CATALAN NUMBERS AND SCHUBERT POLYNOMIALS FOR $w = 1(n+1)\cdots 2$

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Abstract. We show that the Schubert polynomial $\mathfrak{S}_w$ specializes to the Catalan number C_n when $w = 1(n+1)\cdots 2$. Several proofs of this result as well as a q-analog are given. An application to the singularities of Schubert varieties is given.

1. Introduction

The Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ are ubiquitous in combinatorics. Among other things they count Dyck paths, which are lattice paths from $(0,0)$ to (n,n) staying above the main diagonal, and rooted binary trees with n leaves (where each internal node has exactly 2 children). In this paper we present a perviously undiscovered connection between Catalan numbers and certain Schubert polynomials. Our result also has an interpretation in terms of the geometry of certain Schubert varieties.

Let $w \in S_n$ be a permutation. The Schubert polynomial $\mathfrak{S}_w$ is a polynomial representative for the class of the Schubert variety $X_{w_0 w}$ in the flag manifold $GL(n)/B$. A combinatorial formula for Schubert polynomials, which for our purposes can be taken as their definition, was proven independently in [2] and [5]. Diagrams illustrating this formula, called rc-graphs, were introduced in [6] and extensively studied in [1].

For the remainder of this paper, let w_n denote the permutation $1(n+1)n\cdots 2$ in S_{n+1}. Using rc-graphs, we will show that that the principal specialization of the Schubert polynomial for this permutation, $\mathfrak{S}_{w_n}(1, q, \cdots, q^n)$, is equal to $q^{\binom{n}{3}}C_n(q)$, where $C_n(q)$ is the Carlitz-Riordan q-analogue of the Catalan numbers originally introduced in [3]. Section 2 gives the definition of rc-graphs and proves our main theorem via a recurrence counting rc-graphs for w_n. Section 3 gives a bijection to Dyck paths, and Section 4 shows that this bijection can also be described using the Edelman-Greene correspondence. Since w_n is its own inverse in S_{n+1}, transposition is a natural involution on its rc-graphs. In Section 5, we describe our bijection in terms of binary trees, giving a correspondence between features of the rc-graph known as elbow joints and the internal nodes of a binary tree. From this description it will be evident that transposing an rc-graph corresponds to flipping a binary tree around its vertical axis.

Finally we discuss in Section 6 the geometric example which originally motivated this study. Let w'_n be the permutation $(n+2)23\cdots(n+1)1$ in S_{n+2}. We show that the multiplicity of the Schubert variety $X_{w'_n}$ at its most singular point is given by

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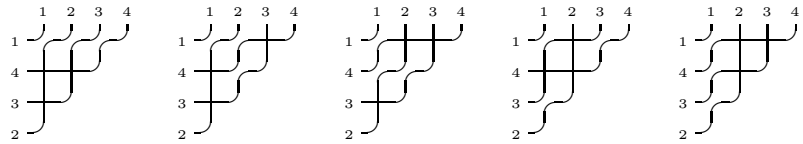

FIGURE 1. The rc-graphs for $w = 1432$

C_n. Experimental evidence [11] suggests that this is higher than the multiplicity of any point on any other Schubert variety of $GL(n+2)/B$.

2. RC-GRAPHS

Let $w \in S_n$ be a permutation. An *rc-graph* for w is a filling of the upper-left half of an $n \times n$ array with cross pieces ($+$) and elbow joints ($\lrcorner$ $\ulcorner$) such that the strand entering the left in row i exits the top in column $w(i)$, with the additional condition that no two strands cross more than once. This second condition can alternatively be stated as there being exactly $l(w)$ cross pieces. For example, the 5 rc-graphs for $w = 1432$ are shown in figure 1. It was shown independently in [2] and [5] that rc-graphs are related to Schubert polynomials by the formula

$$\mathfrak{S}_w(x_1, \cdots, x_n) = \sum_{D \in \mathcal{RC}(w)} \prod_{(i,j) \in C(D)} x_i,$$

where $\mathcal{RC}(w)$ is the set of rc-graphs for w, and $C(D)$ are the locations of the cross pieces in D (indexed so that $(1,3)$ would be in $C(D)$ if D has a cross in the top row and the third column). For example, we have $\mathfrak{S}_{1432} = x_2^2x_3 + x_1x_2x_3 + x_1^2x_3 + x_1x_2^2 + x_1^2x_2$, with the 5 terms corresponding to the 5 rc-graphs in the figure 1 from left to right.

