WHICH SCHUBERT VARIETIES ARE LOCAL COMPLETE INTERSECTIONS?

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Abstract. We characterize by pattern avoidance the Schubert varieties for GL_n which are local complete intersections (lci). For those Schubert varieties which are local complete intersections, we give an explicit minimal set of equations cutting out their neighborhoods at the identity. Although the statement of our characterization only requires ordinary pattern avoidance, showing that the Schubert varieties not satisfying our conditions are not lci appears to require working with more general notions of pattern avoidance. The Schubert varieties defined by inclusions, originally introduced by Gasharov and Reiner, turn out to be an important subclass, and we further develop some of their combinatorics. Applications include formulas for Kostant polynomials and presentations of cohomology rings for lci Schubert varieties.

Contents

1. Introduction

The main purpose of this paper is to characterize the Schubert varieties which are local complete intersections.

Let $G = \mathrm{GL}_n(\mathbb{C})$ and B a Borel subgroup, which we take to be the upper triangular matrices. The quotient G/B is a projective variety known as the **flag variety**; its points correspond to **complete flags**, which are chains of subspaces $F_\bullet = \langle 0 \rangle \subsetneq F_1 \subsetneq \cdots \subsetneq F_{n-1} \subsetneq \mathbb{C}^n$ with $\dim F_i = i$ for all i. The group G, and hence its subgroup B, acts on

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G/B by left multiplication. Given a permutation w, the **Schubert variety** X_w is the closure of the orbit BwB/B of the permutation matrix for w under the action of B.

A local ring R is a **local complete intersection** (**lci**) if it is the quotient of some regular local ring by an ideal generated by a regular sequence. A variety (or, more generally, a scheme) is lci if every local ring is lci. Since regular local rings are automatically lci, smooth varieties are automatically lci. Furthermore, lci varieties are automatically Gorenstein and hence Cohen–Macaulay. Thus, being lci can be viewed as saying that the singularities are in some sense mild.

Following earlier work of Wolper [Wol89] and Ryan [Rya87], Lakshmibai and Sandhya [LS90] found to some amazement at the time that smoothness of the Schubert variety X_w can be characterized by the combinatorial notion of **pattern avoidance**. A permutation $v \in S_m$ **embeds** in $w \in S_n$ if there are some m entries of w, say at indices $i_1 < \cdots < i_m$, in the relative order given by v, meaning that $w(i_j) < w(i_k)$ if and only if $v(j) < v(k)$. If v does not embed in w, then w is said to **avoid** v. Lakshmibai and Sandhya showed that X_w is smooth if and only if w avoids both of the permutations 3412 and 4231 (written in 1-line notation).

More recently, Yong and the second author characterized the permutations w for which X_w is Gorenstein [WY06]. This characterization cannot be given purely in terms of pattern avoidance but requires a more complicated generalization, either **interval pattern avoidance** (called Bruhat-restricted pattern avoidance in the original) or alternatively **bivincular patterns** as explained in [Ú11]. However, the lci Schubert varieties can be characterized by ordinary pattern avoidance. More precisely, we prove the following theorem.

Theorem 1.1. *The Schubert variety X_w is lci if and only if w avoids the six patterns* 53241*,* 52341*,* 52431*,* 35142*,* 42513*, and* 426153*.*

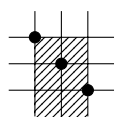
For convenience we work over $\mathbb{C}$ in this paper, but our results and proofs hold over $\mathbb{Z}$ and hence over any field.

Our proof for this theorem carries out the general strategy for any local property suggested by the work of Yong and the second author [WY08]. Let Ω_v denote the **opposite Schubert cell**, which is the orbit B_-vB/B of the permutation matrix v under the opposite Borel group B_- of lower triangular matrices. Furthermore, let $\mathcal{N}_{v,w}$ denote $\Omega_v \cap X_w$. It is a lemma of Kazhdan and Lusztig [KL79, Lemma A.4] that the point vB/B given by a permutation matrix v has a neighborhood in X_w which isomorphic to $\mathcal{N}_{v,w} \times \mathbb{C}^{\ell(v)}$.

For a permutation w avoiding the six patterns, we study explicit equations for $\mathcal{N}_{\mathrm{id},w}$ as a subvariety of Ω_{id} and explicitly find $\operatorname{codim}(X_w)$ generators for its defining ideal, hence showing that X_w is lci at the identity. This suffices since the locus of non-lci points on any scheme is closed; since this locus on a Schubert variety is B-invariant, it must therefore be a union of Schubert subvarieties and hence include the identity. We identify these generators based on the combinatorics of the **essential set** of w, which was originally defined by Fulton [Ful92] to give a minimal set of generators for the ideal defining a matrix Schubert variety. The combinatorics of the essential set were later further studied by Eriksson and Linusson [EL96].

To show that a permutation containing one of the six patterns is not lci, we first identify two infinite familes and eleven isolated pairs (u, v) such that $\mathcal{N}_{u,v}$ is not lci. The two infinite families are generic in the singular locus and were identified independently by Manivel [Man01a] and by Cortez [Cor03]. Now a theorem of Yong and the second author [WY08, Cor. 2.7] implies that if w fails to interval avoid any of these pairs $[u, v]$, then w will not be lci.

We then show that any permutation containing one of our six forbidden patterns must actually contain an interval either from one of the infinite families or from our list of eleven isolated cases. To formulate this proof, we must first translate both sets of avoidance conditions into **marked mesh patterns**, previously defined by the first author [Ú11]. Marked mesh patterns generalize mesh patterns, which were introduced in full generality by Brändén and Claesson [BC11], although special cases had previously been implicitly used, for example in the determination of the singular locus of Schubert varieties carried out independently by Manivel [Man01b], Kassel, Lascoux and Reutenauer [KLR03], Cortez [Cor03], and Billey and Warrington [BW03]. The original motivation for defining mesh patterns was to write various permutation statistics as linear combinations of permutation patterns. They have since been shown to characterize the permutations satisfying various properties. For example, the **simsun** permutations, introduced in [Sun94] and later named after Simion and Sundaram, are characterized by the avoidance of the mesh pattern



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Furthermore, it is not hard to see that every interval pattern can be described as a mesh pattern, as shown by the first author [Ú11, Lemma 22]. Marked mesh patterns are similar to mesh patterns but allow more control over the number of elements occupying a particular region in the graph of a permution.

Another related result is the characterization of Schubert varieties which are **defined by inclusions**, due to Gasharov and Reiner [GR02]. They show that X_w is defined by inclusions if w avoids 4231, 35142, 42513, and 426153. As one can tell from the patterns involved, our theorem implies that Schubert varieties defined by inclusions are lci, which was previously unknown. Indeed, the Schubert varieties defined by inclusions turn out to be an important special case in proving the sufficiency of our pattern avoidance conditions. In particular, we use the essential set to canonically associate a permutation defined by inclusions to any permutation indexing an lci Schubert variety. However, unlike the property of being lci, which is entirely intrinsic to the Schubert variety, it appears from the definition that whether a Schubert variety is defined by inclusions or not may depend on its embedding in the flag variety. It is not known if there is some intrinsic geometric characterization of being defined by inclusions.

More recently, Hultman, Linusson, Shareshian, and Sjöstrand [HLSS09] showed that, given a permutation w, the number of chambers in the inversion arrangement for w is equal to the number of permutations less than or equal to w in Bruhat order if and only if w avoids the same patterns 4231, 35142, 42513, and 426153. This follows earlier work by Sjöstrand [Sjö07] showing that the lower interval of w in Bruhat order corresponds to

rook configurations on a skew partition known as the **right hull** of w if and only if w avoids the same patterns. The connection between these results and that of Gasharov and Reiner is at present a complete mystery.

Hultman [Hul10] has extended this result to other finite reflection groups, but his characterization is in terms of a condition on the Bruhat graph rather than pattern avoidance conditions. We hope that this new result may help in finding a generalization of our theorem to Schubert varieties for other semisimple Lie groups. However, while there is a generalization of interval pattern avoidance for these other Schubert varieties [Woo10], explicit equations for the analogues of $\mathcal{N}_{u,v}$ are not generally known. Alternatively, it may be possible to give a characterization of lci Schubert varieties in terms of the Bruhat graph (possibly with some extra data in the non-simply-laced cases) rather than pattern avoidance.

One could hope to determine explicitly the (non-)lci locus of any Schubert variety. We conjecture that our list of interval patterns fully specifies the non-lci locus. In principle, this conjecture (or its correct version) can be proven by identifying explicit generators for all of the lci Kazhdan–Lusztig varieties $\mathcal{N}_{x,w}$ just as we do here for the case where x is the idenity and w avoids the given patterns. However, at least at first glance, the amount of combinatorial analysis required seems daunting. Another possible extension of our work would be to identify, for each $k > 0$, those Schubert varieties which fail to be lci by at most k excess generators (for the ideal generating $\mathcal{N}_{\mathrm{id},w}$). It would be interesting to know if this property is characterized by pattern avoidance, and furthermore avoidance of a finite number of patterns, for all k.

We remark on several further applications of our results. First we point out how various implications between properties of singularities can be derived purely combinatorially on Schubert varieties by containment of patterns. Also, we can recover results on lci matrix Schubert varieties due to Hsiao [Hsi11]. Furthermore, our proof of sufficiency gives explicit equations for $\mathcal{N}_{\mathrm{id},w}$ when w is lci. We describe two applications of this result. First, we give explicit formulas of the Kostant–Kumar polynomials [KK86, KK90] for both cohomology and K-theory at the identity (which are equivalent to certain specific specializations of the double Schubert and Grothendieck polynomials of Lascoux and Schützenberger [LS82a, LS82b]) in the case where w is lci. This calculation in the smooth case recovers a small part of a result of Kumar [Kum96] characterizing smooth points on Schubert varieties using the Kostant–Kumar polynomials and suggests a possible similar characterization of lci points as well as a potential local definition for being defined by inclusions. Second, we use the result of Akyildiz, Lascoux, and Pragacz [ALP92] identifying the cohomology ring $H^*(X_w)$ with a particular quotient of $\mathcal{N}_{\mathrm{id},w}$ to extend the presentation of $H^*(X_w)$ in the case X_w is defined by inclusions, due to Gasharov and Reiner [GR02], to a presentation of $H^*(X_w)$ for all lci Schubert varieties.

Furthermore, there has been some recent renewed interest in lci varieties in the context of jet schemes spurred by Mustata's theorem [Mus01] that an lci variety has an irreducible jet scheme if and only if the variety has canonical singularities. The lci Schubert varieties should provide a useful class of examples for understanding and possibly extending this theorem since they have a well understood resolution of singularities, the Bott–Samelson resolution [BS58]. (This is only a resolution of singularities in a weak sense since the

image of the exceptional locus contains nonsingular points.) The jet schemes for the special case of determinantal varieties has already been worked out by Mustata [Mus01], Yuen [Yue07], Kosir and Setharuman [KS05], and Docampo [Doc11].

In addition, Anderson and Stapledon [AS09] have recently developed an interpretation of classes in equivariant cohomology of a smooth variety Y as representing subvarieties of the arc space $J_\infty Y$. When X is an equivariant local complete intersection in Y, they show that powers of the cohomology class $[X] \in H_T^*(Y)$ represent certain subvarieties of $J_\infty X$. Our proof shows that the lci Schubert varieties are equivariant local complete intersections; hence they provide an interesting class of examples for their theory. In addition, they relate $[X]^{m+1}$ to the class of the jet scheme $J_m X$. Hence our formula for the Kostant–Kumar polynomials at the identity may help in determining local equations for $J_m X_w$ in the case where X_w is a local complete intersection.

Our paper is organized as follows. Section 2 gives definitions and basic facts about lci varieties, Schubert varieties, equations defining Schubert varieties, and various notions of pattern avoidance. In Section 3, we prove some combinatorial results on the essential sets of permutations which are defined by inclusions as well as permutations avoiding our given patterns. Some of these results may be of independent interest. Section 4 proves that permutations avoiding our given patterns are lci using the combinatorics of Section 3. Section 5 proves that permutations including our given patterns are not lci. We describe various applications in Section 6 and pose a number of open questions in Section 7.

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2. Definitions

Throughout this paper we will use the notation $[\![a, b]\!]$ to denote the set

$$[\![a, b]\!] := \{x \in \mathbb{Z} \mid a \leq x \leq b\}.$$

In particular, $[\![a, a]\!] = \{a\}$, and $[\![a, b]\!] = \emptyset$ when $b < a$. Also, $\#S$ will denote the number of elements in S.

2.1. **Local complete intersections.** Let R be a local ring. The ring R is called a **local complete intersection** (**lci**) if there exists a regular local ring S (meaning one where, letting m be the maximal ideal of S, the dimension of m/m^2 as an S/m vector space is the same as the Krull dimension of S) and an ideal I generated by a regular sequence

on S such that $R = S/I$. Regular rings are trivially local complete intersections, and, by the Kozsul resolution, local complete intersections are Gorenstein and hence Cohen-Macaulay. It turns out that the choice of regular ring S is irrelevant; if $R \cong S/I$ with R a local complete intersection and S any regular local ring, then I will always be generated by a regular sequence. Furthermore, whether R is or is not a local complete intersection can be detected purely by using the Ext functor on R and its residue field k. For details and other basic facts about local complete intersections, see the book by Bruns and Herzog [BH98].

An algebraic variety or scheme X is called a **local complete intersection** (**lci**) if for each point $p \in X$, the local ring $\mathcal{O}_{X,p}$ is a local complete intersection. For any variety, the locus V of points p for which $\mathcal{O}_{X,p}$ is *not* lci is a *closed* set (in the Zariski topology).

Note that if $S = \mathbb{C}[x_1, \ldots, x_n]$, I is an ideal of S generated by k elements, and $\dim \operatorname{Spec} S/I = n-k$, then $\operatorname{Spec} S/I$ is automatically lci, as localization can never increase the number of generators needed for an ideal.

2.2. **Schubert varieties.** Let $G = \mathrm{GL}_n(\mathbb{C})$, which we think of explicitly as the group of invertible $n \times n$ matrices, and let B, B_-, and T denote respectively the subgroups of invertible upper triangular, lower triangular, and diagonal matrices. The **flag variety** is the quotient space G/B; upon a choice of basis for $\mathbb{C}^n$, a point $gB \in G/B$ is naturally identified with a **flag** $F_\bullet : \langle 0 \rangle \subsetneq F_1 \subsetneq \cdots \subsetneq F_{n-1} \subsetneq \mathbb{C}^n$ by taking F_i to be the span of the first i columns of any coset representative g of gB.

Let $w \in S_n$ be a permutation. We think of w as a permutation matrix with 1's at row $w(j)$ (counted from the top) and column j for each j and with 0's everywhere else. We let e_w denote the **Schubert point** which is the coset $wB \in G/B$. The orbit $BwB/B \subset G/B$ is known as a **Schubert cell**, and its closure $X_w = \overline{BwB/B} \subset G/B$ is a **Schubert variety**. The orbit $B_-wB/B \subset G/B$ is known as an **opposite Schubert cell** and denoted Ω_w°. Our conventions are such that X_{id} is a point and X_{w_0} (where w_0 is the long permutation defined by $w_0(i) = n+1-i$ for all i) is all of G/B.

2.3. **Rank conditions for Schubert varieties.** Schubert varieties can be alternatively defined as the set of points representing flags satisfying certain intersection conditions with the standard flag or equivalently as the set of B-cosets with representatives satisfying certain rank conditions on southwest submatrices. For a permutation w, define the **rank function** r_w by

$$r_w(p,q) = \#\{k \leq q \mid w(k) \geq p\}.$$

Let $E_\bullet$ be the flag where E_p is the span of the first p standard basis vectors; this flag is known as the **standard flag**. A flag $F_\bullet$ represents a point gB in the Schubert variety X_w if and only if $\dim(E_p \cap F_q) \geq q - r_w(p+1, q)$ for all $p, q \in [\![1, n]\!]$. This is equivalent to the rank of the southwest $(n+1-p) \times q$ submatrix (consisting of the $n+1-p$ bottommost rows and q leftmost columns) of any coset representative g of gB being at most $r_w(p,q)$ for all p and q.

Many of these rank conditions are redundant, and Fulton [Ful92] showed that the minimal set of conditions defining any Schubert variety are those from what he called the

essential set. The **Rothe diagram** of w is the set of boxes (which we can think of as being drawn over the permutation matrix)

$$\{(p,q) \in [\![1,n]\!] \times [\![1,n]\!] \mid w(q) < p, w^{-1}(p) > q\}.$$

The diagram can be described visually as follows. For each $q \in [\![1,n]\!]$, draw a dot $\bullet$ at $(w(q),q)$. For each dot draw the "hook" that extends north and east of that dot. The boxes not in any hook are the boxes of the diagram. The **essential set** $E(w)$ is the set of boxes in $D(w)$ which are northeast corners in some connected component of $D(w)$. To be precise,

$$E(w) = \{(p,q) \in D(w) \mid (p,q+1) \notin D(w), (p-1,q) \notin D(w)\},$$

and a matrix g represents a point $gB \in X_w$ if and only if the southwest $(n+1-p) \times q$ submatrices of g have rank at most $r_w(p,q)$ for all $(p,q) \in E(w)$. Furthermore, $E(w)$ is the minimal subset of $[\![1,n]\!] \times [\![1,n]\!]$ with this property; no subset of $E(w)$ will correctly define X_w.

Notational Warning 2.1. There are a number of different conventions for the essential set in the literature. In particular, our convention is different from the original one used by Fulton [Ful92] and is known in some sources as the coessential set.

Example 2.2. Let $w = 819372564$. Then the diagram and essential set of w are as in Figure 1. In particular, $E(w) = \{(2,2),(4,4),(4,6),(6,7),(9,2)\}$. □

$D(819372564) =$

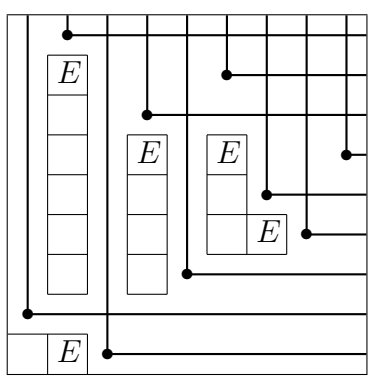


FIGURE 1. Diagram and essential set for $w = 819372564$.

2.4. **Local neighborhoods, Kazhdan–Lusztig varieties, and explicit equations.** We now explain how local properties of X_w can be explicitly calculated. The contents of this section can be found in greater detail (and with proofs) in [WY08, Section 3.2], and the ideas behind it can partially be traced back to Fulton [Ful92].

The opposite Schubert cell $\Omega^\circ_{\mathrm{id}} \subset G/B$ is an open affine neighborhood of e_{id}, and, given any v, $v\Omega^\circ_{\mathrm{id}}$ is an affine neighborhood of the Schubert point e_v. Since B acts on any Schubert variety X_w, it suffices to study one point in each B-orbit, which we take to be the Schubert point e_v, so it suffices to study $X_w \cap v\Omega^\circ_{\mathrm{id}}$. Moreover, by [KL79, Lemma A.4],

$$X_w \cap v\Omega^\circ_{\mathrm{id}} \cong (X_w \cap \Omega^\circ_v) \times \mathbb{C}^{\ell(v)}.$$

Hence, to check if X_w is locally a complete intersection at e_v, it suffices to study the **Kazhdan–Lusztig variety**

$$\mathcal{N}_{v,w} := X_w \cap \Omega_v^\circ.$$

We now give equations which define $\mathcal{N}_{v,w}$ scheme-theoretically. Given a permutation v, let S_v be the polynomial ring whose variables are labelled by the boxes in the diagram of v, so $S_v = \mathbb{C}[z_{p,q}]_{(p,q)\in D(v)}$. Furthermore, let M_v be the matrix with a 1 as the entry at $(v(i), i)$ for each i, $z_{p,q}$ at $(p,q) \in D(v)$, and 0's everywhere else.

For any subsets A and B of $[\![1,n]\!]$ such that both A and B have the same number of elements, let $d^{(v)}_{A,B}$ denote the minor of M_v which is the determinant of the square matrix whose rows are the rows of M_v indexed by elements of A and whose columns are the columns of M_v indexed by elements of B. We will refer to $d^{(v)}_{A,B}$ as a **generalized Plücker coordinate**.

