Delta-Springer varieties

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Flags

Fix a positive integer K and a composition μ of K. A μ-**flag** is a sequence of subspaces

$$V_\bullet = V_1 \subsetneq \cdots \subsetneq V_{\ell(\mu)} = \mathbb{C}^K$$

where $\dim(V_j) = \mu_1 + \cdots + \mu_j$. When $\mu = 1^K$, this is a **complete flag**.

The **flag variety** $\mathrm{Fl}(\mathbb{C}^K)$ is the set of all complete flags and the **partial flag variety** $\mathrm{Fl}_\mu(\mathbb{C}^K)$ is the set of all μ-flags.

Schubert cells

Fix an ordered basis $f_1, \dots, f_K$ for $\mathbb{C}^K$, and let B be the subgroup of matrices that are upper-triangular with respect to this basis.

Given a permutation w, we have a μ-flag $V^{(w)}$ where $V_j^{(w)} = \langle f_{w(1)}, \dots, f_{w(\mu_1+\cdots+\mu_j)} \rangle$. Elements $u, v \in S_n$ give the same flag iff they are in the same left coset of $S_{\mu_1} \times \cdots \times S_{\mu_\ell}$.

The **Schubert cell** C_w is the B orbit of $V^{(w)}$. The Schubert cells give a cell decomposition of $\mathrm{Fl}_\mu(\mathbb{C}^K)$.

Springer fibers

Let N be a nilpotent operator on $\mathbb{C}^K$ of Jordan type λ. The **Springer fiber** $\mathcal{B}^\lambda$ is the subvariety of the flag variety consisting of flags $V_\bullet$ where $NV_i \subseteq V_{i-1}$ for all i.

For an appropriate choice of N, $C_w \cap \mathcal{B}^\lambda$ is always $\mathbb{C}^d$ for some d; this gives an affine paving of $\mathcal{B}^\lambda$.

Cohomology of Springer fibers

Tanisaki gave a presentation for the cohomology ring of a Springer fiber $\mathcal{B}^\lambda$. Let λ' be the conjugate partition to λ, with $\lambda'_j = 0$ for $j > \lambda_1 = \ell(\lambda')$. Define

$$I_\lambda = \langle e_d(S) \mid d > \#S - (\lambda'_{K-\#S+1} + \cdots + \lambda'_K)\rangle,$$

where $S \subseteq \{x_1, \ldots, x_n\}$ and $e_d(S)$ means the d-th elementary symmetric functions in the variables in S.

Then

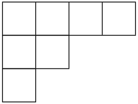
$$H^*(\mathcal{B}^\lambda) = \mathbb{Q}[x_1, \ldots, x_n]/I_\lambda,$$

where $x_i = -c_1(\mathcal{L}_i)$ (the negative of the Chern class of the i-th tautological line bundle).

Cohomology example

Recall

$$I_\lambda = \langle e_d(S) \mid d > \#S - (\lambda'_{K-\#S+1} + \cdots + \lambda'_K) \rangle,$$

For $\lambda =$ , we get that I_λ is generated by all the elementary symmetric functions in 7 variables, e_3, e_4, e_5, e_6 in 6 variables, e_4, e_5 in 5 variables, and e_4 in 4 variables. This means

$$I_\lambda = \langle x_1x_2x_3x_4, x_1x_2x_3x_5, \ldots, x_4x_5x_6x_7,$$
$$x_1x_2x_3 + x_1x_2x_4 + \cdots + x_4x_5x_6,$$
$$\ldots, x_1x_2 + \cdots + x_6x_7, x_1 + \cdots + x_7\rangle.$$

Symmetric group action on cohomology

The group S_n acts on $H^*(\mathcal{B}^\lambda)$ by permuting the variables x_i. Under this action, $H^{2d}(\mathcal{B}^\lambda) \cong V^\lambda$, the irreducible representation usually associated to λ. Let $\mathcal{F}$ denote the graded Frobenius characteristic, which assigns $q^d s_\lambda$ to an instance of V^λ in degree $2d$, it turns out

$$\mathcal{F}(H^*(\mathcal{B}^\lambda) = \tilde{H}_\lambda(\mathrm{x}; q),$$

where $\tilde{H}_\lambda$ is one version of the Hall-Littlewood polynomials.

This is one of the important motivations for studying Springer fibers and their cohomology (and known under more abstract terms before Tanisaki).

