, one definition is that $v \leq w$ if there exist transpositions (switching two arbitrary letters) $t_1, \ldots, t_j$ such that $vt_1 \cdots t_j = w$ and $\ell(v) < \ell(vt_1) < \ell(vt_1t_2) < \cdots < \ell(vt_1 \cdots t_j)$. The **Bruhat graph** of a permutation w is the graph whose vertices are labelled by permutations $v \leq w$ with an edge between u and v if $u = vt$ for some transposition t. This graph encodes much of the geometry of X_w.

A permutation $v \in S_m$ is said to **embed** in $w \in S_n$, with $m \leq n$, if there is some set of positions in w whose entries are in the same relative order as the entries of v. Stated precisely, this means there exist indices $i_1 < i_2 < \cdots < i_m$ such that for all j, k, $1 \leq j, k \leq m$,

$w(i_j) < w(i_k)$ if and only if $v(j) < v(k)$. The permutation w **(pattern) avoids** v if v does not embed in w.

Both Schubert varieties and these various combinatorial ideas generalize respectively to other semisimple Lie groups and their associated Weyl groups. The combinatorics also generalizes to the broader class of Coxeter groups.

3. Questions about Schubert varieties

Schubert varieties are not always smooth; in other words, some points on X_w may not have neighborhoods which are diffeomorphic (or holomorphic) to $\mathbb{C}^{\ell(w)}$. Therefore it is a natural question to ask which Schubert varieties are smooth. Lakshmibai and Sandhya [LS90] showed that Schubert variety X_w is smooth if and only if the permutation w avoids 3412 and 4231. At that time, it was quite surprising that pattern avoidance had anything to do with this question, though explanations for the connection were found later [BB03]. Carrell showed that a Schubert variety X_w is smooth at points corresponding to permutations v for which the corresponding vertex in the Bruhat graph of w has exactly $\ell(w)$ edges [Car95].

In [1] Alexander Yong and I proved a statement in the spirit of that of Lakshmibai and Sandhya combinatorially characterizing w for which X_w is Gorenstein. A variety that is Gorenstein need not be smooth but behaves like a smooth one for many purposes in algebraic geometry.

Further consideration of the conditions we needed in the Gorenstein theorem, as well as consideration of a theorem independently proven by four different groups [BW03, Cor03, KLR03, Man01] characterizing the singular points in Schubert varieties, led to our definition of **interval pattern avoidance** in [2]. This is a generalization of the notion of pattern avoidance using intervals in Bruhat order. We showed that for any local property on Schubert varieties, there is a theorem of the same form as the Lakshmibai–Sandhya result if one uses interval pattern avoidance, although the list of interval patterns may be very long or even infinite. I then generalized this theorem to other semisimple Lie groups in [3] using a notion of pattern avoidance in arbitrary Weyl groups due to Billey and Braden [BB03].

Our original proof that interval pattern avoidance characterizes all local properties involved concretely realizing local neighborhoods of Schubert varieties as subsets of $\mathbb{C}^n$ defined by explicit polynomial equations. Alexander Yong and I proved [6] that our original equations form a Gröbner basis (under a particular term order). This means roughly that they are already in a form suitable for computations that could be the basis of further investigations. These results imply a new determinantal formula for multiplicites of points on Schubert varieties X_w where w belongs to a special class of permutations known as the cograssmannian permutations. Alexander Yong and Li Li recently generalized this formula to the case where w is covexillary [LY10+].

I have also been studying **Kazhdan–Lusztig polynomials** from the viewpoint of Schubert varieties. The Kazhdan–Lusztig polynomial $P_{v,w}(q)$ can be defined as a particular entry in the transition matrix between two bases of a ring known as the Hecke algebra, but it is also one way to measure how singular X_w is on a point of C_v. I proved in [4] a conjecture of Billey and Braden [BB03] characterizing $w \in S_n$ for which $P_{id,w}(q) = 1 + q^h$ for some h (which automatically implies that $P_{v,w}(q) = 0$, $P_{v,w}(q) = 1$ or $P_{v,w}(q) = 1 + q^h$ for all v).