The ring S_v has a grading where $\deg z_{p,q} = p - v(q)$. Note that

$$\deg d^{(v)}_{A,B} = \sum_{p\in A} p - \sum_{q\in B} v(q).$$

Futhermore, $d^{(v)}_{A,B} = 0$ if $\deg d^{(v)}_{A,B} < 0$, and $d^{(v)}_{A,B} \in \{-1, 0, 1\}$ if $\deg d^{(v)}_{A,B} = 0$.

Given $p, q, r \in [\![1,n]\!]$, let $I^{(v)}_{(p,q,r)}$ be the ideal of S_v generated by all $d^{(v)}_{A,B}$ where $A \subseteq [\![p,n]\!]$, $B \subseteq [\![1,q]\!]$, and $\#A = \#B = r+1$; these are all the $r+1$ size minors of the rectangular submatrix consisting of all entries (weakly) SW of (p,q). Given a permutation w, let

$$I_{v,w} = \sum_{(p,q)\in E(w)} I^{(v)}_{(p,q,r_w(p,q))}.$$

The following is a restatement of [WY08, Prop. 3.1]; this Proposition was first stated in a less concise form in [Ful92].

Proposition 2.3. *The Kazhdan–Lusztig variety*

$$\mathcal{N}_{v,w} \cong \operatorname{Spec} S_v/I_{v,w}.$$

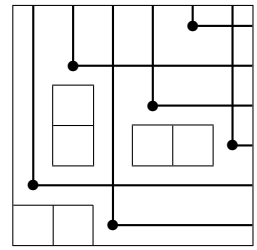
We will be particularly interested in the special case where $v = \mathrm{id}$. Hence, in the remainder of this paper, we will omit v from our notation in this case, so $S = S_v$, $I_w = I_{v,w}$, $d_{A,B} = d^{(v)}_{A,B}$, and $I_{(p,q,r)} = I^{(v)}_{(p,q,r)}$. Note that, in this case, $\deg z_{p,q} = p - q$, and

$$\deg d_{A,B} = \sum_{p\in A} p - \sum_{q\in B} q.$$

Notational Warning 2.4. In [WY08], the variables in matrices are indexed with $z_{p,q}$ being the variable in the p-th row counting from the bottom. This was done for partial compatibility with the conventions for matrix Schubert varieties. Since matrix Schubert varieties play only a marginal role in this paper, we have abandoned that convention and index our matrix variables in the usual way, with $z_{p,q}$ being the entry in row p (counting from the top) and column q (counting from the left).

Example 2.5. Let $v = 215436$ and $w = 526314$. Then

$$M_v = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ z_{3,1} & z_{3,2} & 0 & 0 & 1 & 0 \\ z_{4,1} & z_{4,2} & 0 & 1 & 0 & 0 \\ z_{5,1} & z_{5,2} & 1 & 0 & 0 & 0 \\ z_{6,1} & z_{6,2} & z_{6,3} & z_{6,4} & z_{6,5} & 1 \end{pmatrix} \text{ and } D(w) =$$



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Therefore, $S_v = \mathbb{C}[z_{3,1}, z_{3,2}, z_{4,1}, z_{4,2}, z_{5,1}, z_{5,2}, z_{6,1}, z_{6,2}, z_{6,3}, z_{6,4}, z_{6,5}]$. Also

$$I_{v,w} = I^{(v)}_{(6,2,0)} + I^{(v)}_{(3,2,1)} + I^{(v)}_{(4,5,2)}.$$

Now, $I^{(v)}_{(6,2,0)} = \langle z_{6,1}, z_{6,2} \rangle$. The generalized Plücker coordinate $d_{\{i,6\},\{1,2\}} \in I^{(v)}_{(6,2,0)}$ for all i, so

$$I^{(v)}_{(6,2,0)} + I^{(v)}_{(3,2,1)} = \langle z_{6,1}, z_{6,2}, z_{3,1}z_{4,2} - z_{3,2}z_{4,1}, z_{3,1}z_{5,2} - z_{3,2}z_{5,1}, z_{4,1}z_{5,2} - z_{4,2}z_{5,1} \rangle.$$

Since $d_{\{4,5,6\},\{3,4,5\}} = -z_{6,5}$, and the only generalized Plücker coordinates among the generators of $I^{(v)}_{(4,5,2)}$ that are not either a multiple of $-z_{6,5}$ or in $I^{(v)}_{(6,2,0)} + I^{(v)}_{(3,2,1)}$ are $d_{\{4,5,6\},\{1,3,4\}}$ and $d_{\{4,5,6\},\{2,3,4\}}$,

$$\begin{aligned} I_{v,w} = \langle & z_{6,1}, z_{6,2}, z_{3,1}z_{4,2} - z_{3,2}z_{4,1}, z_{3,1}z_{5,2} - z_{3,2}z_{5,1}, z_{4,1}z_{5,2} - z_{4,2}z_{5,1}, \\ & z_{6,5}, z_{4,1}z_{6,4} + z_{5,1}z_{6,3}(-z_{6,1}), z_{4,2}z_{6,4} + z_{5,2}z_{6,3}(-z_{6,2}) \rangle. \end{aligned}$$

If we let $a = z_{3,1}$, $b = z_{3,2}$, $c = z_{4,1}$, $d = z_{4,2}$, $e = -z_{6,3}$, $f = z_{5,1}$, $g = z_{5,2}$, and $h = z_{6,4}$, then

$$\mathcal{N}_{v,w} \cong \operatorname{Spec} \mathbb{C}[a,b,c,d,e,f,g,h]/\langle ad - bc, ag - bf, cg - df, ch - ef, dh - eg \rangle,$$

which we can think of as the variety of rank 1 "matrices"

$$\begin{pmatrix} a & b & \\ c & d & e \\ f & g & h \end{pmatrix}$$

This is not a local complete intersection, as it is a codimension 3 variety whose ideal requires 5 generators. □

Example 2.6. Let $w = 819372564$, as in Example 2.2, and let $v = \mathrm{id}$. Then

$$I_w = I_{(2,2,1)} + I_{(4,4,2)} + I_{(4,6,3)} + I_{(6,7,3)} + I_{(9,2,0)}.$$

A priori, $I_{(2,2,1)}$ is generated by the $\binom{8}{2}\binom{2}{2}$ generalized Plücker coordinates $d_{A,\{1,2\}}$ where A is a 2-element subset of $\{2, \ldots, 9\}$. Also, $I_{(4,4,2)}$ is a priori generated by $\binom{6}{3}\binom{4}{3}$ generalized Plücker coordinates, $I_{(4,6,3)}$ is generated by $\binom{6}{4}\binom{6}{4}$, $I_{(6,7,3)}$ by $\binom{7}{4}\binom{4}{4}$ (some of which are shared with $I_{(4,6,3)}$), and $I_{(9,2,0)} = \langle d_{\{9\},\{1\}} = z_{9,1}, d_{\{9\},\{2\}} = z_{9,2} \rangle$. However, in our proof that X_w is lci, we will see that only $\binom{n}{2} - \ell(w) = \#D(w) = 16$ of these generators are needed. □

2.5. **Pattern avoidance and generalizations.** As mentioned in the introduction we say that a permutation $v \in S_m$ **embeds** in $w \in S_n$ (or w **contains** v) if there are some m entries of w, say at indices $i_1 < \cdots < i_m$, in the relative order given by v, meaning that $w(i_j) < w(i_k)$ if and only if $v(j) < v(k)$. If v does not embed in w, then w is said to **avoid** v. When we discuss a permutation as being embedded or being avoided by another permutation we usually call it a (**classical**) **pattern**. For example, the pattern 132 has three embeddings in the permutation 526413, namely they are 5**264**13, 5**2**641**3** and 5**2**641**3**. Notice in particular that the indices of an embedding need not be adjacent in the permutation. One of the original motivations of studying permutations as patterns is their relation to sorting algorithms in computer science. Probably the earliest example of such an application is the characterization by Knuth [Knu73] of stack-sortable permutations as the ones avoiding 231. Since then, many extensions of classical patterns have been introduced; the ones relevant here are those of interval patterns and (marked) mesh patterns.

Interval patterns were introduced by Yong and the second author in [WY08] and, we now recall their definition. First recall that the **Bruhat order** on the symmetric group is the reflexive transitive closure of the partial order defined by declaring u to be less than or equal to v if $v = us_{ij}$ and $\ell(v) > \ell(u)$. Here s_{ij} is the transposition that switches the (not necessarily adjacent) positions i and j, and $\ell(v)$ is the number of inversions in the permutation v, or equivalently, the length of any reduced expression for v as a product of simple reflections $s_{i(i+1)}$, called the **Coxeter length** of v. We use the symbol "$\leqslant$" to denote the Bruhat order. Now, if $[u, v]$ and $[x, w]$ are intervals in the Bruhat orders on S_m and S_n respectively, we say that $[u, v]$ (**interval**) **pattern embeds in** $[x, w]$ if there is a common embedding consisting of indices $i_1 < \cdots < i_m$ of u in x and v in w, such that the entries of x and w outside of these indices agree, and additionally, the intervals $[u, v]$ and $[x, w]$ are isomorphic posets. The motivation for these patterns is that they govern any "reasonable" local property, as shown by Yong and the second author [WY08]. Since, given u, v, w, and the indices of the embedding, the permutation x is automatically determined, we can omit x in the notation. Hence we will abuse terminology to say that $[u, v]$ embeds in w or that w avoids $[u, v]$ as appropriate.

Interval patterns are a special case of mesh patterns, which we now define. A **mesh pattern** is a pair (v, R) where v is a permutation (classical pattern) from S_m and R is a subset of the square $[\![0, m]\!] \times [\![0, m]\!]$. An embedding of (v, R) in a permutation w is first of all an embedding of v in w in the usual sense, meaning indices $i_1 < \cdots < i_m$ such that the relative order of $w(i_1), \ldots, w(i_m)$ is given by v. Equivalently, we have order-preserving bijections $\alpha, \beta \colon [\![1, m]\!] \to [\![1, n]\!]$ such that

$$\{(\alpha(i), \beta(j)) \mid (i, j) \in G(v)\} \subseteq G(w),$$

where for any permutation u, $G(u)$ is defined to be the graph

$$G(u) = \{(i, u(i)) : i \in [\![1, n]\!]\}$$

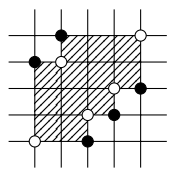
of u. In addition, to be an embedding of (v, R), we further require the following:

$$\text{If } (i, j) \in R \text{ then } R_{ij} \cap G(u) = \emptyset.$$

Here R_{ij} is defined as the rectangle $[\![\alpha(i)+1, \alpha(i+1)-1]\!] \times [\![\beta(j)+1, \beta(j+1)-1]\!]$, where, as a convention, we set $\alpha(0) = 0 = \beta(0)$ and $\alpha(m+1) = n+1 = \beta(m+1)$.

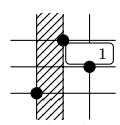
As a simple example, consider the mesh pattern $(12, \{(1,0), (1,1), (1,2)\})$ which can be depicted as follows:

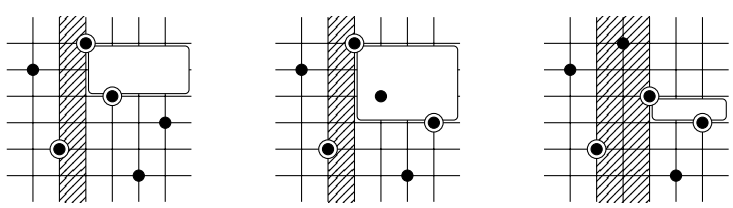
An occurrence of this mesh pattern in a permutation is a non-inversion (an occurrence of the classical pattern 12) with the additional requirement that there is nothing in between the two elements in the occurrence. This is also called a **vincular pattern** and denoted $\underline{12}$. See [Ú11, Subsec. 4.1] for more examples. As one additional example we show how [Ú11, Lemma 22] can be used to translate interval patterns into mesh patterns: Take for example the interval pattern $[14235, 45123]$. This can be translated into the mesh pattern



(The white dots are not part of the mesh pattern; they only indicate the permutation 14235 from the interval.)

The definition of marked mesh patterns given by the first author [Ú11, Subsec. 4.1] extends the definition of mesh patterns and allows another kind of designated regions where a certain number of elements is required to be present. We only review their definition via an example:

Example 2.7. To show that the marked mesh pattern  occurs in the permutation 526413, we first need to find an occurrence of the underlying classical pattern 132. There are three such occurrences, as shown below.



However, only the middle occurrence of 132 is an occurrence of the marked mesh pattern since it is the only occurrence having *at least* one dot in the box marked with "1" in the pattern, as well as having no dots in the shaded vertical strip. □

Notational Warning 2.8. Unfortunately, the customary conventions for writing a permutation in a matrix (stemming ultimately from the conventions for matrix multiplication) is upside down from the customary conventions for the graph of a permutation (stemming ultimately from the conventions for drawing the graph of a function). Furthermore, the convention for indexing entries in a matrix disagree with the cartesian

convention for indexing points on a graph. Hence our conventions for drawing mesh patterns are upside down from our conventions for writing matrices and for drawing Rothe diagrams and essential sets. Our conventions for indexing regions in mesh patterns and regions in Rothe diagrams and essential sets also disagree.

3. Rothe diagrams of lci permutations

Prior to proving the sufficiency of our pattern avoidance conditions, we need some detailed information on the diagrams and essential sets of permutations avoiding the given patterns. This information may be of independent combinatorial interest. We begin by studying the special case of Schubert varieties **defined by inclusions**, which were introduced in a different context by Gasharov and Reiner [GR02]. In particular, the essential sets for permutations indexing these Schubert varieties satisfy certain combinatorial conditions. We weaken these conditions to define what it means for a permutation to be **almost defined by inclusions** and show that every permutation avoiding the given patterns is almost defined by inclusions. To each permutation almost defined by inclusions, we will associate by modifying the diagram a permutation honestly defined by inclusions.

3.1. **Permutations defined by inclusions.** Let $w \in S_n$ be a permutation. We say that w is **defined by inclusions** if, for each box $(p,q) \in E(w)$, $q - r_w(p,q) = \min\{p-1, q\}$. To explain the terminology, note that this condition on the essential set is equivalent to the statement that the intersection conditions defining the Schubert variety are all of the form $E_{p-1} \subset F_q$ or $F_q \subset E_{p-1}$. Gasharov and Reiner proved the following theorem [GR02, Thm. 4.2].

Theorem 3.1. *The following are equivalent:*

(1) *The Schubert variety X_w is defined by inclusions.*
(2) *For every box $(p,q) \in E(w)$, either*
   A: *there are no 1's in the permutation matrix w SW of (p,q) (In other words, there is no k such that $k \leq q$ and $w(k) \geq p$.); or*
   B: *there are no 1's in the permutation matrix w weakly NE of (p,q) (In other words, there is no k such that $k > q$ and $w(k) < p$.)*
(3) *The permutation w avoids* 4231, 35142, 42513, *and* 351624.

For an essential set box $(p,q) \in E(w)$ satisfying condition A, $r_w(p,q) = 0$. This portion of the essential set and of the diagram will not require further combinatorial attention. We now focus on the remainder of the diagram and essential set. Let $D'(w) \subset D(w)$ be the subset consisting of diagram boxes (x,y) where $r_w(x,y) > 0$, and let $E'(w) \subset E(w)$ be the subset of essential set boxes (p,q) satisfying $r_w(p,q) > 0$. We begin with a lemma relating the position (p,q) to $r_w(p,q)$ for $(p,q) \in E'(w)$.

Lemma 3.2. *Let w be a permutation defined by inclusions. Let $(p,q) \in E'(w)$. Then $p \leq q$ and $r_w(p,q) = q - p + 1$.*

Visually, this says that the rank associated to the essential set box (p,q) is its Manhattan distance above the main diagonal plus 1.

Proof. Note that $r_w(p,q)$ is the number of 1's strictly SW of (p,q) in the permutation matrix w. Since there is an 1 in every row and column of w and in particular in every row below row p and every column to the left of column q,

$$(n-p)+(q-1) = s + r_w(p,q) + t + r_w(p,q),$$

where s and t are respectively the number of 1's strictly SE and NW of (p,q). Since $(p,q) \in E'(w)$, there are no 1's strictly to the NE of (p,q). There is one 1 in row p, one 1 in column q, and the remainder are counted once in s, t, or $r_w(p,q)$, so the total number of 1's is $n = s + t + r_w(p,q) + 2$. Therefore,

$$s + t + r_w(p,q) + 2 - p + q - 1 = s + t + 2r_w(p,q),$$

so

$$q - p + 1 = r_w(p,q),$$

as required. □

Now we state a lemma on the relative positions of boxes in $E'(w)$.

Lemma 3.3. *Let w be defined by inclusions, and let (p,q) and (p',q') be two distinct elements of $E'(w)$. Then either $p < p'$ and $q < q'$, or $p > p'$ and $q > q'$.*

In other words, any two boxes in $E'(w)$ are strictly NW and SE of each other. Indeed, we can define a partition λ (drawn in the French manner) whose outer corners are the boxes of $E'(w)$. (Actually, λ should also include any boxes $(p,q) \in E(w) \setminus E'(w)$ which nevertheless satisfy condition B.) We can also define a partition μ whose diagram is $D(w) \setminus D'(w)$; its outer corners are the essential set boxes of rank 0. The skew partition λ/μ is not exactly the right hull of the permutation w as studied by Sjöstrand [Sjö07], but our partition μ is the same as his, and our partition λ is slightly smaller but closely related. It appears that further combinatorial considerations on the diagrams of the permutations may elucidate the connection between the results of Sjöstrand and of Gasharov and Reiner.

Proof. If $p = p'$, then we can assume without loss of generality that $q > q'$. Since (p,q) is in the diagram, $w(q) < p$. Then $w(q) < p'$ and $q > q'$, so (p',q') does not satisfy Condition B.

Otherwise, assume without loss of generality that $p < p'$. If $q \geq q'$, then since (p,q) is in the diagram, $w^{-1}(p) > q$. Therefore, $p < p'$ and $w^{-1}(p) > q'$, so (p',q') does not satistfy Condition B. □

We now describe a partition of $D'(w)$ into rectangular regions, one associated to each box of $E'(w)$. Let k be the number of elements of $E'(w)$. We order the boxes in $E'(w)$ by rank (with ties broken arbitrarily) and label them $(p_1,q_1),\ldots,(p_k,q_k)$, so $r_w(p_1,q_1) \leq \cdots \leq r_w(p_k,q_k)$. Let $R_m \subset D'(w)$ be the set of diagram boxes which are weakly SW of (p_m,q_m) but not weakly SW of $(p_{m'},q_{m'})$ for any $m' < m$. For convenience, let $r_m = r_w(p_m,q_m)$.

Lemma 3.4. *Each region R_m is a rectangle consisting of boxes all from the connected component of $D(w)$ containing (p_m,q_m).*

Proof. Suppose $(x, y) \in D(w)$ is weakly SW of (p_m, q_m) but not in the same component as (p_m, q_m). Since there is one or more crossed out lines between (x, y) and (p_m, q_m) in the drawing of the diagram, $r_w(x, y) < r_w(p_m, q_m)$. If $r_w(x, y) = 0$, then $(x, y) \notin D'(w)$. Otherwise, (x, y) is SW of some essential set box $(p_{m'}, q_{m'})$ in its own connected component. Since $r_w(p_{m'}, q_{m'}) = r_w(x, y) < r_w(p_m, q_m)$, we must have that $m' < m$ by the requirements on our ordering of $E'(w)$. Therefore, by our definition of the region R_m, $(x, y) \notin R_m$.

Because all connected components of $D(w)$ form the diagram of a partition shape, and the essential set boxes in any connected component are the outer corners of the partition, each R_m must be a rectangle. □

For each integer $m \in [\![1, k]\!]$, we define $\mathsf{Wpred}(m)$ as the index of the first region (other than R_m) with a box directly W of (p_m, q_m) and $\mathsf{Spred}(m)$ as the index of the first region (other than R_m) with a box directly S of (p_m, q_m). If no such region exists, we accordingly let $\mathsf{Wpred}(m) = 0$ or $\mathsf{Spred}(m) = 0$. By our definition of the regions R_m, $\mathsf{Wpred}(m) < m$, and $\mathsf{Spred}(m) < m$.

We remark that, in the remainder of the paper, it is not absolutely necessary that our essential set boxes be ordered strictly in increasing rank. Rather, any ordering for which Lemma 3.4 holds will suffice. Explicitly, this means that if $m' < m$, then either $r_{m'} \leq r_m$, or there exists $m'' < m'$ where $(p_{m''}, q_{m''})$ is visually between (p_m, q_m) and $(p_{m'}, q_{m'})$ in the NW–SE ordering of $E'(w)$ on the drawing of the diagram.

