Deodhar sets for cograssmannian permutations

Alexander Woo (joint work with Brant Jones)

AMS Central Sectional Meeting, April 11, 2010

Kazhdan–Lusztig polynomials

Kazhdan and Lusztig showed that there exist unique polynomials $P_{x,w}(q)$ for any $x, w \in S_n$ such that

- The degree of $P_{x,w}$ is at most $(\ell(w) - \ell(x))/2$.
- Let C'_w be defined by

$$C'(w) = q^{-\frac{1}{2}\ell(w)} \sum_x P_{x,w}(q) T_x,$$

  where T_x is the standard basis for the Hecke algebra. Then C'_w is invariant under the bar involution on the Hecke algebra sending $q \mapsto q^{-1}$ and $T_w \mapsto T_{w^{-1}}^{-1}$.

These polynomials $P_{x,w}(q)$ are known as the **Kazhdan–Lusztig polynomials**.

Why Kazhdan–Lusztig polynomials?

The Kazhdan–Lusztig polynomials are known to give:

- Multiplicities for Verma modules (at $q = 1$).
- Bases for interesting representations of S_n. (More precisely, this is given by the C'_w.)
- Poincaré polynomials for intersection cohomology of Schubert varieties.

The last interpretation shows they are positive. However, there is still no known combinatorial proof of positivity despite a number of combinatorial formulas by Brenti, Billera, and others.

Defects

Fix $w \in S_n$ and a reduced word $\mathrm{w} = \mathrm{w}_1\mathrm{w}_2 \cdots \mathrm{w}_\ell(w)$ for w.

A mask σ on w is a 0–1 string of length $\ell(w)$, which we identify with a subword of w. Let $\mathrm{w}^{\sigma[j]}$ denote the product of w_i for all i where $i \leq j$ and $\sigma_i = 1$, and $\mathrm{w}^\sigma = \mathrm{w}^{\sigma[\ell(w)]}$.

A position j in w is a **defect** of σ if multiplying $\ell(\mathrm{w}^{\sigma[j-1]}\mathrm{w}_j) < \ell(\mathrm{w}^{\sigma[j-1]})$. Let $d(\sigma)$ denote the number of defects of σ.

Deodhar's Theorem

Deodhar's theorem states that there exists some set $\mathcal{E}$ of masks such that

$$C'_w = q^{-\frac{1}{2}\ell(w)} \sum_{\sigma \in \mathcal{E}} q^{d(\sigma)} T_{w^\sigma}.$$

He gives a combinatorial interpretation of the bar involution, so one can detect if a given set of masks actually gives C'_w.

Unfortunately, the proof such a set exists and all known general constructions for $\mathcal{E}$ rely on a priori knowledge of the positivity of $P_{x,w}(q)$.

Our goal: Two such sets of masks (constructed from known formulas for $P_{x,w}(q)$) for the case where w is cograssmannian.

Heap diagrams

For any reduced word w for $w \in S_n$, we can draw the **reverse heap** of w as a set of lattice points. Associate to each index j, $1 \leq j \leq \ell(w)$, a point in $\mathbb{N}^2$ such that

- If w_j is the transposition s_i, then the heap point for j is in column x.
- If $j < k$, $\mathrm{w}_j = s_i$, and $\mathrm{w}_k = s_{i-1}$ or $w_k = s_{i+1}$, then the heap point for j is higher than the heap point for k.

