Combinatorics of clans and geometry of B-orbits on G/K

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Definitions

- G is a semisimple Lie group ($G = SL_n$)
- B is a Borel subgroup (upper triangular matrices)
- G/B is a generalized flag variety (moduli of flags of subspaces: $\{0\} \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n$)
- W is the Weyl group of G ($W = S_n$)

The group B acts on G/B, and the orbits are known as **Schubert cells**. They can be naturally indexed by elements of W, with Y_w being the orbit $B \cdot wB/B$ where w is considered as a permutation matrix. The **Schubert variety** is the closure $X_w = \overline{Y_w}$.

Schubert cells and combinatorics

The combinatorics of W is closely connected to the geometry of the Schubert varieties X_w in G/B.

- The length $\ell(w)$ for $w \in W$ (number of inversions for S_n) is the dimension of X_w.
- Bruhat order (generated by $tw \geq w$ if t is a transposition and $\ell(tw) > \ell(w)$) corresponds to closure order: $v \leq w$ if $X_v \subset X_w$.
- NilHecke multiplication (for a simple transposition s, $s \cdot w = w$ if $\ell(sw) < \ell(w)$ and $s \cdot w = sw$ otherwise) is convolution product: $X_{v \cdot w} = X_v B X_w$.
- In particular, covering relations in weak order are given by multiplication by a minimal parabolic: $sw > w$ if $P_s X_w = X_{sw}$, and $sw < w$ if $P_s X_w = X_w$.

Pattern avoidance

Given $v \in S_m$ and $w \in S_n$ with $n \geq m$, we say w **contains** v if there exist indices $1 \leq i_1 < \cdots i_m \leq n$ such that $w(i_j) > w(i_k) \Leftrightarrow v(j) > v(k)$. Otherwise w **avoids** v.

For example (using 1-line notation) $w =$ **46**3**152** contains $v = 3412$, but 532641 avoids 3412.

This notion can be generalized to all Coxeter groups using parabolic decomposition (for not necessarily standard parabolic subgroups). (Billey–Braden '03)

Singularities and pattern avoidance

The closure $X_w = \overline{Y_w}$ of a Schubert cell is known as a **Schubert variety**. Properties of singularities of Schubert varieties are governed by pattern avoidance (or for certain more complicated properties, interval pattern avoidance). For example:

- For $G = SL_n$, X_w is smooth iff w avoids 3412 and 4231 (Lakshmibai–Sandhya '90)
- For smoothness in all finite types, add one element of D_4, one in B_2, several in $B_3 = C_3$, and several in G_2. (Billey–Postnikov '03)
- For $\widetilde{SL_n}$, X_w is rationally smooth iff w avoids 3412 and 4231, or w is a twisted spiral element. (Billey–Crites '12)

Singularities and pattern avoidance, II

The closure $X_w = \overline{Y_w}$ of a Schubert cell is known as a **Schubert variety**. Properties of singularities of Schubert varieties are governed by pattern avoidance (or for certain more complicated properties, interval pattern avoidance). For example:

- For $G = SL_n$, X_w is a local complete intersection iff w avoids 53241, 52431, 52341, 35142, 42513, 426153. (Ulfarsson–W '12)
- For $G = SL_n$, X_w is Gorenstein iff w satisfies 2 infinite series of interval pattern avoidance conditions. (W–Yong '05)

Definition

Let G be a complex semisimple Lie group, θ an involution on G, and $K = G^\theta$ the fixed point set. Then G/K is a **symmetric space**, and all symmetric spaces arise in this manner (by definition).

For example, let $G = SL_n$, $\theta(g) = (g^{-1})^T$. Then $K = SO_n$.

Another example is $G = SL_n$, θ is conjugation by $\begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$. Then $K = S(GL_p \times GL_q)$.

Classification

Ignoring issues of isogeny (i.e. the difference between SL_n and PGL_n), there are 7 classical possibilities and 12 exceptional ones for (G, K):

- (GL_{2n}, Sp_{2n})
- (GL_n, O_n) (this behaves differently for n odd and n even)
- $(GL_n, GL_p \times GL_q)$ where $p + q = n$
- $(O_n, O_p \times O_q)$ where $p + q = n$ (this behaves differently for p and q both odd, both even, and one of each)
- (O_{2n}, GL_n)
- $(Sp_{2n}, Sp_{2p} \times Sp_{2q})$
- (Sp_{2n}, GL_n)

Richardson–Springer map

Let $\mathcal{I}$ be the set of θ-twisted involutions

$$\mathcal{I} = \{w \in W \mid \theta(w) = w^{-1}\}.$$

Richardson and Springer define a map from the set of B-orbits on G/K (equivalently $B \times K$ orbits on G or equivalently K orbits on $B\backslash G$) to $\mathcal{I}$.