First we will show that there are in fact Catalan many rc-graphs for w_n; this will allow us to prove that the combinatorial maps we give from the set of rc-graphs for w_n to other Catalan objects are in fact bijections by only showing that they are injections or surjections.

The Carlitz-Riordan q-Catalan numbers $C_n(q)$ are defined by the recurrence $C_n(q) = \sum_{k=0}^{n-1} q^k C_{n-k-1}(q) C_k(q)$, with $C_0(q) = 1$; under the interpretation of Catalan numbers as counting partitions λ whose Young diagrams fit inside the Young diagram of the staircase partition $\delta_n = n-1, \cdots, 1$, we have $C_n(q) = \sum_\lambda q^{\binom{n}{2} - |\lambda|}$.

Proposition 1.

$$\mathfrak{S}_{w_n}(1, q, \cdots, q^n) = q^{\binom{n}{3}} C_n(q).$$

Proof. We proceed by induction on n. The proposition is clear for $n = 1$.

Given a rc-graph D, the strand D_{n+1} entering the left in row $n+1$ and exiting the top in column $w_n(n+1) = 2$ travels through only the first and second columns. There is a unique k, $1 \le k \le n$, such that D_{n+1} goes through both the square $(k,1)$ and $(k,2)$.

Fixing k, there must be cross pieces at (i,j) for all i,j with $1 \le i \le k-1$, $2 \le j \le n+2-k$; this is true by definition for $j = 2$, and each strand D_l for $k+1 \le l \le n$ crosses D_{n+1} in row l and never travels to the right of column

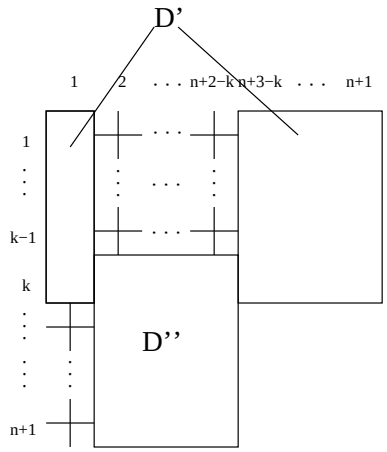

FIGURE 2. Breaking up the rc-graph D

$w(l) = n+3-l$, so they must each go straight through the topmost $k-1$ places in column $n+3-l$, which therefore must all be cross pieces. Note each column of these crosses contributes $\binom{k-1}{2}$ to the weight, for a total of $(n+1-k)\binom{k-1}{2}$.

We must also have cross pieces at $(i,1)$ for $k+1 \le j \le n$, and these contribute $\binom{n}{2} - \binom{k}{2}$ to the weight.

Now let D' be the portion of D consisting of squares (i,j) with $k \le i \le n$ and $2 \le j \le n+2-k$; this is an rc-graph for w_{n-k}. Also, let D'' be the portion of D consisting of the squares (i,j), where $1 \le i \le k$ and either $j = 1$ or $n+3-k \le j \le n+1$, ignoring the intervening cross pieces in columns 2 through $n+2-k$; this is an rc-graph for w_{k-1}.

Figure 2 illustrates the situation; the elbow joints have been left out of the diagram to make it smaller and more readable. The numbers on the left denote in this figure the row number; note that $w_n(k) = n+3-k$.

For convenience, let $\mathrm{wt}(D) = \sum_{(i,j)\in C(D)}(i-1)$. Note $\mathrm{wt}(D)$ is the exponent of q in the term of $\mathfrak{S}_w(1,q,\cdots,q^n)$ corresponding to the rc-graph D. The rc-graph D'' contributes exactly $\mathrm{wt}(D'')$ to $\mathrm{wt}(D)$. The cross pieces in D', however, are each shifted $k-1$ rows down, and there are $\binom{n-k}{2}$ cross pieces in D', so D' contributes $\mathrm{wt}(D') + (k-1)\binom{n-k}{2}$ to the weight of D.

