Interval Pattern Avoidance for $K$-orbit closures

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(joint work with Ben Wyser and Alexander Yong)

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Let:

- $G$ be a semisimple algebraic group ($GL_n$)
- $B$ be a Borel subgroup (upper triangular matrices)
- $W$ be the Weyl group of $G$ ($S_n$)
- $G/B$ be the flag variety.

For each $w \in W$, we have the Schubert variety

$$X_w = \overline{BwB/B} \subseteq G/B,$$

and $\ell(w) = \dim X_w = \#\text{Inv}(w)$
Theorem (Lakshmibai–Sandhya ’90)

For $G = GL_n$, the Schubert variety $X_w$ is smooth if $w$ pattern avoids 4231 and 3412.

Definition by example: 2573461 contains 4231 in 4 ways, by 25\textbf{7}3461, 257\textbf{3}461, 2573\textbf{4}61, 25734\textbf{6}1.

3562714 avoids 4231.
Definition
Given $u, v \in S_m$ and $x, w \in S_n$, we say $[x, w]$ interval contains $[u, v]$ if

- $x$ contains $u$ and $w$ contains $v$ in the same places
- $x$ and $w$ agree in all other places
- $\ell(w) - \ell(x) = \ell(v) - \ell(u)$

The first 2 conditions determine $x$ from $u, v,$ and $w$, so we can talk about $w$ containing or avoiding $[u, v]$.

Example: $35142$ contains $3412$, but $[13245, 35142]$ does not contain $[1324, 3412]$, since $\ell(35142) - \ell(13245) = 5$ but $\ell(3412) - \ell(1324) = 3$. 

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Theorem (W-Yong ’08)

Given any local property of varieties preserved under closure and product with $\mathbb{A}^n$, the set of $w$ such that $X_w$ has this property is characterized by avoiding some set $\{[u_i, v_i]\}$ of interval patterns.

The proof is by showing that, up to products with $\mathbb{A}^n$, a neighborhood of $X_w$ at the point $xB/B$ is isomorphic to a neighborhood of $X_v$ at $uB/B$. 
Let $\theta$ be an involution on $G$, $K = G^\theta$ the subgroup of elements fixed by $\theta$.

Then $K$ acts on $G/B$ with finitely many orbits.
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For $G = GL_n$, $K = GL_p \times GL_q$, (corresponding to the real group $U(p, q)$) orbits can be indexed by $(p, q)$-clans. We denote by $X_\gamma$ the $K$-orbit closure corresponding to the clan $\gamma$. 

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Interval Pattern Avoidance for $K$-orbit closures
A clan is a partial matching on $n$ linearly ordered vertices, with unmatched vertices given a sign (i.e. + or −).

A clan is a $(p, q)$-clan if the number of +’s and the number of matchings adds up to $p$, and the number of −’s and the number of matchings adds up to $q$. 
A clan is a partial matching on \( n \) linearly ordered vertices, with unmatched vertices given a sign (i.e. + or −).

A clan is a \((p, q)\)-clan if the number of +’s and the number of matchings adds up to \( p \), and the number of −’s and the number of matchings adds up to \( q \).

We draw clans like:

\[
+ \quad - \quad , \text{ or } \quad + \quad - \quad + \quad , \text{ or } \quad + \quad - \quad + \quad .
\]
Pattern avoidance on clans

There is a natural notion of pattern avoidance on clans:

\[ +\, -\, + \text{ contains } +\, - \text{ and } -\, +. \]
Pattern avoidance on clans

There is a natural notion of pattern avoidance on clans:

\[ + - + \quad \text{contains} \quad + - \quad \text{and} \quad - + \.
\]

\[ + - + \quad \text{avoids} \quad + - \quad \text{and} \quad + + \.
\]
Smooth $K$-orbit closures

**Theorem (McGovern ’11)**

A $K$-orbit closure $X_\gamma$ is smooth if and only if $\gamma$ avoids $\bigcap$, $\bigcap$, $\bigcap$, $\bigcap$, $\bigcap$, $\bigcap$, and $\bigcap$. 

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Interval Pattern Avoidance for $K$-orbit closures
The length of a clan is

\[ \ell(\gamma) = \left( \sum_{i,j \text{ matched}} j - i \right) - \text{\# crossing pairs} \]

or the number of matchings, plus the number of places enclosed by matchings, except a crossing pair, counts only once, not twice.

We have

\[ \dim X_\gamma = pq + \ell(\gamma). \]
Definition
Given clans \((p, q)\)-clans \(\sigma\) and \(\tau\) and \((p', q')\)-clans \(\delta\) and \(\gamma\) (with \(p' \geq p\) and \(q' \geq q\)), we say \([\delta, \gamma]\) **interval contains** \([\sigma, \tau]\)

- \(\gamma\) contains \(\tau\) and \(\delta\) contains \(\sigma\) in the same places
- \(\delta\) and \(\gamma\) agree in all other places
- \(\ell(\gamma) - \ell(\delta) = \ell(\tau) - \ell(\sigma)\)

The first 2 conditions determine \(\delta\) from \(\sigma, \tau\), and \(\gamma\), so we can talk about \(\gamma\) containing or avoiding \([\sigma, \tau]\).
Universality

Theorem (W-Wyser-Yong)

Given any local property of varieties preserved under closure and smooth morphisms, the set of $\gamma$ such that $X_\gamma$ has this property is characterized by avoiding some set of interval patterns.
We work with $B$-orbit closures on $G/K$.

Given $\delta$, we take a transverse slice $S^\delta$ to $G/K$ at a point of the of $\delta$. There is a smooth morphism $S^\delta \cap X_\gamma \to X_\gamma$ with image a neighborhood of our point.

We show by looking at explicit defining rank conditions on local coordinates that

$$S^\delta \cap X_\gamma \cong S^\sigma \cap X_\tau$$

when $[\delta, \gamma]$ contains $[\sigma, \tau]$. 
Thank you for your attention.
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