About the Delta Conjecture (now theorem)

The Delta Conjecture of Haglund–Remmel–Wilson (now a theorem of D'Adderio–Mellit and Blasiak–Haiman–Morse–Pun–Seelinger) gives a combinatorial formula for the symmetric function $\Delta'_{e_{k-1}} e_n(q,t)$, where Δ' is a family of operators coming from the theory of Macdonald polynomials.

Haglund, Rhoades, and Shimozono showed that the Frobenius characteristic of

$$R_{n,k} = \mathbb{Q}[x_1, \ldots, x_n]/\langle x_1^k, \ldots, x_n^k, e_n, e_{n-1}, \ldots, e_{n-k+1}\rangle$$

is $\Delta'_{e_{k-1}} e_n(q,0)$.

Spanning Line Arrangements

Pawlowski and Rhodes showed that $R_{n,k}$ is the cohomology ring of the space of spanning line arrangements: the set of lines $(L_1, \ldots, L_n) \in \mathbb{P}^{k-1}$ such that the span of $L_1, \ldots, L_n$ is all of $\mathbb{C}^k$.

Generalizing both rings simultaneously

Griffin defined the following generalization of both $R_{n,k}$ and $H^*(\mathcal{B}^\lambda)$. Let $k \leq n$, λ a partition of k, and $s \geq \ell(\lambda)$. Define

$$I_{n,\lambda,s} = \langle e_d(S) \mid d > \#S - (\lambda'_{n-\#S+1} + \cdots + \lambda'_n)\rangle,$$

where $S_\subseteq \{x_1, \ldots, x_n\}$ and $e_d(S)$ means the d-th elementary symmetric functions in the variables S. Define

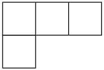
$$R_{n,\lambda,s} = \mathbb{Q}[x_1, \ldots, x_n]/\langle x_1^s, \ldots, x_n^s\rangle + I_{n,\lambda,s}.$$

Griffin gives a basis of standard monomials and calculates the graded Frobenius characteristic in the spirit of Garsia–Procesi. The top degree cohomology is $\mathrm{Ind}_{S_k}^{S_n} V^\lambda$.

Example

Recall

$$I_{n,\lambda,s} = \langle e_d(S) \mid d > \#S - (\lambda'_{n-\#S+1} + \cdots + \lambda'_n) \rangle$$

For $n = 7$, $\lambda =$ , $s = 4$, $I_{n,\lambda,s}$ should have e_4, e_5, e_6, e_7 in 7 variables, e_5, e_6 in 6 variables, e_5 in 5 variables. This means

$$R_{n,\lambda,s} = \mathbb{Q}[x_1, \ldots, x_7]/\langle x_1^4, \ldots, x_7^4, x_1x_2x_3x_4x_5, \ldots, x_3x_4x_5x_6x_7,$$
$$= x_1x_2x_3x_4 + \cdots + x_4x_5x_6x_7 \rangle$$

Special cases

For $n = k = |\lambda|$, $R_{n,\lambda,s} = H^*(\mathcal{B}^\lambda)$.

For $s = k = (1^k)$,$R_{n,\lambda,s} = R_{n,k}$.

Main Question

Is $R_{n,\lambda,s}$ the cohomology ring for some (compact) variety?

The Delta-Springer varieties

Given a positive integer n, a partition λ of k for some $k < n$, and some integer $s \geq \ell(\lambda)$, define

- $\Lambda = (n-k+\lambda_1, \dots, n-k+\lambda_\ell, n-k, \dots, n-k)$ so that Λ has s parts. (Add an $s \times (n-k)$ rectangle to the left of λ)
- $K = s(n-k) + k = |\Lambda|$
- N a nilpotent operator on $\mathbb{C}^K$ of Jordan type Λ.

Define $Y_{n,\lambda,s} \subseteq \mathrm{Fl}_{(1^n,(s-1)(n-k))}(\mathbb{C}^K)$ as the set of partial flags

$$V_\bullet = V_1 \subseteq \cdots \subseteq V_n$$

of type $(1^n, (s-1)(n-k))$ such that $NV_j \subseteq V_{j-1}$ for all j, AND $N^{n-k}\mathbb{C}^K \subseteq V_n$.

Main Theorem

We show

$$H^*(Y_{n,\lambda,s}) = R_{n,\lambda,s},$$

with $x_j = -c_1(\mathcal{L}_j)$.