Cograssmannian heaps

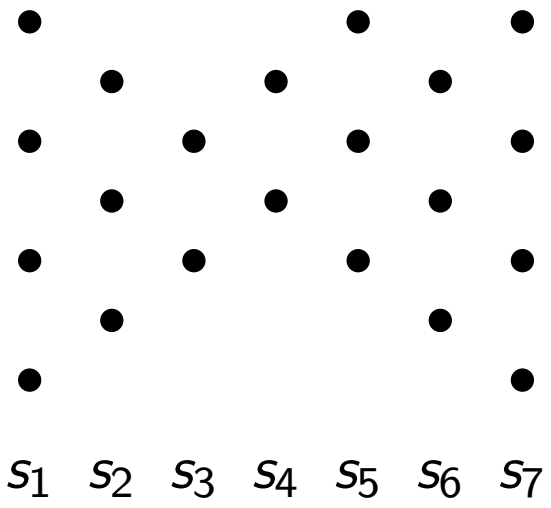
For $w \in S_n$ cograssmannian (i.e. at most one ascent at d), we choose a reduced decomposition $w = vw_0^J$, where v is grassmannian (i.e. at most one descent) and w_0^J is the longest element in the parabolic subgroup W_J generated by $\{s_i \mid i \neq d\}$. We use the decomposition for w_0^J which is the product of

$$(s_1 s_2 \cdots s_{d-1})(s_1 \cdots s_{d-2}) \cdots \text{ and } (s_{n-1} \cdots s_{d+1})(s_{n-1} \cdots s_{d+2}) \cdots .$$

The reverse heap for this reduced decomposition of w (no matter what subword of v is used) has the partition usually associated to v in "Russian style" at the top, and two triangles for the w_0^J part at the bottom, with a gap in the middle bottom.

Cograssmannian heap example

For $w = 86327541$, $v = 23681457$, and the reverse heap is



Masks on heap diagrams

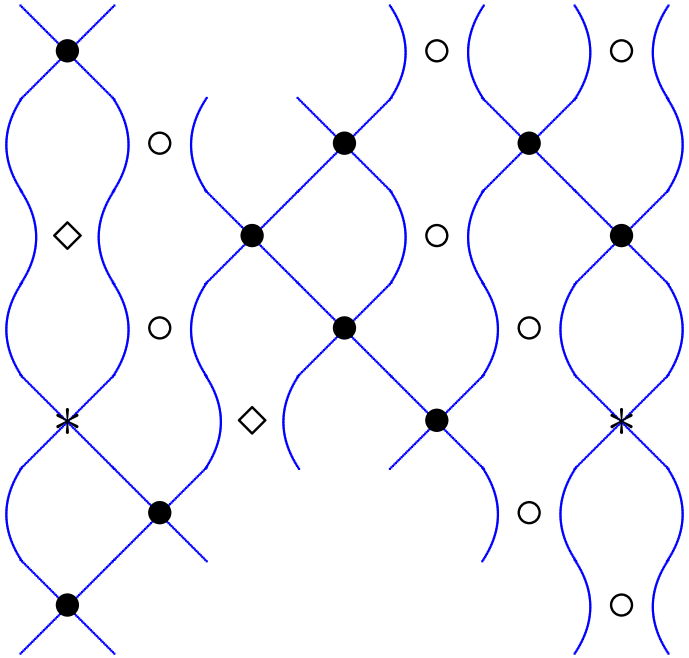
A mask can be represented on a heap by a string diagram. Draw strings starting between the heap columns, crossing them at every 1 in the heap.

A heap point represents a defect if the strings next to it cross an odd number of times above the heap point.

If one labels the strings $1, \ldots, n$ at the top and follows the number to the bottom, then the 1-line notation for w^σ can be read off at the bottom.

Mask example

One mask on $w = 86327541$ is



There are 3 defects and $w^\sigma = 51247368$.