Example 3.5. The permutation $w = 819732654$ is defined by inclusions. The diagram and essential set of w are as in Figure 1. In particular, $E(w) = \{(2, 2), (4, 6), (9, 2)\}$. Furthermore, $E'(w) = \{(2, 2), (4, 6)\}$, $(p_1, q_1) = (2, 2)$ with $r_1 = 1$, and $(p_2, q_2) = (4, 6)$ with $r_2 = 3$. In this case, both R_1 and R_2 are entire connected components of $D'(w)$, but it is possible for a connected component of $D'(w)$ to be partitioned into several regions. □

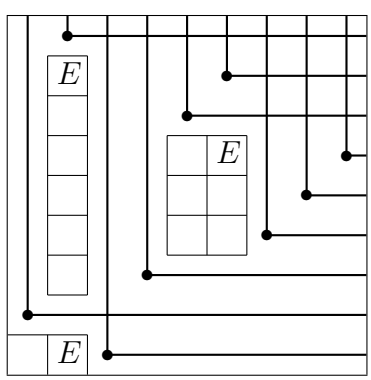
$D(819732654) =$ 

FIGURE 2. Diagram and essential set for $w = 819732654$.

3.2. **Permutations almost defined by inclusions.** Now we define conditions on the diagram of a permutation w which are a weakening of the conditions of Gasharov and Reiner. We say a permutation w is **almost defined by inclusions** if, for all $(p, q) \in$

$E(w)$, either (p,q) satisfies one of the Conditions A and B defined by Gasharov and Reiner, or (p,q) satisfies both one of the following Conditions W and X and one of the following conditions Y and Z. Note Conditions W and X are respectively the mirror images of Conditions Y and Z under reflection across the main antidiagonal (which in terms of permutations takes w to $w_0w^{-1}w_0$).

W: For all $p' < p$, $(p',q) \notin E(w)$ (and hence, for all $p' < p$, $(p',q) \notin D(w)$). Furthermore, either $(p,q-1) \notin D(w)$, or there exists $p' < p$ such that $(p',q-1) \in E(w)$, $(p',q-1)$ satisfies Condition B, and $r_w(p',q-1) = r_w(p,q)$. (The last part of the previous condition is equivalent to $(p',q-1)$ and (p,q) being in the same connected component of $D(w)$.)

X: There exists a unique $p' < p$ such that $(p',q) \in E(w)$. Furthermore, (p',q) satisfies Condition B, and $r_w(p',q) = r_w(p,q)+1$. Finally, if q' is the smallest integer such that $(p',b) \in D(w)$ for all $b \in [\![q',q]\!]$, then $(p,q'-1) \in D(w)$. (Given the first two conditions, the combinatorics of diagrams always implies that $q' > 1$ and that $(p,b) \in D(w)$ for all $b \in [\![q',q]\!]$.)

Y: For all $q' > q$, $(i,q') \notin E(w)$. Furthermore, either $(p+1,q) \notin D(w)$, or there exists $q' > q$ such that $(p+1,q') \in E(w)$, $(p+1,q')$ satisfies Condition B, and $r_w(p+1,q') = r_w(p,q)$.

Z: There exists a unique $q' > q$ such that $(p,q') \in E(w)$. Furthermore, (p,q') satisfies Condition B and $r_w(p,q') = r_w(p,q)+1$. Finally, if p' is the greatest integer such that $(a,q') \in D(w)$ for all $a \in [\![p,p']\!]$, then $(p'+1,q) \in D(w)$.

Note Conditions W and X are mutually exclusive, as are Y and Z. By the **type** of an essential set box failing conditions A and B we mean the pair of conditions among W, X, Y, and Z it satisfies, so a Type WZ essential set box is one that satisfies Conditions W and Z.

Example 3.6. Let $w = 819372564$, as in Example 2.2 and Figure 1. The essential set boxes at $(2,2)$, $(4,6)$, and $(9,2)$ satisfy conditions A and B. The essential set box at $(4,4)$ is of Type WZ, and the essential set box at $(6,7)$ is of Type WY. □

It will turn out that the permutations that are almost defined by inclusions are precisely the ones indexing lci Schubert varieties. In the interest of keeping the logic of the proof clear, we will maintain a distinction between the two notions until we have proved their equivalence.

We now show that avoiding the six given patterns implies that a permutation is almost defined by inclusions.

Theorem 3.7. *If a permutation is not almost defined by inclusions, then it contains one of the patterns* 53241*,* 52341*,* 52431*,* 35142*,* 42513*, and* 351624*.*

Our proof follows Gasharov and Reiner's proof for Theorem 3.1 with some additional complications required in our case.

Proof. Suppose our permutation w is not almost defined by inclusions. Therefore, either there is some box $(p,q) \in E(w)$ that satisfies none of Conditions A, B, W, and X, or there is some box $(p,q) \in E(w)$ that satisfies none of Conditions A, B, Y, and Z. We

will give the details of our proof only in the former case; the latter case can be proved in an entirely identical fashion, except that we switch N and E and switch W and S on diagrams, which corresponds to changing w to $w_0 w^{-1} w_0$.

Our proof strategy will be to split into a number of cases depending on the features of $D(w)$ and the placement of the 1's relative to $D(w)$. In each case we will find one of the stated patterns in w.

Violation of A requires a 1 in the permutation matrix SW of (p, q); we choose c so that $(w(c), c)$ is SW of (p, q). Similarly, we choose c' such that $(w(c'), c')$ is NE of (p, q); the existence of c' is guaranteed by violation of B.

We now split into two cases depending on whether there is a diagram box directly N of (p, q).

(1) For all $p' < p$, the box $(p', q) \notin D(w)$.
Note that $w(j) > w(c')$, as otherwise $(w(c'), j)$ would be in the diagram, contradicting our assumption for this case. Then, since (p, q) violates Condition W, the box $(p, q-1)$ is in the diagram, and it is not S of a box in the same component satisfying Condition B. We now have two cases depending on whether $(p-1, q-1)$ is in the diagram.

(a) The box $(p-1, q-1) \notin D(w)$.
In this case, $w^{-1}(p-1) < q$. Let B be the rectangular box bounded by rows $p-1$ and $w(c)$ and columns q and c', and let B' be the box bounded by rows $w(c')$ and p and columns c and q.
If B does not contain a 1 in its interior, then $w(q+1) > w(c)$ and $w^{-1}(p) > c'$. Therefore, the 1's in columns $c, q, q+1, c'$, and $w^{-1}(p)$ form a 42513 pattern.
If B contains a 1 in its interior, there are two cases. If B' contains a 1 in its interior, then the 1's in B and B' along with the 1's in columns c, q, and c' produce a 52341 or 53241 pattern, depending on whether the 1 in B' is above or below row $w(q)$. If B' does not contain a 1 in its interior, then $w^{-1}(p-1) < c$ and $w(q-1) < w(c')$. Therefore, the 1's in columns $w^{-1}(p-1), c, q-1$ and c' along with the 1 in B produce a 35142 pattern (where the 1 in B represents the 4).

(b) The box $(p-1, q-1) \in D(w)$.
In this case, $w(q) = p-1$. Furthermore, there exists a unique $p' < p$ such that $(p', q-1) \in E(w)$ and $r_w(p', q-1) = r_w(p, q)$. (This is the essential set box in column $q-1$ in the same connected component of $D(w)$.) Since (p, q) does not satisfy Condition W, $(p', q-1)$ does not satisfy condition B. We change our choice of c' if necessary so that $(w(c'), c')$ is NE of both (p, q) and $(p', q-1)$. Now let B be the rectangular box bounded by rows $p-1$ and $w(c)$ and columns q and c', as in Case 1a, and let B' be the box bounded by rows $w(c')$ and p' and columns c and q.
The argument is now the same as in Case 1a, with two minor differences. In the case both B and B' contain a 1, the pattern 52341 is the only one that can be produced. In the case where B contains a 1 but B' does not, p' must be used instead of p.

(2) There exists $p' < p$ with $(p', q) \in D(w)$.

This implies that there exists $p' < p$ with $(p', q) \in E(w)$. We break into the cases where there exists such an $(p', q) \in E(w)$ violating Condition B and where every $(p', q) \in E(w)$ with $p' < p$ satisfies Condition B (which implies that there is only one $(p', q) \in E(w)$ with $p' < p$).

(a) There exists $p' < p$ such that $(p', q) \in E(w)$ and (p', q) violates Condition B.
We may assume, changing our choice of c' if necessary, that $(w(c'), c')$ is NE of (p', q). Now let B be the rectangular box bounded by rows $p-1$ and $w(c)$ and columns q and c', as in Case 1a, and let B' the box bounded by rows $w(c')$ and p and columns c and $q+1$ (and not q as in Case 1a).
If B does not contain a 1 in its interior, then we have a 42513 pattern or a 351624 pattern depending on whether or not B' contains a 1 in its interior. If B contains a 1 in its interior, then if B' contains *two* 1's in its interior, we have a 53241 or a 52341 pattern depending on the arrangement of these two 1's. If B contains a 1 in its interior and B' contains fewer than two 1's in its interior, either $w^{-1}(p-1) < c$, or $(p-1, w^{-1}(p-1))$ is the only 1 in the interior of B. If $w^{-1}(p-1) < c$, then either $w^{-1}(p')$ is E of the 1 in B, in which case $w^{-1}(p-1)$, c, q, the 1 in B, and $w^{-1}(p')$ form a 35142 pattern, or $w^{-1}(p')$ is W of the 1 in B. In this latter case, $w(q+1) > c$, so $w^{-1}(p-1)$, c, q, $q+1$, $w^{-1}(p')$, and the 1 in B form a 351624 pattern. Otherwise, when $(p-1, w^{-1}(p-1))$ is the only 1 in the interior of B, $w^{-1}(p'-1) < c$ and $w(q) < w(c')$, so $w^{-1}(p'-1)$, c, q, the 1 in B, and c' form a 35142 pattern.

(b) For the only $p' < p$ such that $(p', q) \in E(w)$, the essential set box (p', q) satisfies Condition B.
Since (p, q) violates Condition X, either $r_w(p', q) < r_w(p, q) - 1$, or, letting q' denote the smallest integer such that $(p', b) \in D(w)$ for all b with $j' \leq b \leq j$, the box $(p, q'-1) \notin D(w)$. We further split into cases.

(i) The rank conditions satisfy $r_w(p', q) < r_w(p, q) - 1$.
In this case, neither $(p-1, q)$ nor $(p-2, q)$ is in $D(w)$. Let B and B' be as in Case 2a. If B does not have a 1 in the interior, we have a 42513 or 351624 pattern as in Case 2a. Also as in Case 2a, if B' has two 1's in its interior, then we have a 53241 or 52341 pattern. Otherwise, either $w^{-1}(p-1) < c$ or both $w^{-1}(p-2) < c$ and $w(q) < w(c')$, and the remainder of the proof in these cases follows the remaining parts of Case 2a.

(ii) The box $(p, q'-1) \notin D(w)$.
We let $c = j' - 1$; note that, by our assumptions for this case, $(w(c), c)$ is SW of (p, q), as before. In particular, our choice of c now forces $w^{-1}(p-1) < c$. Let B be the rectangular box bounded by rows $p-1$ and $w(c)$ and columns q and c' as before. We now have a 35142 or 351624 pattern depending on whether B contains a 1 in its interior or not. □

We now canonically associate a permutation defined by inclusions to every permutation which is almost defined by inclusions. For a permutation w almost defined by inclusions,

let $E''(w)$ denote the subset of the essential set which satisfies neither Condition A nor Condition B. As before, let $E'(w)$ be the subset of the essential set which satisfies Condition B but not Condition A.

Theorem 3.8. *Let w be a permutation almost defined by inclusions. Then there exists a permutation v such that*

(1) *The essential set $E(v) = E(w) \setminus E''(w)$, and*
(2) *The ranks $r_v(p,q) = r_w(p,q)$ for all $(p,q) \in E(v)$.*

These conditions define a unique permutation v which is defined by inclusions. Furthermore, $\ell(v) - \ell(w)$ is the number of boxes in $E''(w)$.

It would be interesting to know if the existence of such a v in some way gives an alternative characterization, independent of conditions on the diagram, of being almost defined by inclusions. We discuss this further as Question 7.2 in Section 7.

Proof. Formally, we will structure the proof as induction on the number of boxes in $E''(w)$. Informally, one should think of this proof as giving a way to construct v by eliminating boxes of $E''(w)$ from the essential set one at a time, increasing the length of the permutation by 1 at each step. It turns out that the boxes of $E''(w)$ can be removed from the essential set in any order.

The base case is where $E''(w)$ is empty. In this case, w is defined by inclusions, so $v = w$ is such a permutation, and $\ell(v) - \ell(w) = 0$, which is the number of boxes in $E''(w)$.

Let (p,q) be a box of $E''(w)$. We now divide the proof into cases depending on the type of (p,q).

Suppose (p,q) is of type WY. Let $w' = wt$, where t is the transposition switching q and $w^{-1}(p)$. Since there are no essential set boxes directly N of (p,q), $w^{-1}(a) < q$ for all $a \in [\![w(q), p]\!]$. Since there are no essential set boxes directly E of (p,q), $w(b) > i$ for all $in[\![j, w^{-1}(i)]\!]$ with $j < b < w^{-1}(i)$. This implies that $\ell(w') = \ell(w) + 1$ and $D(w') = D(w) \setminus \{(p,q)\}$. Conditions W and Y ensure that $E(w') = E(w) \setminus \{(p,q)\}$ since any diagram box in $(p, q-1)$ or $(p+1, q)$ is respectively S or W of an essential set box of the same rank. Since there are no boxes of $D(w)$ other than (p,q) in the rectangle bounded by rows $w(q)$ and p and columns q and $w^{-1}(p)$, every box of $E(w')$ satisfies exactly the same conditions that it satisfies as an element of $E(w)$. Therefore, w' is almost defined by inclusions, and $E'(w') = E'(w)$. Since $E''(w')$ has one fewer box than $E''(w)$, by the inductive hypothesis, the theorem holds for w', so it holds for w.

Now suppose (p,q) is of type WZ. Let p' and q' be as in Condition Z. It follows that $w(q+1) = p' + 1$. Let $w' = wt$, where t switches q and $q+1$. Since $w(q) < p \leq p' + 1 = w(q+1)$, $\ell(w') = \ell(w) + 1$. Furthermore,

$$D(w') = D(w) \cup \{(p, q+1), \ldots, (p', q+1)\} \setminus \{(p,q), \ldots, (p'+1, q)\}.$$

Since $r_w(p, q') = r_w(p,q) + 1$ by Condition Z, $(p, q+2) \in D(w)$, so $(p, q+2) \in D(w')$. Also, by Condition W, if $(p, q-1) \in D(w)$, then $(p-1, q-1) \in D(w)$, so if $(p, q-1) \in D(w')$, then $(p-1, q-1) \in D(w')$. Therefore, $E(w') = E(w) \setminus \{(p,q)\}$. Every box of $E(w')$ satisfies exactly the same conditions that it satisfies as an element of $E(w)$, so w' is almost

defined by inclusions and $E'(w') = E'(w)$. By the inductive hypothesis, the theorem holds for w', so it holds for w.

Now suppose (p, q) is of type XY. Let q' be as in Condition X; then $w(q' - 1) = p - 1$. Let $w' = wt$ where t switches $q' + 1$ and $w^{-1}(p)$. The argument in this case is entirely analogous to the one in the case where (p, q) is of type WZ.

Finally, suppose that (p, q) is of type XZ. Let p_X and q_X be the p' and q' of Condition X, and p_Z and q_Z the p' and q' of Condition Z. Note that $w(q + 1) = p_Z + 1$ and $w(q_X - 1) = p - 1$. Let $w' = wt$ where t switches $q_X - 1$ and $q + 1$. Since the interior of the rectangle bounded by columns $q_X - 1$ and $q + 1$ and rows $p - 1$ and $p_Z + 1$ consists entirely of boxes in $D(w)$, there is no $a \in [\![q_X - 1, q + 1]\!]$ with $w(a) \in [\![p - 1, p_Z + 1]\!]$, so $\ell(w') = \ell(w) + 1$. Furthermore,

$$\begin{aligned} D(w') =& D(w) \cup \{(p-1, q_X), \dots, (p-1, q)\} \cup \{(p, q+1), \dots, (p_Z, q+1)\} \\ & \setminus \{(p_Z + 1, q_X - 1), \dots, (p_Z + 1, q)\} \setminus \{(p, q_X - 1), \dots, (p_Z + 1, q_X - 1)\}. \end{aligned}$$

By Conditions Z and X respectively, $(p, q + 2)$ and $(p - 2, q)$ are both in $D(w')$, so $E(w') = E(w) \setminus \{(p, q)\}$. Every box of $E(w')$ satisfies exactly the same conditions that it satisfies as an element of $E(w)$, so w' is almost defined by inclusions, and $E'(w') = E'(w)$. By the inductive hypothesis, the theorem holds for w', so it holds for w. □

Example 3.9. If $w = 819372564$, as in Example 2.2, then the permutation v which is associated to w is the permutation $v = 819732654$ given in Example 3.5. □

4. Sufficiency

We now proceed to prove that, if w avoids the six stated patterns, then X_w is lci. First note that the non-lci locus of X_w is closed and invariant under the action of the Borel group B, so the non-lci locus must be a union of Schubert subvarieties of X_w. In particular, the non-lci locus of X_w must contain the point e_{id}. In the case where w avoids the six stated patterns, we show that X_w is lci by showing that it is lci at e_{id}. By Proposition 2.3, we only need to show that the ideal I_w (defined in the paragraph before the Proposition) is generated by $\binom{n}{2} - \ell(w)$ polynomials.

Note that $\binom{n}{2} - \ell(w)$ is precisely the number of boxes in the diagram $D(w)$. Therefore, given a permutation w which is almost defined by inclusions, we will define one polynomial in I_w for each box of $D(w)$. Letting J_w be the ideal generated by these polynomials, we explicitly show that every other generator of I_w is in the ideal J_w and hence that $I_w = J_w$ is a complete intersection.

We begin first with the case where w is defined by inclusions. Following that case, we treat the general case by showing that, if w is almost defined by inclusions, and v is the defined by inclusions permutation associated to w by Theorem 3.8, then the ideal I_w is generated by I_v plus one polynomial for each box of $E''(w)$.

4.1. **The defined by inclusions case.** Let w be a permutation defined by inclusions. Fix a total ordering of the essential set $E'(w)$ in which smaller rank boxes come before larger rank boxes as in the discussion prior to Lemma 3.4. Let k be the number of boxes

in $E'(w)$, $(p_1, q_1), \ldots, (p_k, q_k)$ be the boxes of the essential set in order, $r_m = r_w(p_m, q_m)$ for each $m \in [\![1, k]\!]$, and $R_1, \ldots, R_k$ the rectangular regions of $D'(w)$ defined prior to Lemma 3.4.

For each box $(x, y) \in D(w)$, we define a polynomial $f_{(x,y)}$ in S (which will be a generalized Plücker coordinate) as follows. If $r_w(x, y) = 0$, then let

$$A(x, y) = \{x\} \text{ and } B(x, y) = \{y\}.$$

Otherwise, the box (x, y) is in some rectangle R_m. Let

$$A(x, y) = [\![p_m, p_m + r_m - 1]\!] \cup \{x + r_m\},$$

and let

$$B(x, y) = \{y - r_m\} \cup [\![q_m - r_m + 1, q_m]\!].$$

Now let

$$f_{(x,y)} = d_{A(x,y),B(x,y)}.$$

Let

$$J_w = \langle f_{(x,y)} \rangle_{(x,y)\in D(w)}.$$

Example 4.1. Let $w = 819732654$ as in Example 3.5 and Figure 2. Then

$$\begin{aligned} J_w = \langle & d_{\{2,3\},\{1,2\}}, d_{\{2,4\},\{1,2\}}, d_{\{2,5\},\{1,2\}}, d_{\{2,6\},\{1,2\}}, d_{\{2,7\},\{1,2\}}, d_{\{4,5,6,7\},\{3,4,5,6\}}, \\ & d_{\{4,5,6,8\},\{3,4,5,6\}}, d_{\{4,5,6,9\},\{3,4,5,6\}}, d_{\{4,5,6,7\},\{2,4,5,6\}}, d_{\{4,5,6,8\},\{2,4,5,6\}}, \\ & d_{\{4,5,6,9\},\{2,4,5,6\}}, d_{\{9\},\{1\}}, d_{\{9\},\{2\}} \rangle. \end{aligned}$$

□

Our aim is to prove the following theorem.