Affine paving

Given an injective filling T of Λ by $\{1, \ldots, K\}$, we can define a particular nilpotent operator N_T. If T satisfies certain criteria, then $C_w \cap Y_T$ is always $\mathbb{C}^d$ for some d. There is a combinatorial rule for $\dim(C_w \cap Y_T)$ (including when it is nonempty).

The proof is by induction on n, and we get a recursion for the Hilbert series of $H^*(Y_{n,\lambda,s})$ that matches the recursion for the Hilbert series of $R_{n,\lambda,s}$. The proof is a variant of those of Shimomura, Tymoczko, and Fresse for $\mathcal{B}^\lambda$.

Cell example

Let $n = 5$, $\lambda = (2, 1)$, $s = 3$.

Pick $T = \begin{array}{|c|c|c|c|} \hline 6 & 4 & 2 & 1 \\ \hline 7 & 5 & 3 \\ \cline{1-3} 9 & 8 \\ \cline{1-2} \end{array}$

This means $N_T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Cell example, II

Recall $T = \begin{array}{|c|c|c|c|}\hline 6 & 4 & 2 & 1 \\ \hline 7 & 5 & 3 \\ \cline{1-3} 9 & 8 \\ \cline{1-2} \end{array}$

Pick $w = 38152$. Then

$$C_w \cap Y_T = \begin{pmatrix} a & c & 1 & 0 & 0 \\ 0 & ab & 0 & d & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The $\lambda = \emptyset$ case

When $\lambda = \emptyset$, then we can explicitly construct $Y_{n,\emptyset,s}$ as a tower of $\mathbb{P}^{s-1}$ bundles. We know which bundles these are and can compute directly that $H^*(Y_{n,\emptyset,s}) = \mathbb{Q}[x_1, \ldots, x_n]/\langle x_1^s, \ldots, x_n^2 \rangle$.

For any λ, we have an embedding of $Y_{n,\lambda,s}$ into $Y_{n,\emptyset,s}$. Because the complement of the image is also paved by affines, the induced map $H^*(Y_{n,\emptyset,s}) \to H^*(Y_{n,\lambda,s})$ is a surjection.

Spaltenstein varieties

Let N be a nilpotent operator on $\mathbb{C}^K$ of Jordan type ν. The **Spaltenstein variety** $\mathcal{B}^{\nu}_{\mu}$ is the subvariety of the μ-partial flag variety consisting of μ-flags $V_\bullet$ where $NV_i \subseteq V_{i-1}$ for all i. (Note: $\dim(V_i) - \dim(V_{i-1}) = \mu_i$.)

Recall $Y_{n,\lambda,s}$ paramaterizes $(1^n, (n-k)(s-1))$-flags where $NV_i \subseteq V_{i-1}$ and $N^{n-k}\mathbb{C}^K \subseteq V_n$.

Hence there is a map $\pi : \mathcal{B}^{\Lambda}_{(1^n,(s-1)^{n-k})} \to Y_{n,\lambda,s}$ (by forgetting the last $n-k$ parts of the partial flag).

Cohomology information from Spaltenstein

We show that, given further restrictions on T, over each "cell" of the Delta-Springer variety, the map π is projection from a product (with an appropriate smaller Spaltenstein variety).

This shows the Spaltenstein variety has an affine paving compatible with π and the affine paving of the Delta-Springer fiber, which shows the induced map $\pi^* : H^*(Y_{n,\lambda,s}) \to H^*(\mathcal{B}^{\Lambda}_{1^n,(s-1)^{n-k}})$ is injective.

Brundan and Ostrik have computed the cohomology of the Spaltenstein variety, and the remaining relations generating $I_{n,\lambda,s}$ are among the relations for $H^*(\mathcal{B}^{\Lambda}_{1^n,(s-1)^{n-k}})$.

What I haven't told you

- Explicit description of the irreducible components. There are $\dim(\mathrm{Ind}_{S_k}^{S_n} V^\lambda)$ many of them, each of dimension $n(\lambda) + (s-1)(n-k)$.
- The ind-variety as $s \to \infty$.

Questions

- RSK means what? (Steinberg analogue)
- Equivariant cohomology? (Kumar-Procesi and Goresky-Macpherson analogue)
- Hessenberg analogues?
- Other (classical) types?

Thank you

Thank you for your attention!