Therefore, for an rc-graph D for w_n, we have

$$wt(D) = wt(D'') + (k-1)\binom{n-k}{2} + \mathrm{wt}(D') + (n+1-k)\binom{k-1}{2} + \binom{n}{2} - \binom{k}{2}.$$

Let $F_n(q)$ denote $\mathfrak{S}_{w_n}(1,q,\cdots,q^n) = \sum_{D\in\mathcal{RC}(w_n)} q^{\mathrm{wt}(D)}$.

Then we have

$$\begin{aligned}
F_n(q) &= \sum_{k=1}^{n} q^{(k-1)\binom{n-k}{2}+(n-k+1)\binom{k-1}{2}+\binom{n}{2}-\binom{k}{2}} F_{k-1}(q)F_{n-k}(q) \\
&= \sum_{k=1}^{n} q^{(k-1)\binom{n-k}{2}+(n-k-1)\binom{k-1}{2}+\binom{n}{2}-\binom{k}{2}+\binom{k-1}{3}+\binom{n-k}{3}} C_{k-1}(q)C_{n-k}(q) \\
&= \sum_{k=1}^{n} q^{\binom{n}{3}+n-k} C_{k-1}(q)C_{n-k}(q) \\
&= q^{\binom{n}{3}} \sum_{k=0}^{n-1} q^k C_{n-k-1}(q)C_k(q) \\
&= q^{\binom{n}{3}} C_n(q),
\end{aligned}$$

which proves the proposition.

□

3. A Bijection to Dyck Paths

Given proposition 1, we would like bijections between rc-graphs for w_n and other sets of objects counted by Catalan numbers. One such set is the set of partitions λ whose Young diagrams fit inside the Young diagram for the staircase partition $\delta_n = n-1, \cdots, 1$, or, equivalently, such that $\lambda_k \leq n-k$ for all k. We denote this set $\mathcal{DP}(n)$.

Let D be an rc-graph for w_n, and let $E(D)$ be the set of locations of its elbow joints. We can then associate a partition $\lambda(D)$ to D by requiring that the parts of its conjugate $\lambda'(D)$ be, as a multiset, $\{\{j-1 | (i,j) \in E(D), (i,j) \neq (0,0)\}\}$; in other words, each elbow joint in the i-th row of D should contribute a part of $i-1$ to $\lambda'(D)$.

The proof will involve generalized inverse chute moves, which are local moves first given in [1] that, given an rc-graph for some permutation w, allows one to generate new rc-graphs for w. Let D be any rc-graph for some permutation w. Suppose D has an elbow joint at (i,j), and that the following all hold for some $i' > i$ and some $j' < j$:

(1) For each k, $i < k < i'$, (k,j) is a cross piece, and (i',j) is an elbow joint.
(2) For each k, $j' < k < j$, (i,k) is a cross piece, and (i,j') is an elbow joint.
(3) For each k, $i < k \leq i'$, (k,j') is a cross piece.
(4) For each k, $j' \leq k < j$, (i',k) is a cross piece.

Then the diagram D' obtained from D by changing the elbow joint at (i,j) to a cross piece and the cross piece at (i',j') to an elbow joint is also an rc-graph for w, since in both cases the strand entering this area from the left at (i',j') exits (i,j) on the right, the strand entering from the bottom at (i',j') exits (i,j) on top, all other strands are unchanged, and the number of cross pieces remains the same.

Proposition 2. *The above described map from D to $\lambda(D)$ gives a bijection from $\mathcal{RC}(w_n)$ to $\mathcal{DP}(n)$.*

Proof. We will show that this map is surjective and then appeal to proposition 1.

Begin with the bottom rc-graph for w_n with elbow joints in the first row and cross pieces everywhere else as shown in figure 3; we denote this rc-graph D_{bot}.