Theorem 4.2. *Suppose w is defined by inclusions. Then the ideals I_w and J_w are equal. Hence I_w defines a local complete intersection.*

We will first make a reduction showing I_w is generated by a subset of the original stated generators. Recall that, for $p, q, r \in [\![1, n]\!]$, $I_{(p,q,r)}$ is defined as the ideal generated by the generalized Plücker coordinates $d_{A,B}$ for all $A \subseteq [\![p, n]\!]$ and $B \subseteq [\![1, q]\!]$ where both A and B have $r + 1$ elements. Furthermore, I_w is the ideal

$$I_w = \sum_{(p,q)\in E(w)} I_{(p,q,r_w(p,q))}.$$

Now define $I'_{(p,q,r)}$ to be the ideal generated by $d_{[\![p,p+r-1]\!]\cup\{x+r\},\{y-r\}\cup[\![q-r+1,q]\!]}$ for all x with $x \in [\![p, n - r]\!]$ and all $y \in [\![1 + r, q]\!]$. Let

$$I'_w = \sum_{(p,q)\in E(w)} I'_{(p,q,r_w(p,q))}.$$

Lemma 4.3. *Suppose w is defined by inclusions. Then $I'_{(p,q,r_w(p,q))} = I_{(p,q,r_w(p,q))}$ for all $(p, q) \in E(w)$, and hence $I'_w = I_w$.*

Proof. For all $x \in [\![p, n - r_w(p,q)]\!]$ and all $y \in [\![1 + r_w(p,q), q]\!]$, we have

$$[\![p, p + r_w(p,q) - 1]\!] \cup \{x + r_w(p,q)\} \subseteq [\![p, n]\!]$$

and

$$\{y - r_w(p,q)\} \cup [\![q - r_w(p,q) + 1, q]\!] \subseteq [\![1, q]\!].$$

Furthermore, the size of both these sets is $r_w(p,q) + 1$. Therefore,

$$I'_{(p,q,r_w(p,q))} \subseteq I_{(p,q,r_w(p,q))}.$$

Fix $(p,q) \in E(w)$, and fix $r = r_w(p,q)$. If $r = 0$, then the two generating sets are the same, so we only need to prove the reverse direction in the case where $(p,q) \in E'(w)$. Given $(p,q) \in E'(w)$, we need to show that $d_{A,B} \in I'_{(p,q,r)}$ for all A and B satisfying the conditions that $A \subseteq [\![p,n]\!]$, $B \subseteq [\![1,q]\!]$, and both A and B have $r+1$ elements. We do so by induction on the degree of $d_{A,B}$. Recall that our polynomial ring S is graded so that $\deg d_{A,B} = \sum_{p \in A} p - \sum_{q \in B} q$.

Fix A and B such that $d_{A,B}$ is one of the defined generators of $I_{(p,q,r)}$. If $[\![p, p+r-1]\!] \subseteq A$ and $[\![q-r+1, q]\!] \subseteq B$, then $d_{A,B} \in I'_{(p,q,r)}$ by definition. Otherwise, either the difference $[\![p, p+r-1]\!] \setminus A$ or $[\![q-r+1, q]\!] \setminus B$ is nonempty; we treat the case where $[\![q-r+1, q]\!] \setminus B$ is nonempty and leave the entirely analogous argument in the other case to the reader. Let $b \in [\![q-r+1, q]\!] \setminus B$.

Our proof is by expanding the determinant $d_{A \cup \{b\}, B \cup \{b\}}$ in two different ways. (If $b \in A$, we mean to consider the determinant of the matrix where row b occurs twice. This determinant is identically 0, but we can still consider its expansions formally.) Let b' be the smallest element of B, and note that $b' < q - r + 1$. Now consider the Laplace expansion of $d_{A \cup \{b\}, B \cup \{b\}}$ using column b', which is given by

$$d_{A \cup \{b\}, B \cup \{b\}} = \sum_{a \in A \cup \{b\}} \pm z_{a,b'} d_{A \cup \{b\} \setminus \{a\}, B \cup \{b\} \setminus \{b'\}}.$$

By Lemma 3.2, $p = q - r + 1$. Therefore, $p > b'$, and $a > b'$ for any $a \in A \cup \{b\}$. This implies

$$\begin{aligned} \deg d_{A \cup \{b\} \setminus \{a\}, B \cup \{b\} \setminus \{b'\}} &= \deg d_{A,B} - a + b' \\ &< \deg d_{A,B}. \end{aligned}$$

Furthermore, $p \leq b$, so

$$A \cup \{b\} \setminus \{a\} \subseteq [\![p, n]\!].$$

Now, by the inductive hypothesis,

$$d_{A \cup \{b\} \setminus \{a\}, B \cup \{b\} \setminus \{b'\}} \in I'_{(p,q,r)}$$

for all $a \in A \cup \{b\}$ since they are all of smaller degree. Therefore,

$$d_{A \cup \{b\}, B \cup \{b\}} \in I'_{(p,q,r)}.$$

Now we expand $d_{A \cup \{b\}, B \cup \{b\}}$ along row b. This expansion is given by

$$d_{A \cup \{b\}, B \cup \{b\}} = \pm d_{A,B} + \sum_{b' \in B} z_{b,b'} d_{A, B \cup \{b\} \setminus \{b'\}}.$$

If $b' > b$, then $z_{b,b'} = 0$. If $b' < b$, then

$$\begin{aligned}\deg d_{A,B\cup\{b\}\setminus\{b'\}} &= \deg d_{A,B} - b + b' \\ &< \deg d_{A,B},\end{aligned}$$

so, by the inductive hypothesis,

$$d_{A,B\cup\{b\}\setminus\{b'\}} \in I'_{(p,q,r)}.$$

Therefore,

$$d_{A,B} \in I'_{(p,q,r)},$$

as desired. □

Example 4.4. We illustrate our proof for the case where $p = 4$, $q = 6$, and $r = 3$. (This case comes up for $w = 819732654$ as in Example 3.5.) Here $I_{(p,q,r)}$ is the ideal of all size 4 minors of the matrix

$$\begin{pmatrix} z_{4,1} & z_{4,2} & z_{4,3} & 1 & 0 & 0 \\ z_{5,1} & z_{5,2} & z_{5,3} & z_{5,4} & 1 & 0 \\ z_{6,1} & z_{6,2} & z_{6,3} & z_{6,4} & z_{6,5} & 1 \\ z_{7,1} & z_{7,2} & z_{7,3} & z_{7,4} & z_{7,5} & z_{7,6} \\ z_{8,1} & z_{8,2} & z_{8,3} & z_{8,4} & z_{8,5} & z_{8,6} \\ z_{9,1} & z_{9,2} & z_{9,3} & z_{9,4} & z_{9,5} & z_{9,6} \end{pmatrix},$$

while $I'_{(p,q,r)}$ is the ideal generated by the size 4 minors which use the 3 topmost rows (plus some other row) and the 3 rightmost columns (plus some other column).

Consider

$$d_{\{4,6,7,9\},\{1,2,4,6\}} = \begin{vmatrix} z_{4,1} & z_{4,2} & 1 & 0 \\ z_{6,1} & z_{6,2} & z_{6,4} & 1 \\ z_{7,1} & z_{7,2} & z_{7,4} & z_{7,6} \\ z_{9,1} & z_{9,2} & z_{9,4} & z_{9,6} \end{vmatrix},$$

which has degree 13 in our grading. Our proof writes $d_{\{4,6,7,9\},\{1,2,4,6\}}$ in terms of generalized Plücker coordinates of smaller degree in $I_{(p,q,r)}$, which by induction are in $I'_{(p,q,r)}$, as follows.

In this case, $b = 5$. Hence we consider

$$d_{\{4,5,6,7,9\},\{1,2,4,5,6\}} = \begin{vmatrix} z_{4,1} & z_{4,2} & 1 & 0 & 0 \\ z_{5,1} & z_{5,2} & z_{5,4} & 1 & 0 \\ z_{6,1} & z_{6,2} & z_{6,4} & z_{6,5} & 1 \\ z_{7,1} & z_{7,2} & z_{7,4} & z_{7,5} & z_{7,6} \\ z_{9,1} & z_{9,2} & z_{9,4} & z_{9,5} & z_{9,6} \end{vmatrix}.$$

Since $b' = 1$, we consider the expansion

$$d_{\{4,5,6,7,9\},\{1,2,4,5,6\}} = z_{4,1}\begin{vmatrix} z_{5,2} & z_{5,4} & 1 & 0 \\ z_{6,2} & z_{6,4} & z_{6,5} & 1 \\ z_{7,2} & z_{7,4} & z_{7,5} & z_{7,6} \\ z_{9,2} & z_{9,4} & z_{9,5} & z_{9,6} \end{vmatrix} - z_{5,1}\begin{vmatrix} z_{4,2} & 1 & 0 & 0 \\ z_{6,2} & z_{6,4} & z_{6,5} & 1 \\ z_{7,2} & z_{7,4} & z_{7,5} & z_{7,6} \\ z_{9,2} & z_{9,4} & z_{9,5} & z_{9,6} \end{vmatrix}$$

$$+ z_{6,1}\begin{vmatrix} z_{4,2} & 1 & 0 & 0 \\ z_{5,2} & z_{5,4} & 1 & 0 \\ z_{7,2} & z_{7,4} & z_{7,5} & z_{7,6} \\ z_{9,2} & z_{9,4} & z_{9,5} & z_{9,6} \end{vmatrix} - z_{7,1}\begin{vmatrix} z_{4,2} & 1 & 0 & 0 \\ z_{5,2} & z_{5,4} & 1 & 0 \\ z_{6,2} & z_{6,4} & z_{6,5} & 1 \\ z_{9,2} & z_{9,4} & z_{9,5} & z_{9,6} \end{vmatrix} + z_{9,1}\begin{vmatrix} z_{4,2} & 1 & 0 & 0 \\ z_{5,2} & z_{5,4} & 1 & 0 \\ z_{6,2} & z_{6,4} & z_{6,5} & 1 \\ z_{7,2} & z_{7,4} & z_{7,5} & z_{7,6} \end{vmatrix}.$$

The determinants in the expansion have degrees 10, 9, 8, 7, and 5 respectively, so by induction $d_{\{4,5,6,7,9\},\{1,2,4,5,6\}} \in I'_{(p,q,r)}$.

Now we expand $d_{\{4,5,6,7,9\},\{1,2,4,5,6\}}$ along row $b = 5$. We get

$$d_{\{4,5,6,7,9\},\{1,2,4,5,6\}} = z_{5,1}\begin{vmatrix} z_{4,2} & 1 & 0 & 0 \\ z_{6,2} & z_{6,4} & z_{6,5} & 1 \\ z_{7,2} & z_{7,4} & z_{7,5} & z_{7,6} \\ z_{9,2} & z_{9,4} & z_{9,5} & z_{9,6} \end{vmatrix} - z_{5,2}\begin{vmatrix} z_{4,1} & 1 & 0 & 0 \\ z_{6,1} & z_{6,4} & z_{6,5} & 1 \\ z_{7,1} & z_{7,4} & z_{7,5} & z_{7,6} \\ z_{9,1} & z_{9,4} & z_{9,5} & z_{9,6} \end{vmatrix}$$

$$+ z_{5,4}\begin{vmatrix} z_{4,1} & z_{4,2} & 0 & 0 \\ z_{6,1} & z_{6,2} & z_{6,5} & 1 \\ z_{7,1} & z_{7,2} & z_{7,5} & z_{7,6} \\ z_{9,1} & z_{9,2} & z_{9,5} & z_{9,6} \end{vmatrix} - 1\begin{vmatrix} z_{4,1} & z_{4,2} & 1 & 0 \\ z_{6,1} & z_{6,2} & z_{6,4} & 1 \\ z_{7,1} & z_{7,2} & z_{7,4} & z_{7,6} \\ z_{9,1} & z_{9,2} & z_{9,4} & z_{9,6} \end{vmatrix} + 0\begin{vmatrix} z_{4,1} & z_{4,2} & 1 & 0 \\ z_{6,1} & z_{6,2} & z_{6,4} & z_{6,5} \\ z_{7,1} & z_{7,2} & z_{7,4} & z_{7,5} \\ z_{9,1} & z_{9,2} & z_{9,4} & z_{9,5} \end{vmatrix}.$$

The last term is 0, the next to last term is $d_{\{4,6,7,9\},\{1,2,4,6\}}$, and the determinants in the first three terms have degrees 9, 10, and 12 respectively. Since the first three terms have determinants of degree less than 13, they are in $I'_{(p,q,r)}$. As $d_{\{4,5,6,7,9\},\{1,2,4,5,6\}} \in I'_{(p,q,r)}$,

$$d_{\{4,6,7,9\},\{1,2,4,6\}} \in I'_{(p,q,r)}. \qquad \square$$

Our proof for the theorem requires an ordering of $E'(w)$ and the associated partition of $D'(w)$ as described before Lemma 3.4. We fix here our notation for this partition. Label $E'(w)$ as $(p_1, q_1), \ldots, (p_k, q_k)$ in such a way that $r_w(p_m, q_m) \leq r_w(p_{m'}, q_{m'})$ whenever $m < m'$. Let R_m be the subset of $D'(w)$ defined as those $(x, y) \in D'(w)$ which are (weakly) SW of (p_m, q_m) but not (weakly) SW of $(p_{m'}, q_{m'})$ for any $m' < m$. Let $r_m = r_w(p_m, q_m)$ for all $m \in [\![1, k]\!]$.

Our proof will be by induction and we fix notation for various subideals of I_w and J_w. Let $I_0 \subseteq I_w$ to be the ideal generated by $z_{x,y} (= f_{(x,y)})$ for $(x, y) \in D(w)$ such that $r_w(x, y) = 0$. Let $I_m \subseteq I_w$ be the ideal

$$I_m := I_0 + \sum_{m'=1}^{m} I_{(p_{m'}, q_{m'}, r_{m'})}.$$

Similarly, define $J_0 \subseteq J_w$ to be the ideal generated by $z_{x,y} (= f_{(x,y)})$ for $(x, y) \in D(w)$ such that $r_w(x, y) = 0$, and let

$$J_m := J_0 + \langle f_{(x,y)} \rangle_{(x,y) \in R_{m'}, m' \leq m}.$$

Proof of Theorem 4.2. We show by induction that $I_m = J_m$ for each m. This suffices since $I_w = I_k$ and $J_w = J_k$ by definition.

The ideals I_0 and J_0 are equal by definition, since both are generated by $f_{(x,y)} = d_{\{x\},\{y\}} = z_{x,y}$ for all (x, y) where $r_w(x, y) = 0$.

Assume by induction that $I_{m-1} = J_{m-1}$. By definition, $J_m \subseteq I_m$, so we need to show $I_m \subseteq J_m$. The ideal $I_m = I_{m-1} + I_{(p_m,q_m,r_m)}$, and by Lemma 4.3, $I_{(p_m,q_m,r_m)} = I'_{(p_m,q_m,r_m)}$. Therefore, we only need to show $I'_{(p_m,q_m,r_m)} \subseteq J_m$.

Given $x \in [\![p_m, n - r_m]\!]$ and $y \in [\![1 + r_m, q_m]\!]$, let

$$A(x, y) = [\![p_m, p_m + r_m - 1]\!] \cup \{x + r_m\}$$

and

$$B(x, y) = \{y - r_m\} \cup [\![q_m - r_m + 1, q_m]\!].$$

To show that $I'_{(p_m,q_m,r_m)} \subseteq J_m$, we need to show that $d_{A(x,y),B(x,y)} \in J_m$ for all x and y satisfying the above conditions. We do so by further induction on $\deg d_{A(x,y),B(x,y)}$ (or equivalently on $x - y$).

Let a and b denote the height and width (in terms of the number of boxes) of the rectangular region R_m, so that $(p_m + a - 1, q_m - b + 1)$ is the SW-most box in R_m. If $(x, y) \in R_m$, then $d_{A(x,y),B(x,y)} \in J_m$ by definition. Otherwise, $x \geq p_m + a$, or $y \leq q_m - b$. We give the remaining details of this proof only for latter case where $y \leq q_m - b$ and leave the completely analogous argument in the former case to the reader.

By Lemma 3.4, $r_w(x, y) = r_m$ for all $(x, y) \in R_m$, so, in particular, $r_w(p_m, q_m - b + 1) = r_m$. Note $r_w(p_m, q_m - b + 1)$ is exactly the number of non-diagram boxes directly W of $(p_m, q_m - b + 1)$, and there are exactly $q_m - b$ columns to the left of $(p_m, q_m - b + 1)$. Therefore, there exist $q_m - b - r_m$ diagram boxes directly W of $(p_m, q_m - b + 1)$. In particular, since $1 + r_m \leq y \leq q_m - b$, $q_m - b - r_m \geq 1$, so there exists a diagram box W of $(p_m, q_m - b + 1)$. Let q' be the largest index such that $q' \leq q_m - b$ and $(p_m, q') \in D(w)$. Note that $r_m - r_w(p_m, q')$ is the number of non-diagram columns between q_m and q', so $q_m - b - q' = r_m - r_w(p_m, q')$. Since $y \leq q_m - b$, this implies $y - r_m \leq q' - r_w(p_m, q')$.

Since (p_m, q') is the first diagram box directly W of R_m, either $\mathsf{Wpred}(m) = 0$, in which case $r_w(p_m, q') = 0$, or $(p_m, q') \in R_{\mathsf{Wpred}(m)}$. If $r_w(p_m, q') = 0$, then $r_w(p_m, y - r_m) = 0$, and $z_{p,y-r_m} \in I_0$ for all $p \geq p_m$. Hence, all the entries in an entire column of the submatrix defining $d_{A(x,y),B(x,y)}$ are in I_0, and $d_{A(x,y),B(x,y)} \in I_0$.

In the other case where $\mathsf{Wpred}(m) > 0$, let $m' = \mathsf{Wpred}(m)$, noting that $q' = q_{m'}$ and $r_w(p_m, q') = r_{m'}$, so $y - r_m \leq q_{m'} - r_{m'}$. Now define sets F and G as follows. Let

$$F = [\![p_{m'}, p_{m'} + r_{m'} - 1]\!] \cup [\![p_m, p_m + r_m - 1]\!] \cup \{x + r_m\},$$

and let

$$G = [\![q_{m'} - r_{m'} + 1, q_{m'}]\!] \cup [\![q_m - r_m + 1, q_m]\!] \cup \{y - r_m\}.$$

Note that

$$x + r_m > p_m + r_m - 1 > p_{m'} + r_{m'} - 1$$

and

$$y - r_m < q_{m'} - r_{m'} + 1 < q_m - r_m + 1;$$

hence

$$\#F = r_{m'} + r_m + 1 - \max\{0, p_{m'} + r_{m'} - p_m\},$$

while

$$\#G = r_{m'} + r_m + 1 - \max\{0, q_{m'} - (q_m - r_m)\}.$$

By Lemma 3.2, $r_m = q_m - p_m + 1$ while $r_{m'} = q_{m'} - p_{m'} + 1$. This implies

$$p_{m'} + r_{m'} - p_m = q_{m'} - (q_m - r_m),$$

so F and G have the same number of elements. We let

$$\begin{aligned} c &= r_{m'} + r_m + 1 - \#F \\ &= \max\{0, q_{m'} - (q_m - r_m)\} \\ &= \#\left([\![q_{m'} - r_{m'} + 1, q_{m'}]\!] \cap [\![q_m - r_m + 1, q_m]\!]\right). \end{aligned}$$

Our proof is by expanding the determinant $d_{F,G}$ in two different ways. Let

$$G' = \{y - r_m\} \cup [\![q_{m'} - r_{m'} + 1, q_{m'}]\!],$$

note that G' has $r_{m'}$ elements, and consider the Laplace expansion of $d_{F,G}$ using the columns in G', which is given by

$$d_{F,G} = \sum_{F' \subseteq F} \pm d_{F',G'} d_{F \setminus F', G \setminus G'},$$

where the sum is over all subsets $F' \subseteq F$ of size $r_{m'}$. Since $y - r_m \leq q_{m'}$, $G' \subseteq [\![1, q_{m'}]\!]$, and $F \subseteq [\![q_{m'}, n]\!]$, so $d_{F',G'}$ is in the ideal $I_{m'}$ and hence by the inductive hypothesis in $J_{m'} \subseteq J_m$. Therefore, $d_{F,G} \in J_m$.