Lascoux's Rule

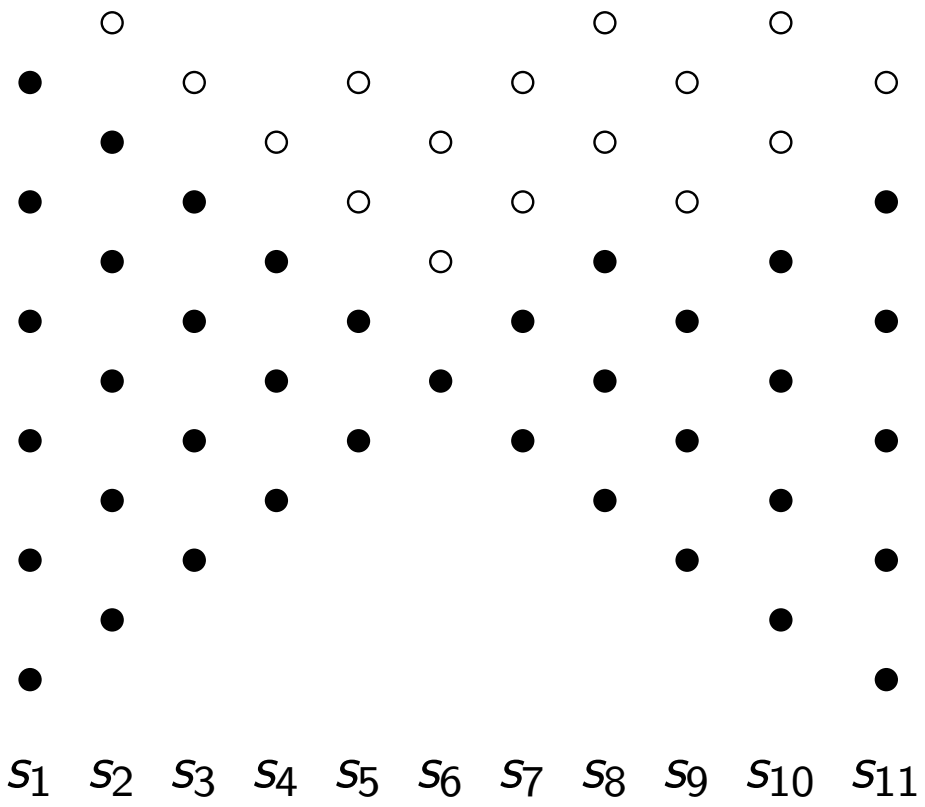
Given a cograssmannian permutation w, Lascoux constructs a tree whose leafs correspond to the valleys of the heap, with the length of the branches reflecting the depths of the valleys.

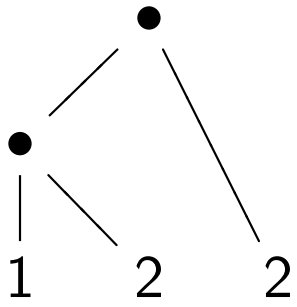
To calculate $P_{x,w}(q)$, one first replaces x with the largest permutation in its W_I-W_J double coset. (Here W_I is the parabolic subgroup generated by the *left* descents of w, which corresponds to the valleys of the heap. As before, W_J is generated by the right descents of w.) Then the heap of x fits inside the heap of w, and the valleys of x are valleys of w.

Label the leafs of the tree with the difference in depth between the valleys of x and w. Then $P_{x,w}(q)$ is q-counted by increasing edge-labelled trees satisfying the given leaf bounds.

Lascoux's Rule example

In S_{12}, let $w = CB9643A87521$ and $x = 965432CBA871$.





First Construction, Reduction to Coset Maximums

Fix a reduced word w, and a set of positions P. Let $\mathcal{F}_{\mathrm{w}}^{P}$ be the set of masks on w with defects exactly at P.

We show that the mapping $\sigma \mapsto \mathrm{w}^{\sigma}$ is an injection from $\mathcal{F}_{\mathrm{w}}^{P} \to S_n$, and its image is a lower order ideal in Bruhat order.

Therefore, if for each W_I-W_J–maximal x (cograssmannian with valleys only in valley columns of w) we construct a set of masks with $\mathrm{w}^{\sigma} = x$ which matches Lascoux's rule, we can declare that the right set of masks for any y in the double coset of x to be the masks which evaluate to y with defects in the same positions.

First Construction, Main Ideas

Fix x and an edge-labelled tree meeting the leaf bounds for x. We construct a mask σ with number of defects equal to the sum of the labels, and $\mathrm{w}^\sigma = x$.

