Interval Pattern Avoidance for K-orbit closures

Alexander Woo
(joint work with Ben Wyser and Alexander Yong)

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Schubert varieties

Let:

- G be a semisimple algebraic group (GL_n)
- B be a Borel subgroup (upper triangular matrices)
- W be the Weyl group of G (S_n)
- G/B be the flag variety.

For each $w \in W$, we have the Schubert variety

$$X_w = \overline{BwB/B} \subseteq G/B,$$

and $\ell(w) = \dim X_w = \#\mathrm{Inv}(w)$

Pattern avoidance on Schubert varieties

Theorem (Lakshmibai–Sandhya '90)

For $G = GL_n$, the Schubert variety X_w is smooth if w pattern avoids 4231 and 3412.

Definition by example: 2573461 contains 4231 in 4 ways, by 2**5**7**34**6**1**, 25**734**6**1**, 25**73**4**61**, 25**7**3**461**.

3562714 avoids 4231.

Interval pattern avoidance

Definition

Given $u, v \in S_m$ and $x, w \in S_n$, we say $[x, w]$ **interval contains** $[u, v]$ if

- x contains u and w contains v in the same places
- x and w agree in all other places
- $\ell(w) - \ell(x) = \ell(v) - \ell(u)$

The first 2 conditions determine x from u, v, and w, so we can talk about w containing or avoiding $[u, v]$.

Example: **35**1**4**2 contains 3412, but $[13245, 35142]$ does not contain $[1324, 3412]$, since $\ell(35142) - \ell(13245) = 5$ but $\ell(3412) - \ell(1324) = 3$.

Universality

Theorem (W-Yong '08)

Given any local property of varieties preserved under closure and product with $\mathbb{A}^n$, the set of w such that X_w has this property is characterized by avoiding some set $\{[u_i, v_i]\}$ of interval patterns

The proof is by showing that, up to products with $\mathbb{A}^n$, a neighborhood of X_w at the point xB/B is isomorphic to a neighborhood of X_v at uB/B.

K-orbit closures

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For $G = GL_n$, $K = GL_p \times GL_q$, (corresponding to the real group $U(p, q)$) orbits can be indexed by (p, q)-clans. We denote by X_γ the K-orbit closure corresponding to the clan γ.

Clans

A **clan** is a partial matching on n linearly ordered vertices, with unmatched vertices given a sign (i.e. $+$ or $-$).

A clan is a (p, q)-clan if the number of $+$'s and the number of matchings adds up to p, and the number of $-$'s and the number of matchings adds up to q.

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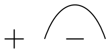
We draw clans like:



, or

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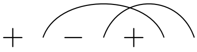

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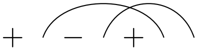
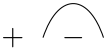
Pattern avoidance on clans

There is a natural notion of pattern avoidance on clans:



contains



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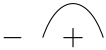

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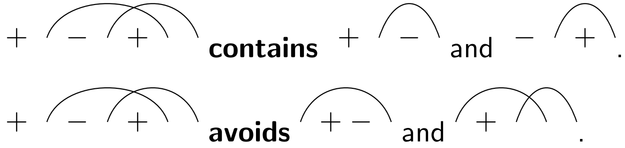
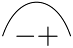


Smooth K-orbit closures

Theorem (McGovern '11)

A K-orbit closure X_γ is smooth if and only if γ avoids ,

, 

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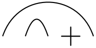
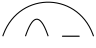
, 

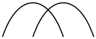
, 

, *and* 

Length of a clan

The length of a clan is

$$\ell(\gamma) = \left(\sum_{i,j \text{ matched}} j - i \right) - \# \text{ crossing pairs}$$

or the number of matchings, plus the number of places enclosed by matchings, except a crossing pair , counts only once, not twice.

We have

$$\dim X_\gamma = pq + \ell(\gamma).$$

Interval pattern avoidance

Definition

Given clans (p, q)-clans σ and τ and (p', q')-clans δ and γ (with $p' \geq p$ and $q' \geq q$), we say $[\delta, \gamma]$ **interval contains** $[\sigma, \tau]$

- γ contains τ and δ contains σ in the same places
- δ and γ agree in all other places
- $\ell(\gamma) - \ell(\delta) = \ell(\tau) - \ell(\sigma)$

The first 2 conditions determine δ from σ, τ, and γ, so we can talk about γ containing or avoiding $[\sigma, \tau]$.

Universality

Theorem (W-Wyser-Yong)

Given any local property of varieties preserved under closure and smooth morphisms, the set of γ such that X_γ has this property is characterized by avoiding some set of interval patterns.

Proof ideas

We work with B-orbit closures on G/K.

Given δ, we take a transverse slice S^δ to G/K at a point of the of δ. There is a smooth morphism $S^\delta \cap X_\gamma \to X_\gamma$ with image a neighborhood of our point.

We show by looking at explicit defining rank conditions on local coordinates that

$$S^\delta \cap X_\gamma \cong S^\sigma \cap X_\tau$$

when $[\delta, \gamma]$ contains $[\sigma, \tau]$.

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Thanks to Kiumars and Frank for organizing this wonderful session.