Cohomology and K-theory classes of regular nilpotent Hessenberg varieties

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Hessenberg varieties

Let $\mathcal{F}_n$ be the variety of complete flags

$$\mathcal{F} := \{F_\bullet = \{0\} \subsetneq F_1 \subsetneq \cdots \subsetneq F_{n-1} \subsetneq \mathbb{k}^n\}.$$

Set an operator $X : \mathbb{k}^n \to \mathbb{k}^n$ and a function

$$h : \{1, \ldots, n\} \to \{1, \ldots, n\}.$$

(Only h with $h(i+1) \geq h(i) \geq i$ for all i will have meaningful results.) Write $\mathbf{h} = (h(1), \ldots, h(n))$.

The **Hessenberg variety** $Y_{X,h}$ is the subvariety of $\mathcal{F}_n$ consisting of all points corresponding to flags $F_\bullet$ with the property that

$$XF_i \subseteq F_{h(i)}.$$

Regular nilpotent Hessenberg varieties

A **regular nilpotent** Hessenberg variety is one where X is regular nilpotent, so X is (conjugate to)

$$X := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Our formula, I

Given h, let $w_h \in S_{2n}$ be the permutation where

- $1, \ldots, n$ appear in order (i.e. $w^{-1}(1) < \cdots < w^{-1}(n)$)
- $n+1, \ldots, 2n$ appear in order
- $w_h(i + h(i)) > n$ for all i and $w(j) < n$ otherwise.

Let $\mathbf{x}_h$ be a list of variables where x_i appears at the $(i + h(i))$-th spot and $x_1, \ldots, x_n$ also appear in order in the other spots.

As a running example, let $n = 5$ and $\mathbf{h} = (3, 4, 4, 5, 5)$. Then

$$w_h = 123647859a$$

and

$$\mathbf{x}_h = (x_1, x_2, x_3, x_1, x_4, x_2, x_3, x_5, x_4, x_5).$$

Our formula, II

Then

$$[Y_{X,h}] = \mathfrak{G}_{w_h}(\mathbf{x}) \in K^0(\mathcal{F}_n)$$

and

$$[Y_{X,h}] = \mathfrak{S}_{w_h}(\mathbf{x}) \in H^*(\mathcal{F}_n)$$

where $\mathfrak{G}$ and $\mathfrak{S}$ are the usual Grothendieck and Schubert polynomials.

For $\mathbf{h} = (3, 4, 4, 5, 5)$,

$$[Y_{X,h}] = \mathfrak{G}_{123647859a}(x_1, x_2, x_3, x_1, x_4, x_2, x_3, x_5, x_4, x_5).$$

Caveats

A few warnings in interpreting the formula:

- The x_i do stand for the usual bundles/Chern classes.
- The formulas are correct only modulo the ideal defining $K^0(\mathcal{F}_n)$ or $H^*(\mathcal{F}_n)$.
- In particular, if one expands the result in terms of Grothendieck/Schubert polynomials, one does not only get Grothendieck/Schubert polynomials for $u \in S_n$, and the coefficients for Grothendieck/Schubert polynomials for $u \in S_m$, $m < n$ may be negative.
- The formula does not agree on the nose with the formula (for cohomology) of Anderson–Tymoczko.

The Knutson–Miller principle, I

Let

$$\pi : GL_n \to \mathcal{F}_n$$

be the map given by sending a matrix g to the flag $F_\bullet$ where F_i is the span of the first i columns of g.

Let M_n be the space of $n \times n$ matrices, with the usual embedding $GL_n \to M_n$.

The torus T of diagonal $n \times n$ matrices acts on M by multiplying on the *right*, which rescales *columns*.

The Knutson–Miller principle, II

If $Y \subseteq \mathcal{F}_n$ is a variety, and $Z \subseteq M_n$ is a variety such that $\pi(Z \cap GL_n) = Y$, then

$$[Y] = [Z]_T,$$

where $[Z]_T$ is the T-equivariant K-theory/cohomology class under the action described above.

Note M_n is affine space with T rescaling coordinates, so $[Z]_T$ can be calculated by combinatorial commutative algebra as K-polynomials or multidegrees.

Matrix equations for Hessenberg variety, I

Suppose

$$g = \begin{bmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & & \vdots \\ z_{n1} & \cdots & z_{nn} \end{bmatrix}$$

represents a flag, so F_i is the span of the first i columns.

Then XF_i is the span of the first i columns of

$$Xg = \begin{bmatrix} z_{21} & \cdots & \cdots & z_{2n} \\ \vdots & & & \vdots \\ z_{n1} & \cdots & \cdots & z_{nn} \\ 0 & \cdots & \cdots & 0 \end{bmatrix}.$$

Matrix equations for Hessenberg variety, II

Then $XF_i \subseteq F_j$ if the matrix

$$\begin{bmatrix} z_{11} & \cdots & z_{1j} & z_{21} & \cdots & z_{2i} \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{(n-1)1} & \cdots & z_{(n-1)j} & z_{n1} & \cdots & z_{ni} \\ z_{n1} & \cdots & z_{nj} & 0 & \cdots & 0 \end{bmatrix}$$

has rank j. (It cannot have rank less than j if g was originally invertible and hence represented a flag.)