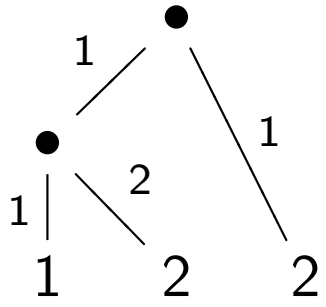
We break the heap of w into segments, one for each valley. The precise division depends on x.

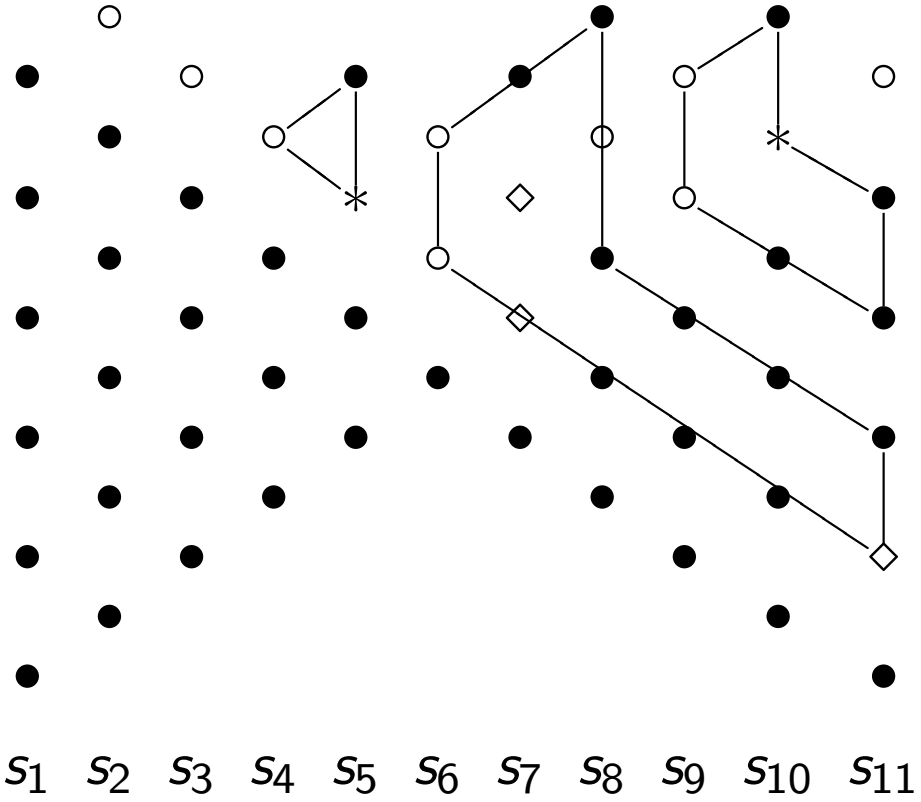
We also break Lascoux's tree into linear chains, one for each leaf.

We treat each segment separately. If λ is the partition whose parts are the labels on the linear chain for the segment, then the i-th diagonal of the segment has λ'_i defects.

First Construction, Example

In S_{12}, let $w = CB9643A87521$ and $x = 965432CBA871$.





Bott–Samelson and Schubert varieties

The Bott–Samelson variety Z_w is (by work of Magyar) the configuration variety defined by the heap. A point in Z_w corresponds to a collection of vector spaces, one for each point in the heap. The dimension of the vector space is the column the point is in. Each vector space is required to contain the ones on its NW diagonal and be contained in the ones on its NE diagonal.

The Schubert variety X_w is the quotient of the Bott–Samelson obtained by forgetting all vector spaces except the bottom one in each column.