Now let

$$F'' = F \setminus A(x,y) = [\![p_{m'}, p_{m'} + r_{m'} - c - 1]\!],$$

and consider the Laplace expansion of $d_{F,G}$ using the rows of F'', which is given by

$$d_{F,G} = \sum_{G'' \subseteq G} \pm d_{F'',G''} d_{A(x,y), G \setminus G''}.$$

Consider the term where

$$G'' = G \setminus B(x,y) = [\![q_{m'} - r_{m'} + 1, q_{m'} - c]\!].$$

By Lemma 3.2, $r_{m'} = q_{m'} - p_{m'} + 1$, so $q_{m'} - r_{m'} + 1 = p_{m'}$, and

$$q_{m'} - c = p_{m'} + r_{m'} - c - 1.$$

Therefore, $F'' = G''$, so $d_{F'',G''} = 1$ for this choice of G''. Our Laplace expansion is therefore

$$d_{F,G} = \pm d_{A(x,y), B(x,y)} + \sum_{G''} \pm d_{F'',G''} d_{A(x,y), G \setminus G''},$$

where the sum is now over all $G'' \subseteq G$ of size $r_{m'} - c$ other than $G \setminus B(x,y)$.

If

$$\sum_{u \in G''} u \geq \sum_{u \in G \setminus B(x,y)} u,$$

then there exists $u' \in G''$ with

$$u' > q_{m'} - c = p_{m'} + r_{m'} - c - 1.$$

Note $u' > p$ for all $p \in F''$, which implies $d_{F'',G''} = 0$ as an entire column of the submatrix is 0.

On the other hand, if

$$\sum_{u \in G''} u < \sum_{u \in G \setminus B(x,y)} u,$$

then

$$\begin{aligned} \deg d_{A(x,y),G\setminus G''} &= \deg d_{A(x,y),B(x,y)} - \sum_{u \in G\setminus B(x,y)} u + \sum_{u \in G''} u \\ &< \deg d_{A(x,y),B(x,y)}. \end{aligned}$$

Since $d_{A(x,y),G\setminus G''} \in I_{(p_m,q_m,r_m)} = I'_{(p_m,q_m,r_m)}$, by the inductive hypothesis for our induction on the degree of $d_{A(x,y),B(x,y)}$,

$$d_{A(x,y),G\setminus G''} \in J_m.$$

Since $d_{F,G} \in J_m$ and

$$d_{F'',G''} d_{A(x,y),G\setminus G''} \in J_m$$

for all $G'' \subseteq G$ of size $r_{m'} - c$ not including $G \setminus B(x,y)$, we have proven $d_{A(x,y),B(x,y)} \in J_m$. □

Example 4.5. Consider $w = 819732654$ as in Example 3.5. For $m = 2$, $(p_2, q_2) = (4, 6)$, and $r_2 = 3$, our proof shows that $d_{\{4,5,6,x+3\},\{y-3,4,5,6\}} \in J_m$ for all x and y with $4 \le x \le 6$ and $4 \le y \le 6$. The only cases where this is not true by definition are those where $y = 4$. Note $m' = \mathsf{Wpred}(2) = 1$.

We illustrate the proof in the case where $x = 5$ and $y = 4$. Consider the determinant

$$d_{\{4,5,6,8\},\{1,4,5,6\}} = \begin{vmatrix} z_{4,1} & 1 & 0 & 0 \\ z_{5,1} & z_{5,4} & 1 & 0 \\ z_{6,1} & z_{6,4} & z_{6,5} & 1 \\ z_{8,1} & z_{8,4} & z_{8,5} & z_{8,6} \end{vmatrix},$$

which has degree 7 in our grading. Our proof writes $d_{\{4,6,7,9\},\{1,2,4,6\}}$ in terms of determinants either of smaller degree in $I_{(p_2,q_2,r_2)}$ or known to be in $I_{m'}$ as follows.

Here, $F = \{2, 4, 5, 6, 8\}$, $G = \{1, 2, 4, 5, 6\}$, and $c = 0$. Hence we consider

$$d_{\{2,4,5,6,8\},\{1,2,4,5,6\}} = \begin{vmatrix} z_{2,1} & 1 & 0 & 0 & 0 \\ z_{4,1} & z_{4,2} & 1 & 0 & 0 \\ z_{5,1} & z_{5,2} & z_{5,4} & 1 & 0 \\ z_{6,1} & z_{6,2} & z_{6,4} & z_{6,5} & 1 \\ z_{8,1} & z_{8,2} & z_{8,4} & z_{8,5} & z_{8,6} \end{vmatrix}.$$

For this example, $G' = \{1, 2\}$. Note that the size 2 minors involving the columns 1 and 2 are in $I_{(2,2,1)} \subseteq I_1$. Therefore, $d_{\{2,4,5,6,8\},\{1,2,4,5,6\}} \in I_1$.

Now, $F'' = \{2\}$. Therefore, we expand $d_{\{2,4,5,6,8\},\{1,2,4,5,6\}}$ along row 2 to get

$$d_{\{2,4,5,6,8\},\{1,2,4,5,6\}} = z_{2,1}\begin{vmatrix} z_{4,2} & 1 & 0 & 0 \\ z_{5,2} & z_{5,4} & 1 & 0 \\ z_{6,2} & z_{6,4} & z_{6,5} & 1 \\ z_{8,2} & z_{8,4} & z_{8,5} & z_{8,6} \end{vmatrix} - 1\begin{vmatrix} z_{4,1} & 1 & 0 & 0 \\ z_{5,1} & z_{5,4} & 1 & 0 \\ z_{6,1} & z_{6,4} & z_{6,5} & 1 \\ z_{8,1} & z_{8,4} & z_{8,5} & z_{8,6} \end{vmatrix} + 0 - 0 + 0.$$

The second term is $-d_{\{4,5,6,8\}\{1,4,5,6\}}$, and the determinant in the first term is in I_2 and has degree $6 < 7$. Hence, by induction, it is in J_2 and $d_{\{4,5,6,8\}\{1,4,5,6\}} \in J_2$ as desired. □

4.2. **The general case.** Now we treat the general case. Let w be almost defined by inclusions, and let $(p, q) \in E''(w)$. Let

$$A'(p, q) = [\![p, p + r_w(p, q)]\!]$$

and

$$B'(p, q) = [\![q - r_w(p, q), q]\!].$$

Define

$$f_{(p,q)} = d_{A'(p,q),B'(p,q)}.$$

We show that I_w is generated by I_v and $f_{(p,q)}$ for $(p, q) \in E''(w)$, where v is the defined by inclusions permutation associated to w by Theorem 3.8. We do so by showing the following Lemma.

Lemma 4.6. *Let w be almost defined by inclusions, v the defined by inclusions permutation associated to w, and let $(p, q) \in E''(w)$. Then*

$$I_{(p,q,r_w(p,q))} \subseteq I_v + \langle f_{(p,q)} \rangle.$$

Proof. Let $r = r_w(p, q)$. The ideal $I_{(p,q,r)}$ is generated by all $d_{A,B}$ where $A \subseteq [\![p, n]\!]$, $B \subseteq [\![1, q]\!]$, and $\#A = \#B = r + 1$. We need to show that

$$d_{A,B} \in I_w + \langle f_{(p,q)} \rangle$$

for all such A and B. We do so by induction on the degree of $d_{A,B}$. If $A = A'(p, q)$ and $B = B'(p, q)$, then $d_{A,B} = f_{(p,q)}$ and we are done. We are left with the cases where $A \neq A'(p, q)$ and where $B \neq B'(p, q)$. We treat only the case where $B \neq B'(p, q)$ and leave the entirely analogous argument in the other case to the reader.

Since w is almost defined by inclusions, (p, q) satisfies either condition W or condition X. We treat these two cases separately.

Suppose (p, q) satisfies condition X. Then there exists $(p', q) \in E(v)$ with $p' < p$ and

$$r_v(p', q) = r_w(p', q) = r + 1.$$

Let $b \in B'(p, q) \setminus B$. Consider the determinant $d_{A\cup\{b\},B\cup\{b\}}$. (If $b \in A$, we mean to consider the determinant of the matrix where row b occurs twice. This determinant is identically 0, but we can still consider its expansions formally.) Note that

$$b \geq q - r = q - r_v(p', q) + 1.$$

Since $(p', q) \in E(v)$ and $r_v(p', q) \neq 0$, by Lemma 3.2,

$$p' = q - r_w(p', q) + 1,$$

so $b \geq p'$. Therefore,

$$A \cup \{b\} \subseteq [\![p', n]\!].$$

Also,

$$B \cup \{b\} \subseteq [\![1, q]\!],$$

and

$$\#A = \#B = r + 1 = r_v(p', q) + 1,$$

so

$$d_{A\cup\{b\},B\cup\{b\}} \in I_v.$$

Now we expand $d_{A\cup\{b\},B\cup\{b\}}$ along row b. This expansion is given by

$$d_{A\cup\{b\},B\cup\{b\}} = \pm d_{A,B} + \sum_{b'\in B} z_{b,b'} d_{A,B\cup\{b\}\setminus\{b'\}}.$$

If $b' > b$, then $z_{b,b'} = 0$. If $b' < b$, then

$$\deg d_{A,B\cup\{b\}\setminus\{b'\}} = \deg d_{A,B} - b + b' < \deg d_{A,B},$$

so by the inductive hypothesis,

$$d_{A,B\cup\{b\}\setminus\{b'\}} \in I_w + \langle f_{(p,q)} \rangle.$$

Therefore, since every term of its expansion is in $I_v + \langle f_{(p,q)} \rangle$,

$$d_{A,B} \in I_v + \langle f_{(p,q)} \rangle.$$

Next suppose (p, q) satisfies Condition W. Since $B \neq B'(p, q)$, $q > r+1$, so there exists $s < q$ with $(p, s) \in D(w)$. Let q' be the largest such s. If $q' = q - 1$, Condition W states that there exists $p' < p$ where $(p', q') \in E(v)$.

If $q' < q - 1$, then (p, q') is not in $E(w)$, because otherwise (p, q') would violate both Conditions Y and Z. Since $(p, q' + 1) \notin D(w)$, this implies that there exists $p' < p$ with $(p', q') \in E(w)$. Now $(p' + 1, q') \in D(v)$ but $(p' + 1, q' + 1) \notin D(w)$, so (p', q') does not satisfy Condition Y.

Furthermore, since $q' < w^{-1}(p-1) \leq q$ (as $(p-1, q') \in D(w)$ but $(p-1, q) \notin D(w)$) but $w(s) > p$ for all s with $q' < s < q$ (as $(p, s) \in D(w)$), it must be that $w(q) = p - 1$. Now if there exists a unique $q'' > q'$ with $(p', q'') \in E(w)$, then $q'' > q$ (since $w(s) \geq p - 1 \geq p'$ for all s with $q' < s \leq q$). Note that $(p', q' + 1)$ and (p', q) are two distinct boxes not in $D(w)$ but between (p', q') and (p', q''), so $r_w(p', q'') - r_w(p', q') \geq 2$. Therefore, (p', q') does not satisfy Condition Z. Since w is almost defined by inclusions, it must be the case that (p', q') satisfies Condition A or B, and hence $(p', q') \in E(v)$. Let $r' = r_v(p', q') = r_w(p', q')$.

Let $G = B \cap [\![1, q']\!]$. Note that, by the definition of q', every box between (p, q') and (p, q) is not in $D(v)$, and $r - r'$ is the number of such boxes, so $r - r' = q - q' - 1$. Since $B \subseteq [\![1, q]\!]$, this implies

$$\#G \geq r + 1 - (q - q') = r'.$$

If $\#G \geq r' + 1$, then $d_{F,G} \in I_v$ for any $F \subseteq A$ with $\#G$ elements. Since, by Laplace expansion using the columns of G,

$$d_{A,B} = \sum_{F\subseteq A} \pm d_{F,G} d_{A\setminus F, B\setminus G},$$

it must follow that $d_{A,B} \in I_v$.

Otherwise, G has exactly r' elements. Let

$$b \in [\![q' - r', q']\!] \setminus G.$$

We complete the proof by expanding the determinant $d_{A\cup\{b\},B\cup\{b\}}$ in two ways. First let $G' = G \cup \{b\}$, and consider the Laplace expansion of $d_{A\cup\{b\},B\cup\{b\}}$ using the columns of G', which is given by

$$d_{A\cup\{b\},B\cup\{b\}} = \sum_{F' \subseteq A\cup\{b\}} \pm d_{F',G'} d_{A\cup\{b\}\setminus F',B\cup\{b\}\setminus G'},$$

where the sum is over all subsets $F' \subseteq A \cup \{b\}$ of size r'. By Lemma 3.2, $p' = q' - r' + 1$, so $b \geq p'$. Therefore, $A \cup \{b\} \subseteq [\![p', n]\!]$, and

$$d_{F',G'} \in I_{(p',q',r')} \subseteq I_v$$

for all choices of F'. Hence

$$d_{A\cup\{b\},B\cup\{b\}} \in I_v.$$

Now we expand $d_{A\cup\{b\},B\cup\{b\}}$ along row b and use the last part of the argument in the case where (p, q) satisfies Condition X to show that

$$d_{A,B} \in I_v + \langle f_{(p,q)} \rangle. \qquad \square$$

Example 4.7. Consider $w = 819372564$ as in Example 2.2. In this case, $v = 819732654$ is the permutation associated to w by Theorem 3.8. Lemma 4.6 states that

$$I_{(4,4,2)} \subseteq I_v + \langle d_{\{4,5,6\},\{2,3,4\}} \rangle$$

and that

$$I_{(6,7,3)} \subseteq I_v + \langle d_{\{6,7,8,9\},\{4,5,6,7\}} \rangle.$$

To illustrate the proof, consider $d_{\{4,5,7\},\{1,3,4\}} \in I_{(4,4,2)}$, so $(p, q) = (4, 4)$ and $B'(p, q) = \{2, 3, 4\}$. Note the essential set box at $(4, 4)$ is of type WZ. Here $(p', q') = (2, 2)$, so $r' = 1$. Hence $G = \{1\}$, and $b = 2$.

Therefore we consider

$$d_{\{2,4,5,7\},\{1,2,3,4\}} = \begin{vmatrix} z_{2,1} & 1 & 0 & 0 \\ z_{4,1} & z_{4,2} & z_{4,3} & 1 \\ z_{5,1} & z_{5,2} & z_{5,3} & z_{5,4} \\ z_{7,1} & z_{7,2} & z_{7,3} & z_{7,4} \end{vmatrix}.$$

Since the size 2 minors involving the columns 1 and 2 are in $I_{(2,2,1)} \subseteq I_v$, $d_{\{2,4,5,7\},\{1,2,3,4\}} \in I_v$.

Then we expand $d_{\{2,4,5,7\},\{1,2,3,4\}}$ along row $b = 2$, getting

$$d_{\{2,4,5,7\},\{1,2,3,4\}} = z_{2,1} \begin{vmatrix} z_{4,2} & z_{4,3} & 1 \\ z_{5,2} & z_{5,3} & z_{5,4} \\ z_{7,2} & z_{7,3} & z_{7,4} \end{vmatrix} - 1 \begin{vmatrix} z_{4,1} & z_{4,3} & 1 \\ z_{5,1} & z_{5,3} & z_{5,4} \\ z_{7,1} & z_{7,3} & z_{7,4} \end{vmatrix} + 0 - 0.$$

The second term is $-d_{\{4,5,7\},\{1,3,4\}}$, and the first term involves $d_{\{4,5,7\},\{2,3,4\}}$, which is in $I_{(4,4,2)}$ and has smaller degree. Hence by induction $d_{\{4,5,7\},\{1,3,4\}} \in I_v + \langle d_{\{4,5,6\},\{2,3,4\}} \rangle$. $\square$

Finally we finish the proof of half of our theorem.

Theorem 4.8. *Suppose w avoids* 52431*,* 52341*,* 53241*,* 35142*,* 42513*, and* 351624*. Then X_w is a local complete intersection.*

Proof. By Theorem 3.7, w is almost defined by inclusions. Now by Theorem 3.8, there exists a permutation v which is defined by inclusions such that $\ell(v)-\ell(w)$ is the number of boxes in $E''(w)$. By Theorem 4.2, I_v is generated by $\binom{n}{2}-\ell(v)$ polynomials. Furthermore, by Lemma 4.6, I_w is generated by I_v and $\ell(v)-\ell(w)$ polynomials, so I_w is generated by $\binom{n}{2}-\ell(w)$ polynomials. Hence, by Proposition 2.3, X_w is a local complete intersection in a neighborhood of e_{id}. Hence X_w is a local complete intersection. □

5. Necessity

Our strategy for the proof of the reverse direction is as follows. We identify two infinite families and eleven isolated intervals $[u,v]$ such that the Kazhdan–Lusztig variety $\mathcal{N}_{u,v}$ is not lci, and hence X_v is not lci at e_u. It follows from [WY08, Thm. 2.6] that X_w is not lci if w interval contains $[u,v]$. We show that, if w contains one of the six given patterns, then w will interval contain either one of the eleven intervals or an interval from one of the two infinite families. This is accomplished using marked mesh patterns, which were previously defined by the first author in [Ú11, Subsec. 4.1].

We now list our two infinite families and eleven isolated intervals, along with drawings as mesh patterns.

Family A consists of intervals of the form

$$[(a+1)a\cdots 1(a+b+2)\cdots(a+2),(a+b+2)(a+1)a\cdots 2(a+b+1)\cdots(a+2)1],$$

where $a,b>0$ and $a>1$ or $b>1$. We list the first few members of the family in Figure 3.

Family B consists of intervals of the form

$$\begin{aligned}&[(a+1)\cdots 1(a+3)(a+2)(a+b+4)\cdots(a+4),\\&(a+3)(a+1)\cdots 2(a+b+4)1(a+b+3)\cdots(a+4)(a+2)],\end{aligned}$$

where $a,b\geq 0$ and $a+b\geq 1$. We list the first few members of the family in Figure 4.

We list the exceptional intervals in Figure 5.

Note the set of intervals listed is invariant under two symmetries. The **reverse complement** of a permutation v, an interval pattern $[u,v]$, or a mesh pattern (v,R) is obtained by rotating the graph $G(v)$ along with the conditions from R (or from u) by 180 degrees, or equivalently by conjugating both u and v by w_0 and moving R accordingly. The **inverse** of a permutation v is the permutation v^{-1}; this is equivalent to reflecting the graph $G(v)$ along with the conditions across the line $y=x$. One can of course also apply both symmetries. Geometrically, one can explain this invariance by noting that $\mathcal{N}_{u,v}\cong\mathcal{N}_{w_0uw_0,w_0vw_0}\cong\mathcal{N}_{u^{-1},v^{-1}}\cong\mathcal{N}_{w_0u^{-1}w_0,w_0v^{-1}w_0}$. Note that w avoids v (or $[u,v]$ or (v,R)) if and only if the reverse complement of w avoids the reverse complement of v (or of $[u,v]$ or of (v,R)), and the same holds for the inverse and also for the inverse of the reverse complement. We will appeal to these symmetries in the remainder of this section.