This map is usually not injective or not surjective, but we can think of the B-orbits as being indexed by twisted involutions decorated with additional data.

(GL_n, O_n)

For (GL_n, O_n), twisted involutions are permutations w such that $w_0 w w_0 = w^{-1}$. Multiplying (on the right) by w_0 gives a bijection between twisted involutions and actualy involutions. The Richardson–Springer map is actually a bijection. Bruhat order is upside-down Bruhat order on the subset of involutions.

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NilHecke multiplication by s_i is multiplication by s_i when that gives an involution, and conjugation by s_i otherwise. This means weak order is NOT induced weak order.

Clans

For $G = GL_n$, $K = GL_p \times GL_q$, the orbits are indexed by (p, q)-clans. A **clan** is a partial matching on n linearly ordered vertices, with unmatched vertices given a sign (i.e. $+$ or $-$).

A clan is a (p, q)-clan if the number of $+$'s and the number of matchings adds up to p, and the number of $-$'s and the number of matchings adds up to q.

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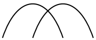
We draw clans like:

Length of a clan

The length of a clan is

$$(\sum_{i,j\text{matched}} j-i) - \text{crossingpairs}$$

or the number of matchings, plus the number of places enclosed by matchings, except a crossing pair , counts only once, not twice.

The Richardson-Springer map sends each clan to the involution associated to the unsigned matching, and length is the average of inversions and excedences on this involution (Incitti).

The dimension of the orbit indexed by a (p, q)-clan γ is $\binom{p}{2} + \binom{q}{2} + \ell(\gamma)$.

W and NilHecke actions

The symmetric group S_n acts on clans by permuting entries.

The NilHecke algebra acts on clans with s_i acting as follows:

- If entry i and $i+1$ are a $+$ and a $-$ (or vice versa), s_i turns them into a matched pair
- If entries i and $i+1$ are the same sign or a matched pair, s_i does nothing
- Otherwise, s_i switches the i and $i+1$-th entries if it increases length, and does nothing otherwise.

Weak order

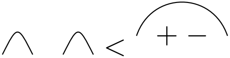
As in the Schubert case, the NilHecke action is convolution product: $Y_{w\cdot\gamma} = Y_w B Y_\gamma$ (when Y_γ is thought of as a B-orbit in G/K).

Weak order can be defined as reachability order under NilHecke multiplication, so a covering relation in weak order is multiplication by a minimal parabolic: $s \cdot \gamma$ covers γ if $P_s Y_\gamma = Y$

Weak order is graded by length. The minimal length elements (corresponding to closed orbits) are the ones with no matchings.

Bruhat order

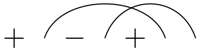
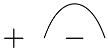
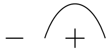
Bruhat order (defined geometrically by $\delta \leq \gamma$ if $X_\delta \subseteq X_\gamma$) has the same moves as weak order (but possibly at a distance), plus the

move 

The Richardson-Springer map is a poset map in Bruhat order (but not weak order!), but Bruhat order is not induced by its image.

Pattern avoidance on clans

There is a natural notion of pattern avoidance on clans:

 **contains**  and .

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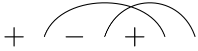
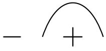
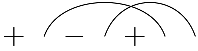
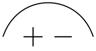
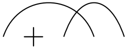
 **contains**  and .

 **avoids**  and .

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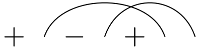
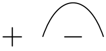
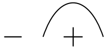
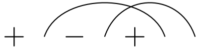
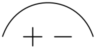
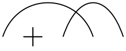
 **contains**  and .

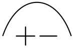
 **avoids**  and .

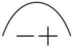
A **non-crossing** clan is one that avoids .

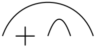
Smoothness for non-crossing clans

McGovern proved the following:

Theorem

Let γ be a clan. Then X_γ is NOT smooth if and only if γ contains one of the following 8 clans: , 

, 

, 

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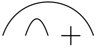
, 

, 

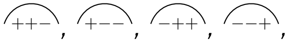
, *and* 

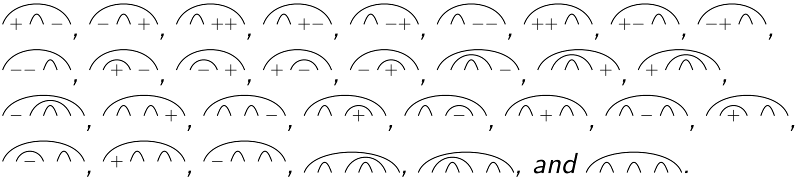
Partial LCI theorem

We have a partial result for local complete intersections:

Theorem

Let γ be a non-crossing clan. Then X_γ is NOT lci if and only if γ contains one of the following 35 clans:





Noncrossing clans and Richardson varieties

Given a noncrossing clan γ, Wyser showed that there are inverse cograssmannian and inverse grasmmannian permutations $u(\gamma)$ and $v(\gamma)$ such that $X_\gamma = X_{u(\gamma)} \cap X^{v(\gamma)}$, where $X_{u(\gamma)}$ and $X^{v(\gamma)}$ are Schubert and opposite Schubert varieties. Here X_γ is thought of as the closure of a K-orbit on $B\backslash G$ rather than a B-orbit on G/K.

All the local properties of the B-orbit closure on G/K are the same as for the K-orbit closure on $B\backslash G$, and local properties of X_u are the same as those for $X_{u^{-1}}$.

Understanding local properties of Richardson varieties completely reduces to understanding them on Schubert varieties, and local properties on Grassmannian Schubert varieties are mostly well understood.

Clans to skew partitions

There is a canonical bijection between (co)grassmannian permutations and partitions, where Bruhat order becomes containment.

The bijection $\gamma \mapsto (u(\gamma), v(\gamma))$ can be described in terms of the skew partition for u^{-1} and v^{-1}:

At the i-th step:

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- If the i-th entry of γ is a $-$, the paths go right.

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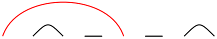
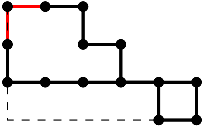
- If the i-th entry of γ starts a matching, the paths go away from each other.
- If the i-th entry of γ is a $+$, the paths go down.
- If the i-th entry of γ is a $-$, the paths go right.
- If the i-th entry of γ ends a matching, the paths go towards each other.

Clan bijection example

For example,

 corresponds to 

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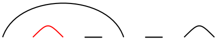
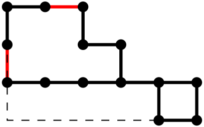
 corresponds to 

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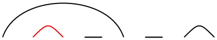

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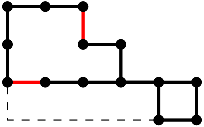

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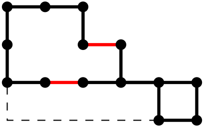

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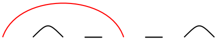
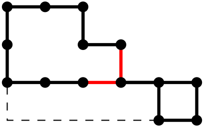
 corresponds to 

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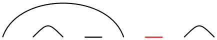

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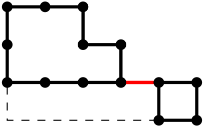

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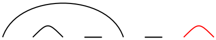

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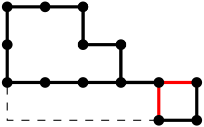

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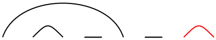

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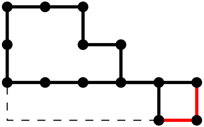

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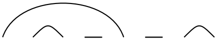
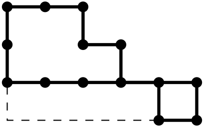
 corresponds to 

If we want to continue the bijection to permutations, we get $12734568 < 85376421$.

Smoothness for skew partitions

Assume u has at most one descent at p, and v has at most one ascent, also at p. Then X_u^v is smooth if and only if none of the "components" of $\lambda(v)/\lambda(u)$ has an **internal inner corner**.

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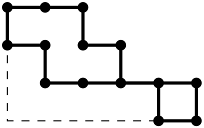
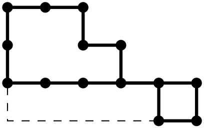
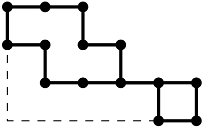
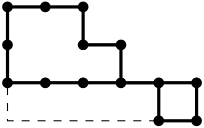
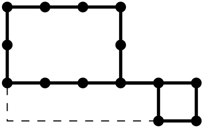
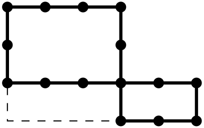
 and  are not smooth.

Smoothness for skew partitions

Assume u has at most one descent at p, and v has at most one ascent, also at p. Then X_u^v is smooth if and only if none of the "components" of $\lambda(v)/\lambda(u)$ has an **internal inner corner**.

For example,

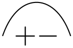
 and  are not smooth.

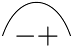
 and  are smooth.