The Zelevinsky variety

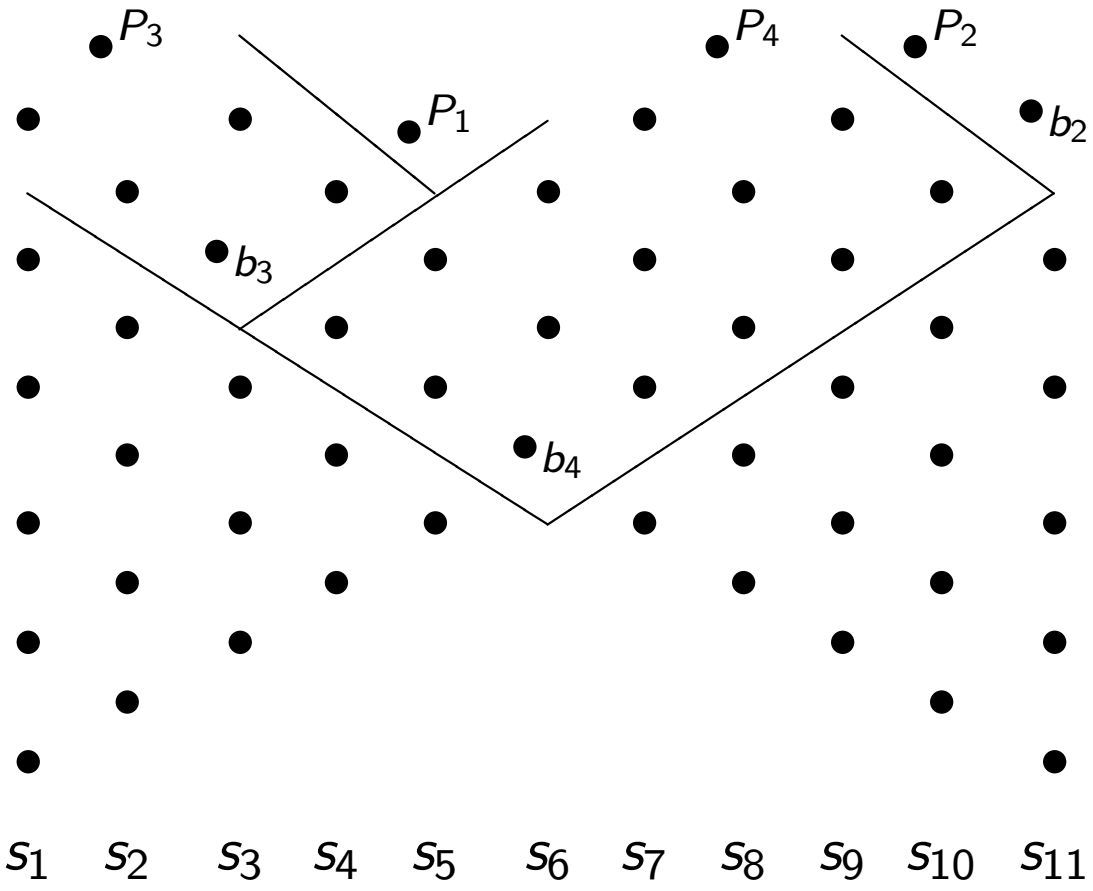
Fix an order on the **peaks**, which are maximal elements, of the heap. Label them $P_1, \ldots, P_k$ according to this order, which we denote $\mathbf{P}$.

Let R_i be the subset of the grasmmanian part of the heap consisting of points below P_i but not below P_j for any $j > i$. Each R_i is a rectangle. Let b_i be the heap point at the bottom of R_i.

The Zelevinsky variety $Z_{\mathbf{P}}$ is the quotient of the Bott–Samelson obtained by forgetting all vector spaces except the b_i and the bottom one in each column. There is a further quotient map $\pi_{\mathbf{P}} : Z_{\mathbf{P}} \to X_w$.

Zelevinsky variety example

In S_{12}, let $w = CB9643A87521$.



The Zelevinsky resolution and Kazhdan–Lusztig polynomials

Zelevinsky showed that, if the ordering $\mathbf{P}$ of the peaks is "neat", then the map $\pi_{\mathbf{P}}$ is "small". A general theorem about intersection homology then implies that, if $\mathbf{P}$ is neat,

$$P_{x,w}(q) = \sum_i \dim(H_{2i}(\pi_{\mathbf{P}}^{-1}(p_x))),$$

where p_x is any point in the Schubert cell $C_x \subset X_w$.