Matrix equations for Hessenberg variety, Example

We can put all these rank conditions together as rank conditions for a single matrix. For our running example ($n = 5$, $\mathbf{h} = (3, 4, 4, 5, 5)$, or $XF_1 \subseteq F_3, XF_3 \subseteq F_4$), we get

$$\begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{21} & z_{14} & z_{22} & z_{23} & z_{15} & z_{24} & z_{25} \\ z_{21} & z_{22} & z_{23} & z_{31} & z_{24} & z_{32} & z_{33} & z_{25} & z_{34} & z_{35} \\ z_{31} & z_{32} & z_{33} & z_{41} & z_{34} & z_{42} & z_{43} & z_{35} & z_{44} & z_{45} \\ z_{41} & z_{42} & z_{43} & z_{51} & z_{44} & z_{52} & z_{53} & z_{45} & z_{54} & z_{55} \\ z_{51} & z_{52} & z_{53} & 0 & z_{54} & 0 & 0 & z_{55} & 0 & 0 \end{bmatrix}$$

where the first 4 columns have rank 3 and the first 7 columns have rank 4.

Let $Z_{X,h} \subseteq M_n$ be the scheme of $n \times n$ matrices satisfying these equations, so $Y_{X,h} = \pi(Z_{X,h} \cap GL_n)$.

Matrix Schubert varieties

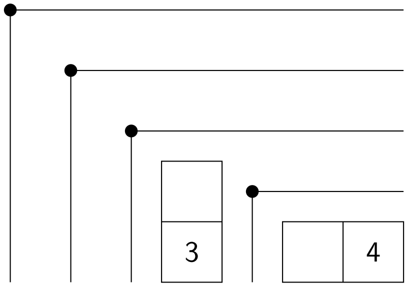
Let w be an $a \times b$ **partial permutation matrix**, meaning w is a 0/1 matrix with at most one 1 in each row or column.
Given a partial permutation matrix w, we have its **rank matrix**, with $r_{pq}(w)$ being the number of $1's$ (weakly) NW of (p, q).

$$w = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} r = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 3 & 4 & 4 & 4 \end{bmatrix}$$

The **matrix Schubert variety** A_w is the set of all $a \times b$ matrices whose NW rank $p \times q$ submatrices have rank $r_w(p, q)$.

Essential sets

Not every rank condition defining A_w is necessary. Fulton showed that the **essential set** is the unique minimal set of conditions that suffices.



Hessenberg equations and matrix Schubert varieties

The essential rank conditions above are precisely the conditions enforcing $XF_1 \subseteq F_3$, $XF_3 \subseteq F_4$ on

$$\begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{21} & z_{14} & z_{22} & z_{23} & z_{15} & z_{24} & z_{25} \\ z_{21} & z_{22} & z_{23} & z_{31} & z_{24} & z_{32} & z_{33} & z_{25} & z_{34} & z_{35} \\ z_{31} & z_{32} & z_{33} & z_{41} & z_{34} & z_{42} & z_{43} & z_{35} & z_{44} & z_{45} \\ z_{41} & z_{42} & z_{43} & z_{51} & z_{44} & z_{52} & z_{53} & z_{45} & z_{54} & z_{55} \\ z_{51} & z_{52} & z_{53} & 0 & z_{54} & 0 & 0 & z_{55} & 0 & 0 \end{bmatrix}$$

This means specializing the matrix Schubert variety to variables as in the matrix above gives $Z_{X,h}$.

Knutson–Miller theorem

Knutson and Miller proved in 2003

Theorem

Let A_w be a matrix Schubert variety. Then

- *The K-theory/cohomology class of A_w is given by $\mathfrak{G}_{\overline{w}}$ or $\mathfrak{S}_{\overline{w}}$, each evaluated in the torus weights of the columns. (Here, $\overline{w}$ is the permutation (in a larger matrix) whose essential set is the same as the essential set of w.)*
- *A_w is Cohen–Macaulay.*

(There is more to this theorem.)

Variable specialization and K-polynomials

Let $R = \Bbbk[\mathbf{a}]$, $S = \Bbbk[\mathbf{z}]$, and $f : R \to S$ be a surjective, T-equivariant map that sends variables to variables. Let $A = R/I$. Then

$$[A]_R = \sum_i (-1)^i [\overset{i}{\mathrm{Tor}}(A, S)]_S,$$

where $[A]_R$ is the K-polynomial of A (as an R-module).
If $\ker f$ is generated by (part of) a regular sequence on A, then $\mathrm{Tor}^i(A, S) = 0$ for $i > 0$, and $[A]_R = [f_*(A)]_S$.
If A is Cohen–Macaulay, then any regular system of parameters on A is a regular sequence, so it would suffice to show $\operatorname{codim} \operatorname{Spec}(A) \cap \operatorname{Spec}(S) = \operatorname{codim} \operatorname{Spec}(A) + \operatorname{codim} \operatorname{Spec}(S)$.

Unfortunately..., but...

As a subvariety of M_n, $Z_{w,h}$ does NOT have the right codimension (except in small cases).
However, $Y_{X,h}$ and hence $Z_{X,h} \cap GL_n$ DOES have the right codimension. In other words $Z_{X,h}$ fails to have the right codimension only because of some "junk" components contained in $M_n \setminus GL_n$.
This means that $\mathrm{Spec}(Tor^i(A_{w,h}, S)) \cap GL_n$ is supported only on the locus of singular matrices $M_n \setminus GL_n$.
For the Knutson–Miller principle, we don't need to worry about anything on $M_n \setminus GL_n$, so our formula works.

What we couldn't figure out

The scheme $Z_{X,h}$ has junk components that are not in $\overline{\pi^{-1}(Y_{X,h})}$. Can we find the additional equations that will cut out $\overline{\pi^{-1}(Y_{X,h})}$? Unfortunately, this is probably at least as complicated as the analogous problem of Berget–Fink for matrix torus orbit closures.

The Hessenberg variety $Y_{X,h}$ has a natural decomposition given by intersections with certain specific Schubert cells. Can we find classes for the closures of these cells? The same technique works for some of these cells, but doesn't work in general. There are tricks that can get us some more cells.

Thank you

Thank you for your attention.