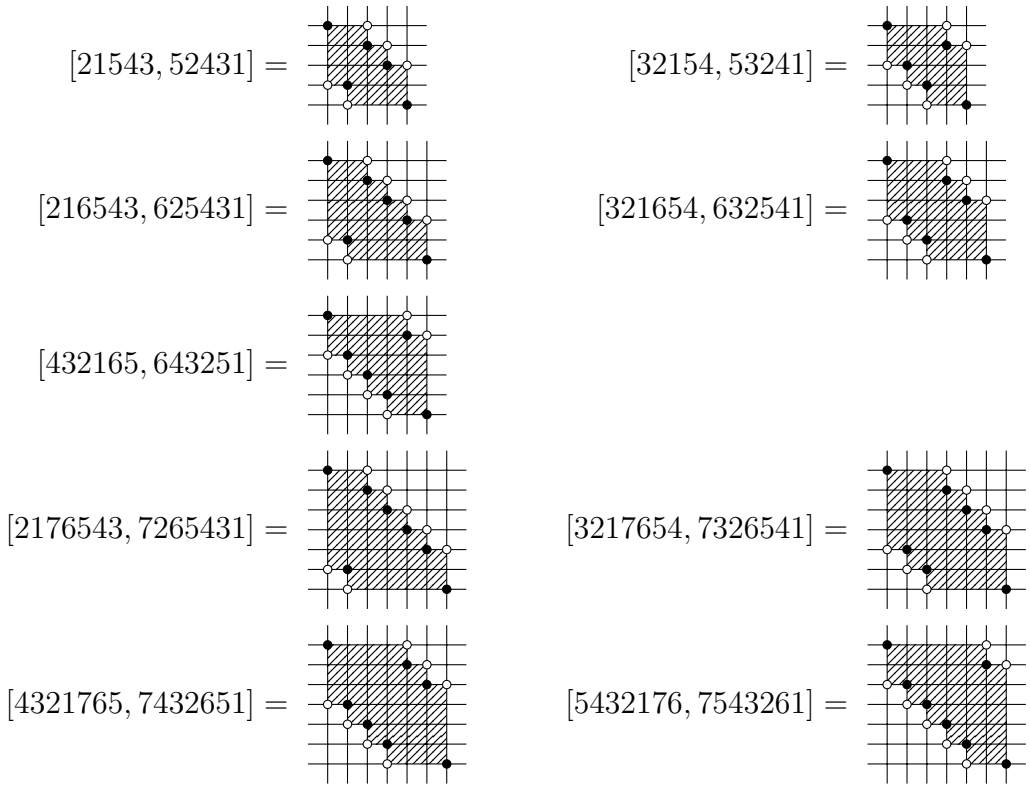


FIGURE 3. The first few members of the family A

Proposition 5.1. *For each of the intervals $[u, v]$ in Family A, Family B, and listed in Figure 5, the variety $\mathcal{N}_{u,v}$ is not lci (precisely only at the origin $\mathbf{0}$).*

Proof. The varieties $\mathcal{N}_{u,v}$ for the infinite families A and B were determined independently by Cortez [Cor03] and Manivel [Man01a]. For Family A, they are the varieties of $(a+1) \times (b+1)$ matrices of rank at most 1; these are not lci except in the case where $a = b = 1$. For Family B, they are the varieties of $(a + b + 2) \times 2$ matrices of rank at most 1; these are not lci unless $a + b + 2 = 2$. In the case of Family A, this is immediate from the definition of the ideal $I_{v,w}$ given in Subsection 2.4. For Family B, this description requires a change in coordinates changing half the variables by a sign. Note that for both of these families, $\mathcal{N}_{u,v}$ is only singular at the origin, and X_u is in fact an irreducible component of the singular locus of X_v, so some generic singularity of X_v is not lci.

For the intervals in C, D, E, and F, the variety $\mathcal{N}_{u,v}$ is isomorphic (after respectively 0, 2, 0, and 1 change in sign) to the variety of 3×3 "matrices" with a corner cut out of "rank" 1. In other words, their coordinate ring is generated by the 2×2 "determinants" of the "matrix"

$$\begin{pmatrix} a & b & \\ c & d & e \\ f & g & h \end{pmatrix},$$

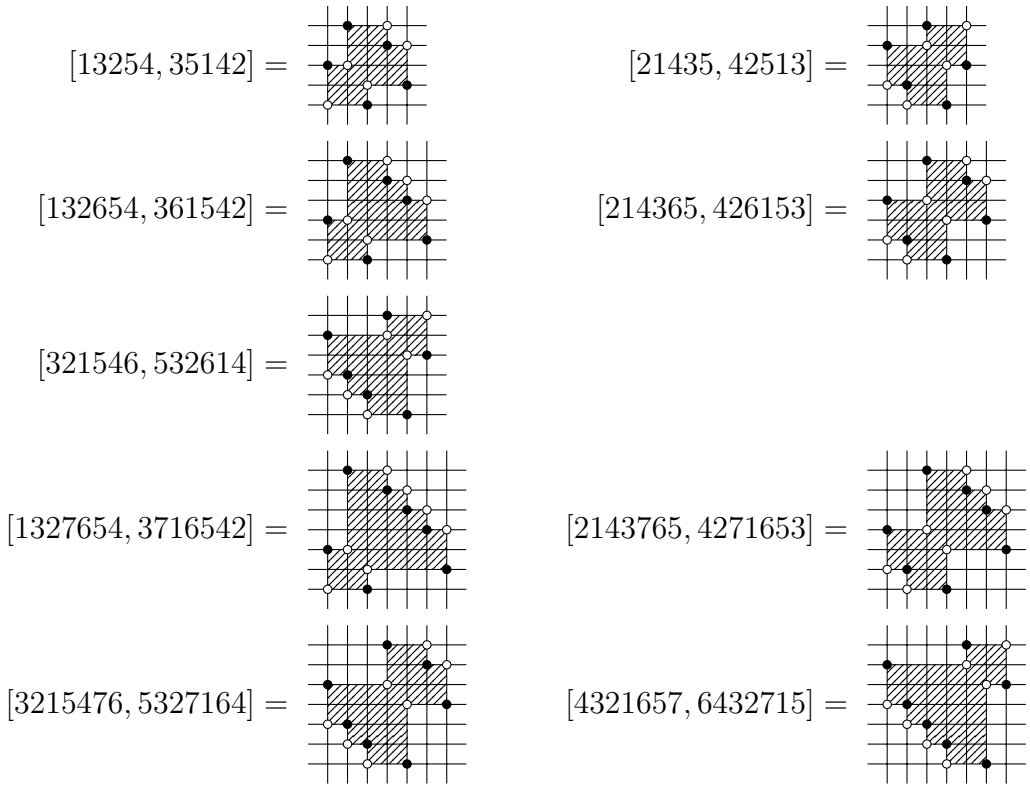


FIGURE 4. The first few members of the family B

namely $ad-bc$, $ag-bf$, $cg-df$, $ch-ef$, and $dh-eg$. This is a codimension 3 subvariety which is the vanishing locus of 5 linearly independent degree 2 polynomials and hence is not lci. Note that this variety has two components in its singular locus which meet at $\mathbf{0}$.

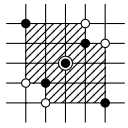
For the two intervals in G, the variety $\mathcal{N}_{u,v}$ is isomorphic to one defined by equations $ae-bd$, $af-cd$, $bf-ce$, $ag+bh+ci$, and $dg+eh+fi$ over the variables

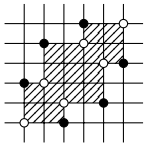
$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

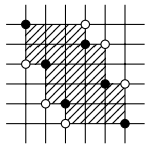
The equations state that the first two rows are dependent and both orthogonal to the third row. This is a codimension 3 subvariety which is the vanishing locus of 5 polynomials and hence not lci.

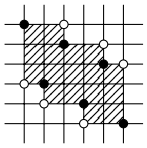
For the interval in H, the variety $\mathcal{N}_{u,v}$ is isomorphic to one defined by equations $ae-bd$, $af-cd$, $bf-ce$, $gk-hj$, $gl-ij$, $hl-ik$, $ag+bh+ci$, $dg+eh+fi$, $aj+bk+cl$, and $dj+ek+fl$ over the variables

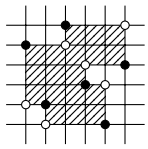
$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \\ j & k & l \end{pmatrix}.$$

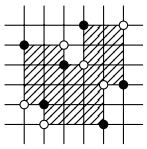
$C = [21354, 52341] =$ 

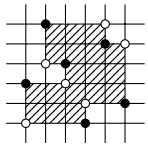
$D = [132546, 351624] =$ 

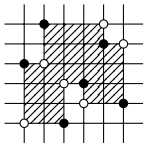
$E = [421653, 642531] =$ 

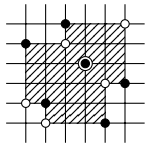
$E^{\mathrm{i}} = [326154, 635241] =$ 

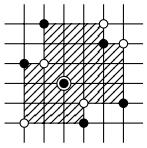
$F = [215436, 526314] =$ 

$F^{\mathrm{i}} = [215436, 524613] =$ 

$F^{\mathrm{rc}} = [143265, 364152] =$ 

$F^{\mathrm{irc}} = [143265, 461352] =$ 

$G = [215436, 526413] =$ 

$G^{\mathrm{rc}} = [143265, 463152] =$ 

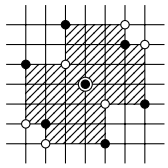
$H = [2154376, 5274163] =$ 

FIGURE 5. The exceptional intervals

The equations say that the first two rows are dependent, the last two rows are dependent, and the first two rows are orthogonal to the last two rows. This is a codimension 5 subvariety which is the vanishing locus of 10 polynomials and hence not lci. □

The calculations in the proof can be verified by hand as in Example 2.5 or by using the Macaulay 2 code accompanying [WY08] currently available at the both authors' websites.

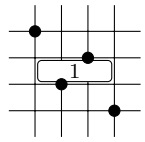
We believe that this list is not merely helpful in proving our theorem but is actually the complete list specifying exactly the non-lci loci of all Schubert varieties in the sense of [WY08, Thm. 2.6].

Theorem 5.2. *If the permutation w contains one of the patterns* 53241*,* 52341*,* 52431*,* 35142*,* 42513*, and* 351624*, then the Schubert variety X_w is not a local complete intersection.*

In Lemmas 5.3, 5.4, and 5.5, we show that, if a permutation w contains one of the six classical patterns in the theorem, then it also interval contains one of the intervals in

family A or B or one of the exceptional intervals in Figure 5. Then the Schubert variety X_w is not lci by Proposition 5.1 and [WY08, Thm. 2.6].

Note that a permutation w contains at one of the first three patterns in the theorem if and only if it contains the marked mesh pattern

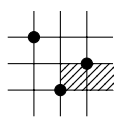


(5.1)

where the area marked with a 1 must contain at least one element.

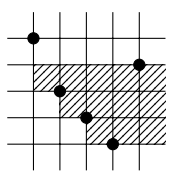
Lemma 5.3. *A permutation w contains the marked mesh pattern* (5.1) *if and only if it contains a mesh pattern from family A, the exceptional C, or one of the exceptionals E, E^{i}.*

In the proof below we will talk about "staircasing" boxes. Depending on context, this will mean picking either all of the NE-most or all of the SW-most elements in the box. A NE-most element i is one which has no elements to its NE in the box; to be specific, this means for all j in the box, either $j < i$ or $w(j) < w(i)$. Similarly, a SW-most element i is one with no elements to its SW, so for all j in the box, either $j > i$ or $w(j) > w(i)$. For example, suppose that we have an occurrence of the mesh pattern



,

in a permutation. Assume that for this particular occurrence there are exactly four elements in the box $(1, 1)$, and furthermore, that these four elements have the pattern 1423. By "staircasing" this box we mean adding the elements corresponding to 3 and 4 (in the box) and disregarding the other elements. This produces the pattern



.

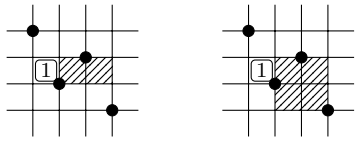
In the following three lemmas we will need to use the "shading lemma" of Hilmarsson, Jónsdóttir, Sigurðardóttir, Úlfarsson and Viðarsdóttir [HJS$^+$11], which gives conditions under which we can shade extra boxes in a particular mesh pattern without changing the set of permutations that avoid the pattern. We will use this lemma so frequently in the proofs below that when we add more shading to a pattern without giving a particular reason, the reader can assume this lemma is being used. The arguments for this lemma are elementary.

Proof. The "if" part is easily verified. We consider two cases for the "only if" part.

(1) There exists an occurence of (5.1) in which the second and third boxes of the marked region are empty or in which the first and second boxes of the marked region are empty.

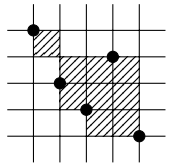
The case where the first and second boxes of the marked region are empty follows from the other case by the reverse complement symmetry, since the set of stated mesh patterns is invariant under reverse complement.

Therefore we can assume that we have an occurrence of the pattern on the left below, which implies an occurrence of the pattern on the right.

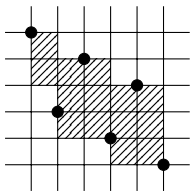


(5.2)

If we choose the right-most element in the marked box in the occurrence of the pattern on the right, we have an occurrence of the pattern



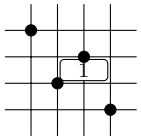
Consider the box with lower left corner with coordinates $(2, 4)$. If this box is empty, we can staircase boxes labelled $(1, 3)$ and $(3, 4)$, which will produce an occurrence of a member of family A. If box $(2, 4)$ is not empty, we choose the lowest element in that box. This will produce an occurrence of the pattern



If all of the boxes $(1, 3)$, $(2, 5)$ and $(4, 4)$ in this pattern are empty, we have an occurrence of the exceptional E^{i}; otherwise we staircase these boxes and produce a member of the family A.

(2) In every occurence of (5.1), either the second or third box of the marked region is non-empty and either the left or right box of the marked region is non-empty.

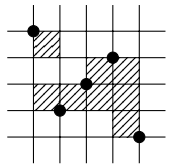
Since the second or third box is non-empty, we contain the pattern below.



(5.3)

We choose the lowest element in these two boxes and we consider two cases – depending on whether this element is in the first or second box.

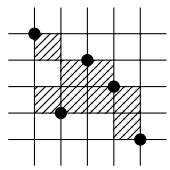
(a) If the lowest element is in box $(2, 2)$ of the pattern in (5.3) we produce an occurrence of the pattern



(5.4)

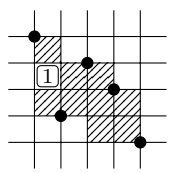
Here we can assume the boxes $(1,3)$ and $(2,3)$ are empty since otherwise we have an occurrence of the pattern on the left in (5.1), which we handled above in case (1). Similarly, we can assume that the boxes $(2,1)$ and $(3,1)$ are empty. Finally the boxes $(2,4)$ and $(3,4)$ can be assumed to be empty; otherwise we are back in case (1) (with the first and second boxes empty). Having shaded these extra boxes in the pattern (5.4), we have produced the exceptional C.

(b) If the lowest element is in box $(3,2)$ of the pattern (5.3) we produce an occurrence of the pattern

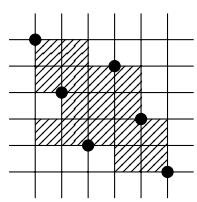


Now we can assume that the box $(3,1)$ is empty since otherwise we would have an occurrence of the pattern in (5.2). We consider two cases, depending on whether the box $(1,3)$ is empty or not.

(i) The box $(1,3)$ is not empty. This produces an occurrence of the pattern

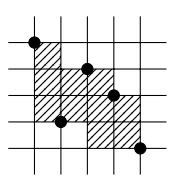


We can assume the marked box has a unique element that is both right-most and top-most because we would otherwise produce a member of family A. We therefore have an occurrence of the pattern



If the box $(3,1)$ is non-empty, we can staircase it and produce a member of family A. If this box is empty, but the box $(3,5)$ is non-empty, we can staircase it and produce a member of family A. Finally, if both of these boxes are empty we have the exceptional E.

(ii) Both of the boxes $(1,3)$ and $(3,1)$ are empty. This produces an occurrence of the pattern



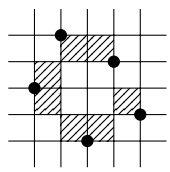
If both of the boxes $(2,1)$ and $(2,4)$ are empty, we have a member of family A. The same is true if one of the boxes are empty. If both boxes

contain elements, we staircase them (taking NE-most elements in $(2,1)$ and SW-most elements in $(2,4)$). Two things can occur: either the stair produced in $(2,1)$ is completely to the left of the stair in $(2,4)$, which produces a pattern from family A, or the stairs will cross at some point, in which case the first element in the stair for $(2,4)$ that is to the left of an element in the lower stair can be used to form a pattern from the family A. □

Lemma 5.4. (1) *If a permutation w contains the pattern* 35142*, then it contains at least one of the mesh patterns in family A or B or one of the exceptionals C, E, E^{i}, F^{rc}, F^{irc}, G, or H.*

(2) *If a permutation w contains the pattern* 42513*, then it contains at least one of the mesh patterns in family A or B or one of the exceptionals C, E, E^{i}, F, F^{i}, G^{rc}, or H.*

Proof. It suffices to prove part (1) since the other part can be obtained by applying reverse-complement to everything. If a permutation w contains the pattern 35142, then it will also contain the mesh pattern

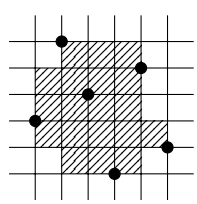


Consider the region made up of the boxes $(2,2)$, $(2,3)$, $(3,2)$ and $(3,3)$. If this region contains more than one element, we will have an occurrence of the pattern in (5.1). Then Lemma 5.3 would imply what we are trying to prove. We can therefore assume that there is at most one element in this region.

(1) If the region has no elements then by staircasing boxes $(1,1)$ and $(4,3)$ we produce an interval from family B.

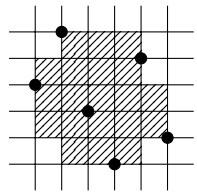
(2) If there is a single element in the region we need to look at four cases, depending on the box it belongs to. We can actually do just the cases where the element is in boxes $(2,2)$, $(2,3)$ and $(3,3)$ since the case of box $(3,2)$ follows from applying the inverse symmetry to the case involving box $(2,3)$.

(a) The element is in box $(2,3)$. We get the mesh pattern



If box $(5,4)$ is not empty, we get the marked mesh pattern in (5.1). If box $(5,3)$ or $(1,1)$ is not empty, we get a mesh pattern from family B by staircasing both boxes. If all four boxes are empty, we get the exceptional F^{rc}.

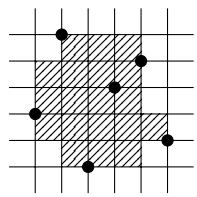
(b) The element is in box $(2, 2)$. We get the mesh pattern



.

If box $(5, 4)$ is not empty, we get the mesh pattern in (5.1). We can therefore assume that box is empty. If box $(1, 1)$ is not empty, and staircasing the box only adds one element to the pattern, we get the exceptional H. If staircasing adds more than one element, then we get an interval in family A. If both boxes are empty, we get the exceptional G.

(c) The element is in box $(3, 3)$. We get the mesh pattern



.

If the boxes $(1, 1)$ and $(5, 3)$ are empty, we get an interval from family B. If these boxes are not empty, the elements occupying them can also be used to produce an interval from the same family. □

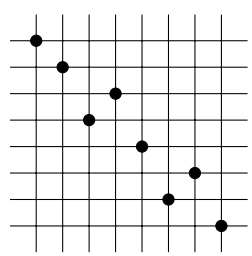
Lemma 5.5. *If a permutation w contains the pattern* 351624*, then it contains at least one of the mesh patterns in family A or B or one of the exceptionals C, D, E, F, G or H (or their symmetries under inverse and reverse-complement).*

To prove this lemma, we need the following definition. A **slab permutation** is a permutation that avoids the patterns $213, 123$, and 132 or equivalently avoids the marked mesh pattern

.

A typical permutation of this sort is

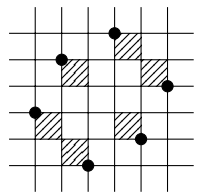
87564231 =



.

These permutations are counted by the Fibonacci numbers, as was first shown by Simion and Schmidt [SS85, Prop. 15].

Proof. If a permutation w contains the pattern 351624, then it will also contain the mesh pattern



.

Let α be the region consisting of the boxes $(3,3)$, $(3,4)$, $(4,3)$, $(4,4)$, and β be the region consisting of the boxes $(2,2)$, $(2,3)$, $(3,2)$, $(3,3)$. If w contains the mesh pattern (5.1) we are done by Lemma 5.3 above. We therefore assume w does not contain that mesh pattern. This implies that the elements in $\alpha \cup \beta$ must form a slab permutation. If the slab lies entirely in α or entirely in β, then we get a member of family B. (The boxes $(1,1)$ and $(5,5)$ may need to staircased.) Otherwise we can assume that the slab starts in the upper left corner of α and ends in the lower right corner of β. (For the other possibility we can use the inverse symmetry.) This will give us an occurrence of the pattern 35142 which implies what we want by Lemma 5.4. Finally $\alpha \cup \beta$ could be empty. If the boxes $(1,1)$ and $(5,5)$ are both empty, we get the exceptional D. If either of these boxes is non-empty, we staircase both and get an interval in family B. □

6. Remarks and applications

6.1. **Singularity implications with patterns.** In this section, we show that each of the pattern avoidance criteria that determine the geometric properties we have discussed imply one another.