Cell decomposition for the Zelevinsky variety

There is a cell decomposition of $Z_{\mathbf{P}} = \bigcup_\tau C_\tau$. Here, τ consists of the data of one partition at the bottom of each rectangle R_i along with a permutation $v_\tau \leq w_0^J$.

Under the map $\pi_{\mathbf{P}}$, each cell C_τ is mapped to a Schubert cell C_x for some x, which we denote $\mathbf{P}^\tau$ (parallel to the notation w^σ).

The previous formula becomes

$$P_{x,w}(q) = \sum_{\tau : \mathbf{P}^\tau = x} q^{\dim(C_\tau) - \ell(x)}.$$

The dimension of C_τ is the sum of the sizes of the partitions plus the length of v_τ.

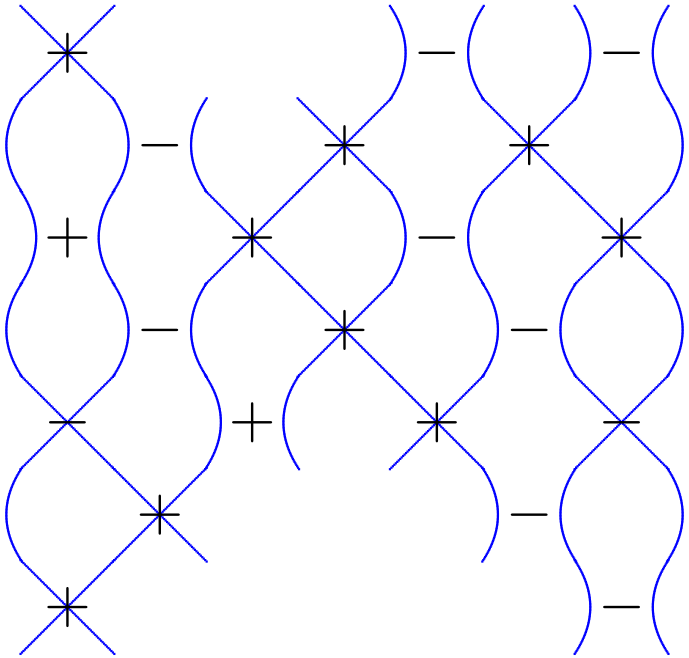
The Plus–Minus labelling of a mask

Instead of describing a mask by 0's and 1's, we can describe it by $+$'s and -'s. Label all the strings of the string diagram $1, \ldots, n$ at the top. Instead of having a 1 at every heap point where strings cross and a 0 where they don't, we have a $+$ where the higher-labelled string leaves to the left and a $-$ where the higher-labelled string leaves to the right.

It turns out that the number of $+$'s for a mask σ is the same as $d(\sigma) + \ell(\mathrm{w}^{\sigma})$.

Mask example with Plus–Minus labelling

Our earlier mask on $w = 86327541$ is



There are 10 plusses, 3 defects and $w^\sigma = 51247368$.