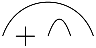
McGovern's Theorem

This gives a new proof of a weak version of McGovern's theorem:

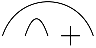
Theorem

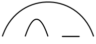
Let γ be a non-crossing clan. Then X_γ is NOT smooth if and only if γ contains one of the following 7 clans: 

, 

, 

, 

, 

, 

, *and* 

LCI for skew partitions

By a theorem of Darayon, X_u^v is a **local complete intersection** iff each component of the skew partition has at most one internal inner corner on each side, and the internal inner corners have arm and leg lengths 1.

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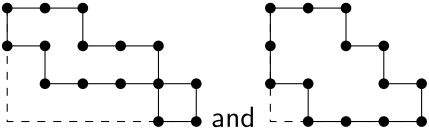

and are not lci.

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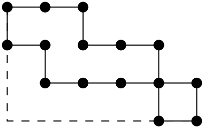
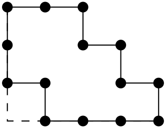
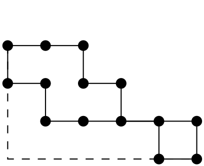
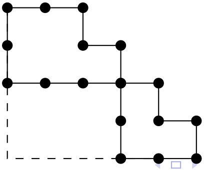
 and  are not lci.

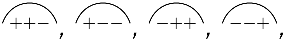
 and  are lci.

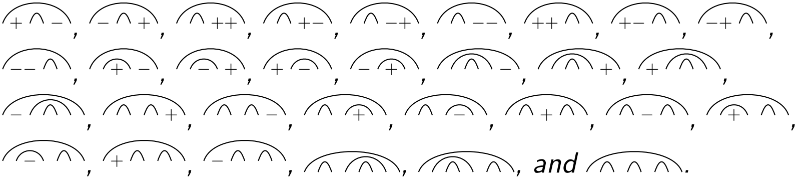
LCI in terms of non-crossing clans

Translating this into clans gives:

Theorem

Let γ be a non-crossing clan. Then Z_γ is NOT lci if and only if γ contains one of the following 35 clans:





Other singularity invariants

Similar theorems can be deduced for the Gorenstein property or for multiplicity using what is known for (co)grassmannian Schuberts.

There are also some other pairs (G, K) where the analogue of non-crossing clans gives Richardson varieties indexed by Weyl group elements inverse to cominuscule parabolic elements.

Hecke algebra action

The Hecke algebra $\mathcal{H}_W$ interpolates between the monoid/group algebras of the NilHecke monoid (at $q = 0$) and W (at $q = 1$). Hence we get an action of $\mathcal{H}_W$ on a module $\mathcal{M}_{G,K}$ that is a free $\mathbb{C}[q, q^{-1}]$-module with basis indexed by orbits (or, if we wish, orbits plus local systems).

Kazhdan–Lusztig–Vogan polynomials

Analogous to Kazhdan–Lusztig polynomials, one can define Kazhdan–Lusztig–Vogan polynomials.

- One definition of them is as entries in the transition matrix between a "canonical basis" on $\mathcal{M}_{G,K}$ and the standard basis indexed by orbits.
- They can also be defined purely combinatorially (without reference to the Hecke algebra).
- They are important invariants in the analogue of category $\mathcal{O}$ of real Lie groups.

For non-crossing clans, the formula of Lascoux for cograssmannian permutations can be used to give a similar formula for KLV polynomials.

Interval embeddings

Given intervals $[\sigma, \tau]$ and $[\gamma, \delta]$ in Bruhat order, we can say $[\gamma, \delta]$ **contains** $[\sigma, \tau]$ if δ contains τ and γ contains σ using the same locations, AND $\ell(\tau) - \ell(\sigma) = \ell(\delta) - \ell(\gamma)$.

We can say $[\gamma, \delta]$ avoids $[\sigma, \tau]$ otherwise.

A single clan (or permutation) δ avoids $[\sigma, \tau]$ if $[\gamma, \delta]$ avoids $[\sigma, \tau]$ for ALL $\gamma \leq \delta$.

Interval avoidance and singularities

We think we can prove interval pattern avoidance governs singularities on $(GL_n, GL_p \times GL_q)$ by working with explicit coordinates, but the general geometric explanation of pattern avoidance of Billey–Braden for Schubert varieties fails here.

Conclusion

- B-orbits on symmetric spaces are indexed by combinatorial objects that can be thought of as decorated involutions
- Notions from the combinatorics of Coxeter groups have analogues on these indexing sets, and most combinatorial questions have not been extensively explored.

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Thank you for your attention.