We fix some notation for the patterns we will be using. Define the **smooth patterns** as

$$s = 3412, \quad s_c = 4231;$$

the **factorial patterns** as

$$f = 3\underline{41}2, \quad f_c = 4231;$$

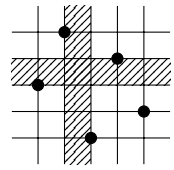
the **dbi patterns** as

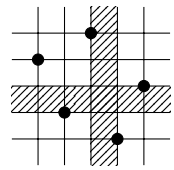
$$d_1 = 35142, \quad d_2 = 42513, \quad d_3 = 351624,$$
$$d_c = 4231;$$

the **lci patterns** as

$$\ell_1 = 35142, \quad \ell_2 = 42513, \quad \ell_3 = 351624,$$
$$\ell_{c_1} = 53241, \quad \ell_{c_2} = 52341, \quad \ell_{c_3} = 52431;$$

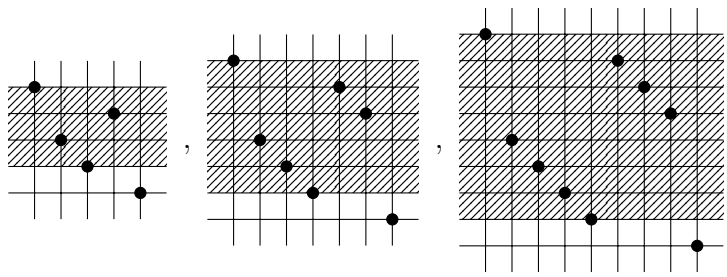
and the **Gorenstein patterns** as

$$g_1 = \begin{smallmatrix} 12\overline{3}45 \\ 3\underline{5}142 \end{smallmatrix} = 31524\,(1\leftrightarrow 5),\ (2\leftrightarrow 3) =$$ 

$$g_2 = \begin{smallmatrix} 1\overline{2}345 \\ 42\underline{5}13 \end{smallmatrix} = 42513\,(1\leftrightarrow 5),\ (3\leftrightarrow 4) =$$ 

Finally we define the two **Gorenstein corner families**. Let

$$\mathcal{G}_1 = \left(\begin{smallmatrix} 1\overline{2345} \\ 53241 \end{smallmatrix}, \begin{smallmatrix} 1\overline{234567} \\ 7432651 \end{smallmatrix}, \begin{smallmatrix} 1\overline{23456789} \\ 954328761 \end{smallmatrix}, \dots\right)$$

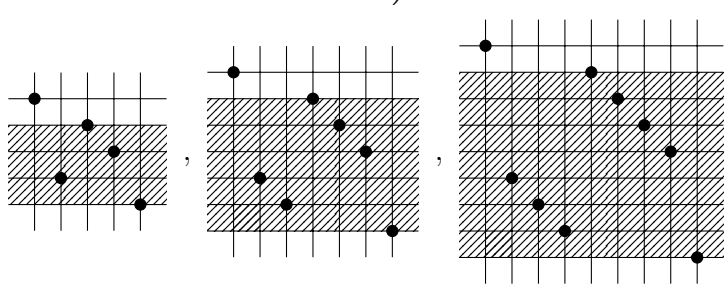
$$= \Bigg($$  $$\Bigg).$$

The general member of this family is of the form

$$\begin{smallmatrix} 1\overline{2\cdots\cdots k} \\ k\ell\cdots 2\cdots \ell{+}11 \end{smallmatrix},$$

where $\ell = (k-3)/2$. Also, let

$$\mathcal{G}_2 = \left(\begin{smallmatrix} \overline{12345} \\ 52431 \end{smallmatrix}, \begin{smallmatrix} \overline{1234567} \\ 7326541 \end{smallmatrix}, \begin{smallmatrix} \overline{123456789} \\ 943287651 \end{smallmatrix}, \dots\right)$$

$$= \Bigg($$  $$\Bigg).$$

The general member of this family is of the form

$$\begin{smallmatrix} \overline{12\cdots\cdots k} \\ k\ell\cdots 2\cdots \ell{+}11 \end{smallmatrix},$$

where $\ell = (k-1)/2$.

Then the Schubert variety X_w is

(1) smooth if and only if w avoids the smooth patterns [LS90],
(2) factorial if and only if w avoids the factorial patterns [BMB07],
(3) defined by inclusions if and only if w avoids the dbi patterns [GR02],
(4) a local complete intersection if and only if w avoids the lci patterns (Theorem 1.1),

(5) Gorenstein if and only if w avoids the Gorenstein patterns, and every associated Grassmannian permutation avoids every member of the corner families [WY06].

It is clear that avoidance of the smooth patterns implies avoidance of the factorial patterns.

Lemma 6.1. *Avoidance of the factorial patterns implies avoidance of the dbi patterns.*

Proof. Since $d_c = f_c$ it suffices to show that containment of d_i, for $i = 1, 2, 3$, implies containment of f or f_c. The cases $i = 1$ and $i = 2$ are similar so we only do the former.

Assume we have an occurrence of $d_1 = 35142$ in a permutation w. If the letters 5_w and 1_w are adjacent in the occurrence, then we have an occurrence of f. (Here we write "5_w" instead of "the letter in w corresponding to 5".) So assume they are not adjacent, and let the letter to the right of 5_w be a. If $4_w < a$ or $a < 2_w$, we have an occurrence of f. So assume that $2_w < a < 4_w$. Then the letters 5_w, a, 4_w, 2_w form an occurrence of f_c.

Now assume we have an occurrence of $d_3 = 351624$ in a permutation w. We prove this case by induction on the size of the gap between 6_w and 2_w. If these letters are adjacent we are done. If $5_w < a$, we can move 6_w to a and shorten the gap. If $a < 4_w$, then the letters 3_w, 5_w, 6_w, a, 4_w form an occurrence of f. Therefore we can assume that $4_w < a < 5_w$. Then 3_w, 5_w, 1_w, a, 2_w form an occurrence of d_1, and we know that this implies containment of either f or f_c. □

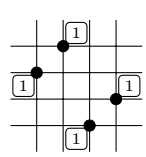
Clearly we have that avoidance of the dbi patterns implies avoidance of the lci patterns. So all that remains is:

Lemma 6.2. *Avoidance of the lci patterns implies avoidance of the Gorenstein patterns and the corner families for the associated Grassmannians.*

Proof. The containment of g_i implies the containment of ℓ_i for $i = 1, 2$. Also if a permutation contains one of the patterns in the Gorenstein corner family $\mathcal{G}_1$ then it contains ℓ_{c_1}. Finally if a permutation contains one of the patterns in the Gorenstein corner family $\mathcal{G}_2$, then it contains ℓ_{c_3}. □

Figure 6 gives another way of viewing some of the patterns above. Where possible, the classical patterns have been grouped together into a single marked mesh pattern to emphasize the underlying classical patterns 3421 and 4231 that characterize the smooth varieties.

Since we are emphasizing the underlying patterns 3421 and 4231 it is worth noting that the first pattern is present in two other known properties of permutations. Billey and Warrington [BW01] introduced 321-hexagon avoiding permutations as those permutations that avoid 321 as well as four classical patterns from S_8. Alternatively, these can be characterized as avoiding 321 and



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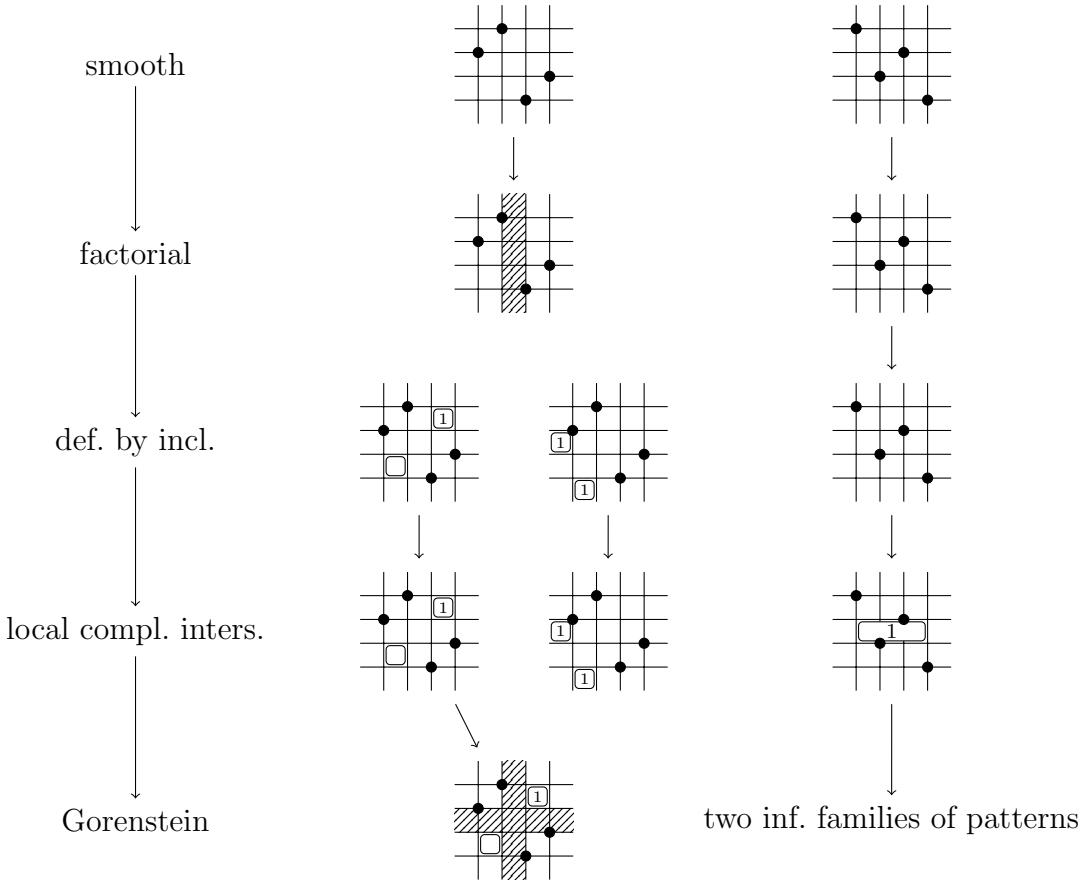
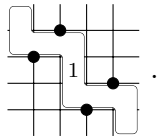


FIGURE 6. Properties of Schubert varieties described with pattern avoidance. An arrow between two patters means that avoidance of the first pattern implies avoidance of the second

Tenner [Ten11] studies permutations with the property that the number of repeated letters in the reduced decomposition equals the number of occurrences of 321 and 3412. She shows that these are exactly the permutations avoiding 4321 and 9 classical patterns from S_5. It is easy to verify that these are the permutations that avoid 4321 and



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6.2. **Lci matrix Schubert varieties.** Let $\pi : \mathrm{GL}_n \to G/B$ be the natural quotient map, and let $i : \mathrm{GL}_n \to M_n$ be the inclusion of GL_n into the affine space of $n \times n$ matrices. The **matrix Schubert variety** Y_w is the closure

$$Y_w := \overline{i(\pi^{-1}(X_{w_0 w}))};$$

it was introduced by Fulton in [Ful92].

As Fulton notes, Y_w is also the Kazhdan–Lusztig variety $\mathcal{N}_{v_n,\widetilde{w}}$, where $v_n \in S_{2n}$ is the permutation defined by $v(i) = n+1-i$ if $i \leq n$ and $v(i) = n + (2n+1-i)$ if $i > n$ and $\widetilde{w}$ is defined by $\widetilde{w}(i) = n + w_0 w(i)$ if $i \leq n$ and $\widetilde{w}(i) = 2n+1-i$ if $i > n$. Furthermore, looking at the matrices, it is clear that $\mathcal{N}_{v_n,\widetilde{w}} \times \mathbb{C}^{n(n-1)} \cong \mathcal{N}_{\mathrm{id},\widetilde{w}}$, since the generalized Plücker coordinates defining $I_{\widetilde{w}}$ do not involve any of the additional variables found in $M^{(\mathrm{id})}$ but not in $M^{(v_n)}$. Therefore, Y_w is lci if and only if $X_{\widetilde{w}}$ is. As a consequence, we have the following corollary.

Corollary 6.3. *The matrix Schubert variety Y_w is a local complete intersection if and only if w avoids 1342, 1432, 1423, 31524, 24153, and 426153.*

One can also reformulate this statement in terms of the diagram of w. Doing so recovers a theorem of Jen-Chieh Hsiao [Hsi11, Theorem 5.2].

6.3. **Local K-theory and cohomology classes at the identity.** Given a smooth variety X with an action of an algebraic torus $T = (\mathbb{C}^*)^n$ satisfying certain conditions (including that the fixed points are isolated), Goresky, Kottwicz, and Macpherson [GKM98] show that the map on equivariant cohomology

$$H_T^*(X) \to \bigoplus_{p \in X^T} H_T^*(p)$$

induced by the inclusion of the fixed points into X is an injection and describe its image. This result was extended by Knutson and Rosu [Ros03, Thm. A.5] to K-theory and the map

$$K_T^*(X) \to \bigoplus_{p \in X^T} K_T^*(p).$$

(All cohomology and K-theory will be with $\mathbb{Q}$ coefficients.) Since $H_T^*(p)$ is isomorphic to $\mathbb{Q}[t_1, \ldots, t_n]$ and $K_T^*(p)$ is isomorphic to $\mathbb{Q}[t_1^{\pm}, \ldots, t_n^{\pm}]$, this theory provides in many cases an easier method for calculating in the cohomology or K-theory ring. Indeed, for the case of the flag variety G/B, this theory was implicitly anticipated by work of Kostant and Kumar [KK86, KK90], who furthermore gave recursive formulas in $\bigoplus_{p \in X^T} H_T^*(p)$ and $\bigoplus_{p \in X^T} K_T^*(p)$ for sets of elements of $H_T^*(G/B)$ and $K_T^*(G/B)$ which form bases for these rings as free $H_T^*(p)$ and $K_T^*(p)$ modules. Kumar [Kum96] later showed that these basis elements are actually the classes of Schubert varieties.

The T-fixed points of G/B are the Schubert points e_u for all $u \in S_n$. Hence, the classes $[\mathcal{O}_{X_w}]\,|_{e_u}$ (both in $H_T^*(e_u)$ and $K_T^*(e_u)$) determine the cohomology and K-theory class of the Schubert variety X_w. In addition, these classes encode significant geometric information about X_w. Our aim in this section is to describe the consequences of our explicit equations for lci Schubert varieties on the class $[\mathcal{O}_{X_w}]\,|_{e_{\mathrm{id}}}$ in the case where X_w is lci. We recover in part half of a theorem of Kumar in the case where X_w is smooth. We will need to use the algebraic machinery of K-polynomials and multidegrees, which we briefly describe here but are described in greater detail in [MS05, Chap. 8] and [KM05, Section 2.3].

Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ denote an integral basis for the weight lattice of $T = (\mathbb{C}^*)^n$, which we identify with $\mathbb{Z}^n$. An action of T on a polynomial ring $S = \mathbb{C}[x_1, \ldots, x_k]$ (assuming the

x_i are eigenvectors for the action) corresponds to the grading assigning each variable x_i a degree $\lambda^{(i)} = \sum_{j=1}^n \lambda_j^{(i)} \mathbf{a}_j \in \mathbb{Z}^n$, where $\lambda^{(i)}$ is the weight of the action of T on x_i. In the case where our grading is positive, meaning that the $\lambda^{(i)}$ generate a pointed cone in $\mathbb{Z}^n$, any finitely generated graded S-module M has a **Hilbert series** (also known as formal T-character)

$$\mathcal{H}(M) = \sum_{\lambda \in \mathbb{Z}^n} \dim M_\lambda \mathbf{t}^\lambda,$$

where the sum is over all weights of T, M_λ is the weight space for λ, and $\mathbf{t}^\lambda$ denotes $\prod_{j=1}^n t_j^{\lambda_j}$ where $\lambda = \sum_{j=1}^n \lambda_j \mathbf{a}_j$.

The K**-polynomial** of M can be defined as

$$\mathcal{K}(M) = \mathcal{H}(M) \prod_{i=1}^k (1 - \mathbf{t}^{\lambda^{(i)}}).$$

Given a finite $\mathbb{Z}^n$ graded free resolution

$$0 \leftarrow M \leftarrow E_0 \leftarrow \cdots \leftarrow E_L \leftarrow 0$$

of M with

$$E_k = \bigoplus_j S(-\lambda^{(j,k)}),$$

the K-polynomial satisfies

$$\mathcal{K}(M) = \sum_{k=1}^L (-1)^k \sum_j \lambda^{(j,k)},$$

so $\mathcal{K}(M)$ is a representative for the class of M in $K_T^*(\operatorname{Spec} S)$.

Define the **multidegree** of M, denoted $\mathcal{C}(M)$, as the sum of the lowest degree terms of $\mathcal{K}(M, \mathbf{1} - \mathbf{t})$. (This means we substitute $1 - t_i$ for t_i for each $i \in [\![1, n]\!]$.) Taking the lowest degree terms in this way is, up to a sign change which conveniently agrees with a difference between the usual conventions for Grothendieck and Schubert polynomials, the same as taking the Chern map from K-theory to cohomology, so $\mathcal{C}(M)$ can be regarded as a representative for the class of M in $H_T^*(\operatorname{Spec} S)$.

Given a graded complete intersection S/I, where I is generated by $f_1, \ldots, f_L$, the K-polynomial $\mathcal{K}(S/I)$ and multidegree $\mathcal{C}(S/I)$ are easily seen to be

$$\mathcal{K}(S/I) = \prod_{i=1}^L (1 - \mathbf{t}^{\deg f_i})$$

and hence

$$\mathcal{C}(S/I) = \prod_{i=1}^L \left(\sum_{j=1}^n \langle \mathbf{a}_j, \deg f_i \rangle t_j \right),$$

since the Koszul resolution is a free resolution of S/I.

Now we consider specifically the case of T-invariant subvarieties of G/B, and more specifically Ω_{id}. In this case, $S = S_{\mathrm{id}}$, and since T under its usual action on G/B acts on

the matrix entry at (i,j) with weight $\mathbf{a}_i - \mathbf{a}_j$ (where $\mathbf{a}_i$ denotes as usual the homomorphism from T to C^* picking out the i-th diagonal entry), it acts on the variable $z_{i,j}$, which is the coordinate function for the matrix entry at (i,j), with weight $\mathbf{a}_j - \mathbf{a}_i$. Moreover, $\operatorname{Spec} S$ equivariantly retracts onto e_{id}, so we can identify classes in $H_T^*(\operatorname{Spec} S)$ and $K_T^*(\operatorname{Spec} S)$ respectively with classes in $H_T^*(e_{\mathrm{id}})$ and $K_T^*(e_{\mathrm{id}})$.

Hence we can identify $[\mathcal{O}_{X_w}]\,|_{e_{\mathrm{id}}}$ with $\mathcal{C}(S/I_w)$ (for cohomology) or $\mathcal{K}(S/I_w)$ (for K-theory). Furthermore, it is a well-known folklore theorem (see [Gol01, WY09] for proofs for which none of the authors claim originality) that, in the coordinates for the weight space we use,

$$[\mathcal{O}_{X_w}]\,|_{e_v} = \mathfrak{S}_{w_0 w}(t_{v(1)}, \ldots, t_{v(n)}; t_n, \ldots, t_1)$$

in cohomology, and

$$[\mathcal{O}_{X_w}]\,|_{e_v} = \mathfrak{G}_{w_0 w}(t_{v(1)}, \ldots, t_{v(n)}; t_n, \ldots, t_1)$$

in K-theory, where $\mathfrak{S}_w$ and $\mathfrak{G}_w$ are the double Schubert and Grothendieck polynomials of Lascoux and Schützenberger [LS82a, LS82b]. Hence we can and will state our results purely as an identity between polynomials.

Since we have explicit generators for the ideal defining $\mathcal{N}_{\mathrm{id},w}$ whenever X_w is lci, we can calculate their degrees to obtain explicit formulas for the local K-theory and cohomology classes of X_w at the identity.

Proposition 6.4. *Suppose X_v is defined by inclusions, $(x,y) \in D(v)$, $r = r_v(x,y)$, and $f_{(x,y)}$ is the generator of I_v defined in Section 4. Then $\deg f_{(x,y)} = \mathbf{a}_{x+r} - \mathbf{a}_{y-r}$ (regardless of the choices made in defining $f_{(x,y)}$).*

Proof. In our grading, $z_{i,j}$ has degree $\mathbf{a}_j - \mathbf{a}_i$. Hence the degree of $d_{A,B}$ is $\sum_{j\in B} \mathbf{a}_j - \sum_{i \in A} \mathbf{a}_i$.