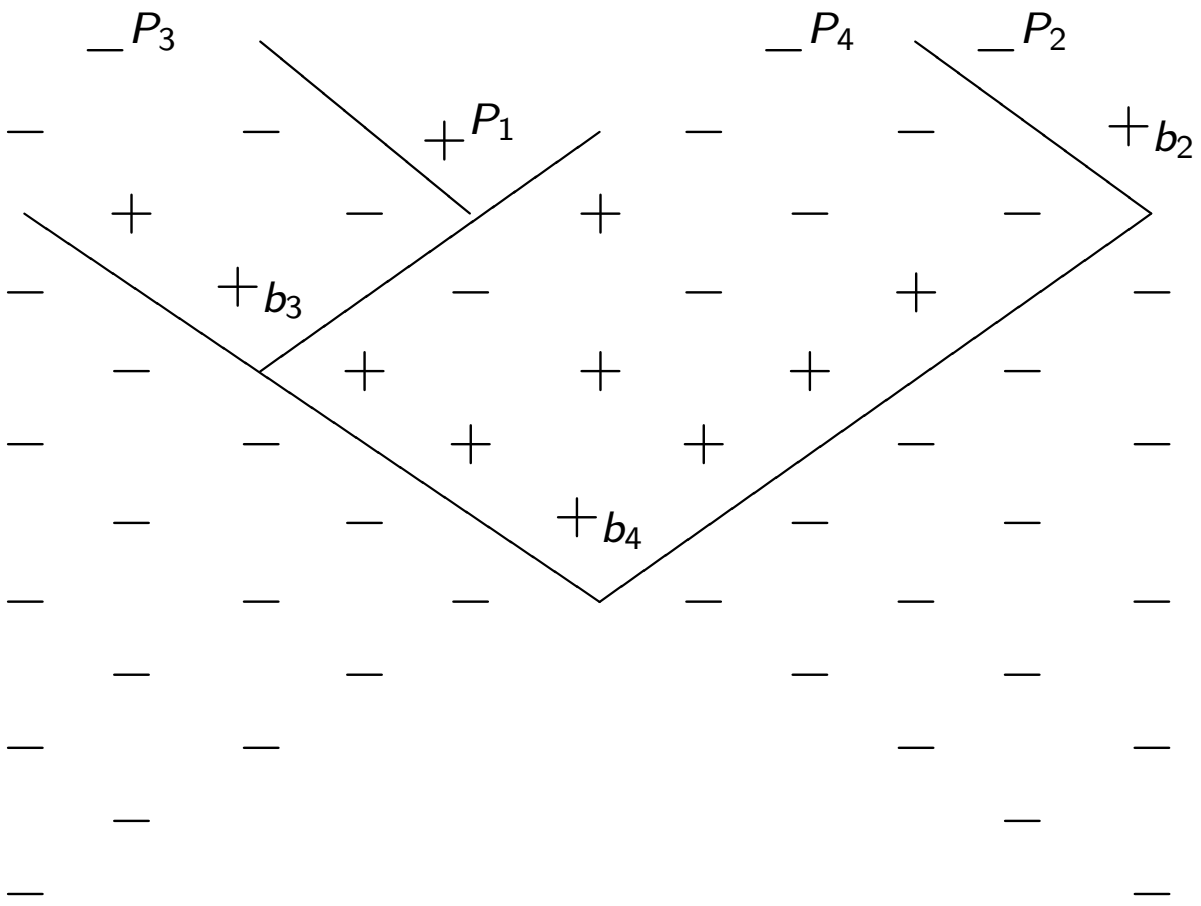
Second Construction, Main Ideas

We pick a neat ordering of the peaks $\mathbf{P}$ and hence a neat Zelevinsky resolution.

For each τ indexing a cell on $Z_{\mathbf{P}}$, we construct a mask σ so that the number of $+$'s in each rectangle equals the size of the partition, the number of $+$'s in the bottom part equals the length of v_τ, and $\mathrm{w}^\sigma = \mathbf{P}^\tau$.

Second Construction, Example

In S_{12}, let $w = CB9643A87521$. Let $\tau = (1, 1, 11, 422)$, $v_\tau = \mathrm{id}$.



The Two Constructions Differ!

The two constructions are different. Every mask in the second construction maps to a different cell in the Zelevinsky resolution. There are examples of the first construction where this does not happen.

This suggests that the Lascoux rule might have some genesis independent of the Zelevinsky resolution.

The covexillary case?

Lascoux's rule was extended to covexillary permutations by Lascoux and Schützenberger. Is there a way to extend either of our constructions? Zelevinsky's resolution does NOT always extend to a small resolution in the covexillary case, so the first construction may be better for this purpose.

The End

Thank you for your attention!