Lascoux and Schützenberger gave a formula for Kazhdan–Lusztig polynomials $P_{v,w}(q)$ in the case where w is cograssmannian [LS81], and Zelevinsky [Zel83] derived a similar formula from geometric considerations. Brant Jones and I have ongoing work giving interpretations of these formulas in terms of Deodhar sets, which is a purely combinatorial approach to understanding Kazhdan–Lusztig polynomials introduced in [Deo90].

Vic Reiner, Alexander Yong, and I have studied the cohomology rings of Schubert varieties and in particular proved some results which lead to shorter presentations of these rings [5]. Although this question has links to explicit geometry as demonstrated by Akyildiz, Lascoux, and Pragacz [ALP92], our work mostly involved the algebra and combinatorics found in Schubert calculus.

4. Ongoing and future work

Alexander Yong and I are working to extend our Gröbner basis results to other simple Lie groups. In contrast to the case of Gl_n mentioned above, the correct equations for local neighborhoods cannot be easily deduced from known results, so proving a Gröbner basis result in this case involves finding the correct equations. Furthermore, such a result might shed more light on the combinatorics of Schubert calculus for these groups.

Our result that essentially all local properties are characterized by interval pattern avoidance suggests a program to actually find such characterizations. In particular, I am working with Henning Ulfarsson on characterizing the Schubert varieties (for Gl_n) which are local complete intersections. Other potential projects include extending the characterization of Gorenstein Schubert varieties and the characterization of singular points, as well as the characterization of factorial Schubert varieties by Bousquet–Mélou and Butler [BB07], to other Lie groups.

Hessenberg varieties form another general class of subvarieties of the flag variety; this class includes Springer fibers and Peterson varieties as special cases. In general, much less is known about these than about Schubert varieties, and I am interested in questions about both them and their intersections with Schubert varieties. This includes algebraic questions about their equations, geometric questions about their singularities, and combinatorial questions involving an analogue of Schubert calculus for these varieties. In particular, I am working with Erik Insko and Julianna Tymoczko on understanding the cohomology classes of Hessenberg varieties associated to a regular nilpotent matrix. This would extend work of Anderson and Tymoczko [AT10] and possibly give geometric explanations to some mostly combinatorial work of Harada and Tymoczko [HT10+].

A potential line of research I am interested in is to study analogous questions for affine Kac–Moody groups, an infinite-dimensional generalization of Lie groups.

5. Undergraduate research

This past summer I worked with Chris Conklin, a student at Saint Olaf, on graph-theoretic properties of Bruhat graphs. He proved that a permutation w has a Bruhat graph which can be drawn without crossings in the plane if and only if the length $\ell(w) < 4$ and w avoids 321. He also showed that this was characterized by avoiding a finite list of patterns and found the list via computer search. In addition, he investigated which Bruhat graphs can be drawn without crossings on the torus and Bruhat graphs for other Coxeter groups. We are writing a short paper for publication.

There are other graph-theoretic questions one can ask about Bruhat graphs, and almost none of them have known answers. For example, one can investigate their chromatic polynomials, which is the polynomial giving for every k the number of proper k-colorings of the graph. (Bruhat graphs are always 2-colorable since every edge is between an odd permutation and an even permutation.) One can also count the number of matchings on these graphs, or calculate the eigenvalues of the adjacency matrix. Answers to these graph theoretic questions should be related to properties of the original permutation. One can also generalize these questions to other Coxeter groups. These problems are suitable for research by undergraduate students.

My original motivation for looking into planarity of Bruhat graphs was to begin investigating if certain properties of permutations could be characterized by avoidance of finitely many patterns. While there can be properties of permutations which are characterized by avoidance of infinitely many patterns, it may be that properties which depend only on the Bruhat graph (and are given by pattern avoidance) require only finitely many patterns. A number of accessible concrete questions, such as characterizing which permutations have a particular Bruhat graph, shed light on this more abstract question.

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