Given $(x,y) \in D(v)$, $f_{(x,y)}$ is defined to be $d_{A(x,y),B(x,y)}$ where

$$A(x,y) = [\![p, p+r-1]\!] \cup \{x+r\},$$
$$B(x,y) = \{y-r\} \cup [\![q-r+1, q]\!],$$

and (p,q) is some essential set box NE of (x,y) and in the same connected component of the diagram. At first glance it appears that the degree of $f_{(x,y)}$ depends on the choice of (p,q), but Lemma 3.2 tells us that (except in the case where $r=0$ and this is irrelevant), $r = q-p+1$, so $p = q-r+1$ and $p+r-1 = q$. Hence,

$$\deg f_{(x,y)} = \mathbf{a}_{y-r} + \sum_{i=0}^{r-1} \mathbf{a}_{q-i} - \mathbf{a}_{x+r} - \sum_{i=0}^{r-1} \mathbf{a}_{p+i} = \mathbf{a}_{y-r} - \mathbf{a}_{x+r}. \qquad \square$$

Proposition 6.5. *Suppose X_w is lci, $(p,q) \in E''(w)$, $r = r_w(p,q)$, and $f_{(p,q)}$ is as defined in Section 4.2. Then*

$$\deg f_{(p,q)} = \sum_{i=0}^{r} (\mathbf{a}_{q-r} - \mathbf{a}_{p+r}).$$

Proof. The polynomial $f_{(p,q)}$ is defined as $d_{A'(p,q),B'(p,q)}$ where

$$A'(p,q) = [\![p, p+r]\!]$$

and

$$B'(p,q) = [\![q-r,q]\!].$$

The proposition follows immediately. □

The corollaries on follow immediately from the propositions and the discussion above.

Corollary 6.6. *Suppose X_v is defined by inclusions. Then*

$$\mathcal{K}(S/I_v) = \mathfrak{G}_{w_0v}(t_1,\dots,t_n;t_n,\dots,t_1) = \prod_{(x,y)\in D(v)} (1 - t_{y-r_v(x,y)}/t_{x+r_v(x,y)}),$$

and

$$\mathcal{C}(S/I_v) = \mathfrak{S}_{w_0v}(t_1,\dots,t_n;t_n,\dots,t_1) = \prod_{(x,y)\in D(v)} (t_{y-r_v(x,y)} - t_{x+r_v(x,y)}).$$

Corollary 6.7. *Suppose X_w is lci, and let v be the permutation defined by inclusions associated to w by Theorem 3.8. Then*

$$\begin{aligned}\mathcal{K}(S/I_w) &= \mathfrak{G}_{w_0w}(t_1,\dots,t_n;t_n,\dots,t_1)\\ &= \mathfrak{G}_{w_0v}(t_1,\dots,t_n;t_n,\dots,t_1) \prod_{(p,q)\in E''(w)} \left(1 - \prod_{i=0}^{r_w(p,q)} t_{q-i}/t_{p+i}\right),\end{aligned}$$

and

$$\begin{aligned}\mathcal{C}(S/I_w) &= \mathfrak{S}_{w_0w}(t_1,\dots,t_n;t_n,\dots,t_1)\\ &= \mathfrak{S}_{w_0v}(t_1,\dots,t_n;t_n,\dots,t_1) \prod_{(p,q)\in E''(w)} \left(\sum_{i=0}^{r_w(p,q)} t_{q-i} - t_{p+i}\right).\end{aligned}$$

For j and i with $1 \le j < i \le n$, let $s_{ji} \in S_n$ be the transposition switching j and i. For the case where X_v is smooth, the following is a theorem of Kumar [Kum96], restated in our language.

Theorem 6.8. *The following are equivalent:*

(1) *X_v is smooth.*
(2)
$$\mathcal{K}(S/I_v) = \mathfrak{G}_{w_0v}(t_1,\dots,t_n;t_n,\dots,t_1) = \prod_{(i,j):s_{ji}\not\le v} (1 - t_j/t_i)$$
(3)
$$\mathcal{C}(S/I_v) = \mathfrak{S}_{w_0v}(t_1,\dots,t_n;t_n,\dots,t_1) = \prod_{(i,j):s_{ji}\not\le v} (t_j - t_i).$$

Comparing Theorem 6.8 and Corollary 6.6 tells us (because $\mathbb{Q}[t_1,\dots,t_n]$ is a unique factorization domain) that, in the case where X_v is smooth, the map

$$\begin{aligned}D(v) &\to \{(i,j) \mid s_{ji} \not\le v\}\\ (x,y) &\mapsto (x + r_v(x,y), y - r_v(x,y))\end{aligned}$$

is a bijection with the claimed image. Indeed, $D(v)$ has $\binom{n}{2} - \ell(v)$ elements, and $\{(i,j) \mid s_{ji} \not\leq v\}$ has $\binom{n}{2} - \ell(v)$ elements whenever X_v is smooth by a theorem of Carrell [Car94]. We believe this statement can be proved purely combinatorially.

In addition, this map from $D(v)$ to the set of transpositions (or equivalently the set of positive roots) makes sense for any v. We believe the image of this map always contains $\{(i,j) \mid s_{ji} \not\leq v\}$ and equals this set precisely when v is defined by inclusions.

6.4. **Cohomology rings of lci Schubert varieties.** Let $Z \subseteq \Omega^\circ_{\mathrm{id}}$ be the scheme theoretic vanishing locus of a principal nilpotent vector field on G/B. In our coordinates on $\Omega^\circ_{\mathrm{id}}$, the scheme Z is defined by the ideal $\langle a_{i,j} \rangle_{1 \leq j < i \leq n} \subseteq S$, where

$$a_{i,j} = z_{i+1,j} - z_{i,j-1} + z_{i,j}(z_{j,j-1} - z_{j+1,j}).$$

Akyildiz, Lascoux, and Pragacz [ALP92] show that, for any Schubert variety X_w, the cohomology ring $H^*(X_w, \mathbb{C})$ is actually the coordinate ring of the scheme-theoretic intersection $X_w \cap Z$.

In earlier work leading to this theorem, Akyildiz and Akyildiz [AA89] show that the isomorphism $\phi : \mathbb{C}[Z] \to H^*(G/B)$ is explicitly given by

$$\phi(z_{i,j}) = h_{i-j}(x_1, \ldots, x_j),$$

where $h_{i,j}$ denotes the complete homogeneous symmetric function in the given variables, and x_i is explicitly the first Chern class $c_1(L_i^\vee)$ of the i-th dual tautological line bundle. Moreover, Akyildiz, Lascoux, and Pragacz [ALP92] show that

$$H^*(X_w) = H^*(G/B)/\phi(I_w)$$

not only as abstract rings but explicitly as $H^*(G/B)$-modules under the interpretation $x_i = c_1(L_i^\vee)$.

Gasharov and Reiner [GR02] gave a presentation for the cohomology ring $H^*(X_v)$ (and indeed, its projection onto any partial flag variety) in the case where X_v is defined by inclusions. Given $(p,q) \in E(v)$ with $r_v(p,q) = 0$, let

$$K_{(p,q)} = \langle e_k(x_{q+1}, \ldots, x_n) \rangle_{k=p-q}^{n-q},$$

where e_k denotes the k-th elementary symmetric function in the given variables. Given $(p,q) \in E'(w)$, let

$$K_{(p,q)} = \langle e_k(x_1, \ldots, x_q) \rangle_{k=q-p+2}^{n-q}.$$

Then Gasharov and Reiner show the following [GR02, Thm. 3.1].

Theorem 6.9. *Let v be defined by inclusions. Then*

$$H^*(X_v) = H^*(G/B) / \sum_{(p,q) \in E(v)} K_{(p,q)}$$

as an $H^*(G/B)$ module where $x_i = c_1(L_i^\vee)$.

Indeed, the original aim of Gasharov and Reiner was to investigate $H^*(X_v)$ in the case where X_v is smooth, and they defined the class of Schubert varieties defined by inclusions because it is the more general class for which their formula holds. Using our explicit generators for I_v, one can calculate explicit generators for $\phi(I_v)$ and hence a

presentation for $H^*(X_v)$. The presentation we obtain is different from that of Gasharov and Reiner; recovering their result from ours via the theorem of Akyildiz, Lascoux, and Pragacz requires some use of determinantal and symmetric function identities.

In general, their presentation (which is not always minimal) can require fewer than $\binom{n}{2} - \ell(w)$ generators, so this shows I_v can in some situations require more generators than $\phi(I_v)$. (Indeed, this is obvious from considering the one dimensional Schubert varieties X_{s_i} where s_i is the permutation switching i and $i+1$.)

In the more general case where X_w is lci, one can obtain a presentation for $H^*(X_w)$ by using the theorem of Gasharov and Reiner along with the following proposition.

Proposition 6.10. *Suppose X_w is lci, $(p,q) \in E''(w)$, and $r = r_w(p,q)$. Then*

$$\phi(f_{(p,q)}) = \phi(d_{A'(p,q),B'(p,q)}) = s_{(p-q+r)^{r+1}}(x_1, \ldots, x_q),$$

where $s_{(p-q+r)^{r+1}}$ is the Schur function (in the given variables) corresponding to the rectangular partition with $r+1$ parts of size $p-q+r$.

Proof. Note that $A'(p,q) = [\![p, p+r]\!]$ and $B'(p,q) = [\![q-r, q]\!]$. Hence,

$$\phi(d_{A'(p,q),B'(p,q)}) = \begin{vmatrix} h_{p-q+r}(x_1, \ldots, x_{q-r}) & \cdots & h_{p-q}(x_1, \ldots, x_q) \\ \vdots & & \vdots \\ h_{p+r-q+r}(x_1, \ldots, x_{q-r}) & \cdots & h_{p+r-q}(x_1, \ldots, x_q) \end{vmatrix}.$$

As noted by Akyildiz and Akyildiz [AA89], one can repeatedly apply column operations using the symmetric function identity

$$h_a(x_1, \ldots, x_b) = h_a(x_1, \ldots, x_{b-1}) + x_b h_{a-1}(x_1, \ldots, x_b)$$

to show that

$$\phi(d_{A'(p,q),B'(p,q)}) = \begin{vmatrix} h_{p-q+r}(x_1, \ldots, x_q) & \cdots & h_{p-q}(x_1, \ldots, x_q) \\ \vdots & & \vdots \\ h_{p+r-q+r}(x_1, \ldots, x_q) & \cdots & h_{p+r-q}(x_1, \ldots, x_q) \end{vmatrix}.$$

The proposition than follows immediately from the Jacobi–Trudi identity, a standard identity in the theory of symmetric functions. □

We obtain the following corollary from the proposition and the discussion above.

Corollary 6.11. *Suppose X_w is lci, and let v be the permutation defined by inclusions associated to w by Theorem 3.8. Then*

$$H^*(X_w) = H^*(X_v) / \langle s_{(p-q+r_w(p,q))^{r_w(p,q)+1}}(x_1, \ldots, x_q) \rangle_{(p,q) \in E''(w)}$$

as an $H^*(G/B)$ module where $x_i = c_1(L_i^\vee)$.

Using Theorem 3.1 and Corollary 6.11, one can obtain an explicit presentation of the cohomology ring $H^*(X_w)$ whenever X_w is lci.

7. Questions

We conclude with a list of questions for future research. We begin with two purely combinatorial problems.

Question 7.1. Enumerate the permutations $w \in S_n$ for which X_w is lci. An ideal answer would provide an explicit generating function.

For smooth Schubert varieties, the analogous question was answered in unpublished work of Haiman [Hai92]. (A proof of this formula appears in [BMB07].) Bousquet-Mélou and Butler [BMB07] gave a generating function for the number of factorial Schubert varieties. On the other hand, the analogous question for Schubert varieties defined by inclusions and for Gorenstein Schubert varieties are still open.

This question has motivation beyond mere curiosity. The generating function for the number of smooth Schubert varieties reflects the earlier theorem of Ryan [Rya87] that all smooth Schubert varieties are iterated Grassmannian bundles. One might hope that a generating function for the number of lci Schubert varieties could lead to a similar structure theorem.

We expect that a generating function for the Schubert varieties defined by inclusions could possibly be obtained by an argument similar to the one for smooth Schubert varieties. Answering the following more specific combinatorial question may help in deriving the generating function enumerating lci Schubert varieties from a (currently unknown) generating function for Schubert varieties defined by inclusions.

Question 7.2. Determine if the converse to Theorem 3.8 is true. More precisely, suppose w is a permutation with essential set $E(w)$, and suppose $E''(w) \subset E(w)$ is the set of essential set boxes that are not defined by inclusions. If $E(w) \setminus E''(w)$ is the essential set for some permutation v (necessarily defined by inclusions) such that $r_v(p,q) = r_w(p,q)$ for all $(p,q) \in E(w) \setminus E''(w)$ and $\ell(v) - \ell(w) = \#E''(w)$, then is w necessarily almost defined by inclusions (or equivalently lci)?

We now proceed to questions of a more geometric nature.

Question 7.3. Find a geometric explanation for the appearance of ordinary pattern avoidance in our theorem.

A priori, one would expect that ordinary pattern avoidance is not sufficient for characterizing the Schubert varieties that are lci. Indeed, the weaker conditions of being factorial [BMB07] and of being Gorenstein [WY06] cannot be characterized by pattern avoidance since there exist examples where X_v is not factorial (respectively not Gorenstein), w pattern contains v, and X_w is factorial (respectively Gorenstein). Instead some more general form of pattern avoidance is required in the statement of those theorems.

On the other hand, the pattern map of Billey and Braden [BB03] (which was also described by Bergeron and Sottile [BS98]) gives a geometric explanation of why the smooth Schubert varieties can be characterized by pattern avoidance. Their explanation relies on the smoothness of the T-fixed locus (with the reduced scheme structure) of any smooth variety, which was originally shown (in fact for any linearly reductive group T over a field

of characteristic 0) by Fogarty and Norman [FN77]. To use the explanation of Billey and Braden, we would need that the T-fixed locus of any lci variety is lci. It seems to be not known if this is true in general.

Question 7.4. Determine the lci locus for any Schubert variety.

We conjecture that the intervals given in Proposition 5.1 determine precisely the non-lci locus. Proving this conjecture would answer this question.

The smooth locus was characterized independently by Billey and Warrington [BW03], Cortez [Cor03], Kassel, Lascoux, and Reutenauer [KLR03], and Manivel [Man01b] following earlier work of Gasharov [Gas01]. The proofs in [BW03, KLR03, Man01a] are similar and use a criterion on the Bruhat graph due to Carrell [Car94], while [Cor03] uses more geometric methods depending on partial resolutions of singularities. Perrin [Per09] used similar geometric methods to characterize the Gorenstein locus in the specific case of Schubert varieties coming from a (co)minscule parabolic.

Question 7.5. Characterize the lci Schubert varieties for the other simple Lie groups.

In our view, the ideal answer to this question would be a characterization in terms of the definition of pattern avoidance via root subsystems given by Billey and Postnikov [BP05]. However, a criterion in terms of the Bruhat graph similar to that of Carrell [Car94] would be of interest both for this question and the previous one. Hence we pose the following question.

Question 7.6. Determine if X_w being lci depends solely on the Bruhat graph of w. If so, find reasonable properties of the Bruhat graph that characterize when X_w is lci. Similarly, determine if X_w being lci at e_v depends solely on the Bruhat graph between v and w and find properties of the graph that characterize when X_w is lci at e_v.

Another possible further generalization of our work is the following.

Question 7.7. For each value of $k \geq 1$, characterize the Schubert varieties X_w for which I_w can be generated by at most $\operatorname{codim}(X_w) + k$ generators. In particular, is this set given by classical pattern avoidance, and, if so, by avoidance of finitely many patterns?

The condition of failing to be lci by at most k generators is, like the property of being lci (which is the case $k = 0$), an intrinsic homological property of the ring which holds on Zariski open sets and does not depend on the embedding.

One can also ask for characterizations of the lci locus both for GL_n and for other Lie groups in terms of the local K-polynomial or multidegree.

Question 7.8. Determine if the converse to Corollary 6.7 is true, both at the identity and for a general v. This means determining if X_w is automatically lci at e_v whenever the $\mathcal{K}(N_{v,w})$ is a product of $\ell(w) - \ell(v)$ terms of the form $(1 - \mathbf{t}^\lambda)$ for some weight λ or whenever $\mathcal{C}(N_{v,w})$ is a product of $\ell(w) - \ell(v)$ terms of the form $\sum_{j=1}^n \langle \mathbf{a}_j, \lambda \rangle t_j$.

Note that Kostant and Kumar [KK90, KK86] give general recursive algebraic formulas for $\mathcal{K}(N_{v,w})$ and $\mathcal{C}(N_{v,w})$, so one can in principle apply this criterion without knowing any geometry.

Theorem 6.8 is an equivalence and extends (in the original formulation by Kumar) to local neighborhoods, so X_w is smooth at e_v whenever $\mathcal{K}(\mathcal{N}_{v,w})$ is a particular specific product of $\ell(w) - \ell(v)$ terms of the form $(1 - \mathbf{t}^\lambda)$ or equivalently whenever $\mathcal{C}(\mathcal{N}_{v,w})$ is a particular product of $\ell(w) - \ell(v)$ linear terms (which in this case are all given by roots).

Finally we focus on some questions related specifically to the Schubert varieties defined by inclusions.

Question 7.9. Is being defined by inclusions equivalent to some intrinsic geometric property of X_w which does not depend on its embedding into G/B?

Note that K-theory and cohomology classes are not intrinsic, so Corollary 6.6 does not answer this question. However, Corollary 6.6 does give a specific form for the K-polynomial and multidegree when X_w is defined by inclusions. Hence we can ask the following.

Question 7.10. Determine if the converse to Corollary 6.6 is true. In particular, determine if X_w is automatically defined by inclusions whenever $\mathcal{K}(S/I_w)$ is a product of $\ell(w)$ terms of the form $(1 - \mathbf{t}^\lambda)$ for a *root* λ, or whenever $\mathcal{C}(S/I_w)$ is a product of $\ell(w)$ terms of the form $\sum_{j=1}^n \langle \mathbf{a}_j, \lambda \rangle t_j$ where λ is a *root*.

We can show combinatorially that a positive answer to Question 7.8 implies a positive answer to this question.

If the answer to this question is positive, it could possibly be used to define what it means for X_w to be locally defined by inclusions at e_v. Since the roots appearing in Corollary 6.6 satisfy the condition on the roots appearing in Theorem 6.8 (but not with multiplicity 1), the conditions in Kumar's theorem specifying the roots that appear in $\mathcal{K}(\mathcal{N}_{v,w})$ (which are the roots appearing in $\mathcal{C}(\mathcal{N}_{v,w})$) when $\mathcal{N}_{v,w}$ is smooth may be helpful. One can then ask the following.

Question 7.11. Define a local notion of being defined by inclusions using K-polynomials or multidegrees, extending to other Lie groups if possible. If possible, link this definition to conditions defining Richardson varieties or to (relative) cohomology rings of Schubert or Richardson varieties.

An answer to this question may help solve a major mystery regarding the Schubert varieties defined by inclusions. Hultman, Linusson, Shareshian, and Sjöstrand [HLSS09] studied for any element w of any Coxeter group a hyperplane arrangement $\mathcal{A}_w$ known as the inversion arrangement of w. They showed that $re(w)$, the number of chambers of $\mathcal{A}_w$, is always at most $br(w)$, the number of elements of G which are less than or equal to w in Bruhat order. For the Coxeter group $G = S_n$, they also showed that $re(w) = br(w)$ if and only if X_w is defined by inclusions. Their proof only shows that the permutations satisfying $re(w) = br(w)$ are given by the same pattern avoidance conditions, and no explanation for this coincidence is known.

Hultman [Hul10] later showed that, for an arbitrary Coxeter group, $re(w) = br(w)$ if and only if the Bruhat graph of w satisfies a particular criterion. There are various ways to extend this criterion to intervals $[v, w]$ in Bruhat order rather than a single permutation

w. Indeed, we hope such a criterion may be useful in answering the previous question. Hence we ask the following.

Question 7.12. Extend Hultman's criterion to intervals in Bruhat order, and link this criterion to some hyperplane arrangement associated to inversions or to a local notion of being defined by inclusions as in Question 7.11.

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