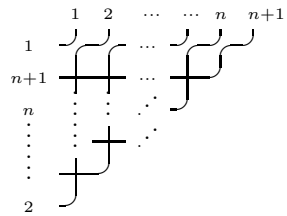

FIGURE 3. The pipe dream D_{bot}

Note that $\lambda(D_{\text{bot}}) = \emptyset$. Now we construct the rc-graph D with $\lambda(D) = \lambda$. Let λ'_i denote the i-th part of λ', with the parts in decreasing order. Now, for each i starting from 1, if $\lambda'_i = k$, take the rightmost cross piece in row $k+1$ that is not under a cross piece in the top row, and let l be the column it is in. There exists such a cross piece since, as $\lambda_j < n-j$, λ' has at most $n-j$ parts of size greater than or equal to j, and D_{bot} has $n-j$ cross pieces in row $j+1$. Now let m be the leftmost column to the right of l such that $(1,l)$ has an elbow joint. Then moving the cross piece at $(k+1,l)$ to $(1,m)$ is a generalized chute move as follows. Condition 2 follows from the definition of m, and conditions 1 and 3 follow from D_{bot} having only cross pieces in rows 2 through k, none of which have been moved. Condition 4 holds because the leftmost elbow joint in row $k+1$ to the right of column l must either be at the end of the row, or the result of an immediately preceding move from row $k+1$, and therefore in column m. Performing these corresponding generalized chute moves for all the parts of λ' in decreasing order constructs an rc-graph for w_n that goes to λ under the given map. Therefore the map is a surjection and hence by our earlier count of rc-graphs a bijection.

□

4. Edelman-Greene Correspondence

Our bijection has a second description in terms of the Edelman-Greene correspondence, a variant of the usual RSK correspondence, originally introduced in [4] and extended to the semi-standard case used here in [2]. This correspondence associates to each rc-graph a pair (P,Q) of column-strict Young tableaux of the same shape in such a way that if two rc-graphs have the same P-tableau, they must be rc-graphs for the same permutation.

The Edelman-Greene correspondence works as follows. First convert the rc-graph into a sequence $((a_1,\alpha_1),\cdots,(a_{l(w)},\alpha_{l(w)}))$ of pairs of numbers to put into the tableaux as follows. Reading each row of the rc-graph from left to right and starting with the top row, if the k-th cross piece is encountered at (i_k,j_k), let $a_k = i_k$ and $\alpha_k = i_k + j_k$. Then we insert the α_k one by one to create the P tableau using Edelman-Greene insertion; this is identical to RSK insertion except that, when inserting the letter i into a row with both an i and an $i+1$, that row remains unchanged and an $i+1$ is bumped into the next row. As in RSK insertion, after all the bumping associated with inserting a single letter α_k is completed, a_k is added to the Q tableau so that it has the same shape as P. When we are finished, P will be both row and column-strict, but Q will only be row-strict. Since column-strict tableaux are customary, we transpose both tableaux.

$$\begin{array}{ccccc} 1 & 2 & \cdots & \cdots & n \\ 2 & \ddots & \ddots & n & \\ \vdots & \ddots & n & & \\ \vdots & n & & & \\ n & & & & \end{array}$$

FIGURE 4. P tableau for $w = n \cdots 1$

$$\begin{array}{ccccc} 2 & 3 & \cdots & \cdots & n+1 \\ 3 & \ddots & \ddots & n+1 & \\ \vdots & \ddots & n+1 & & \\ \vdots & n+1 & & & \\ n+1 & & & & \end{array}$$

FIGURE 5. P tableau for w_n

Edelman and Greene showed that the P tableau for the long word $n \cdots 1$ is always the one in figure 4. As a trivial corollary, the P tableau for any rc-graph for w_n must be the one in figure 5. In the case of the long word, they also gave an inverse to their insertion procedure similar to *evacuation*, an operation on tableaux originally due to Schützenberger[10]. Take the Q-tableau, and find the box on the outer boundary with the biggest label, breaking ties by preferring the southernmost such box. Let $a_{\binom{n}{2}}$ be the label of this box, and $\alpha_{\binom{n}{2}}$ the row this box is in. Now remove the box and do jeu de taquin to fill the space, eventually leaving a "hole" in the northwest corner of the tableau. Repeat to recover $a_{\binom{n}{2}-1}$ and $\alpha_{\binom{n}{2}-1}$ and so on until the tableau is empty. To adjust this so that it works for w_n rather than the long word, we simply increase each α_k by 1.

Proposition 3. *The Edelman-Greene correspondence sends an rc-graph for w_n to a Q-tableau in which the label i occurs only in rows $i-1$ and i.*

Proof. Since Q is column-strict, the label i cannot occur in any row strictly below than row i.

Now suppose the label i occurs in row j for some $j < i-1$. Then the rightmost entry in row j must have label k for some $k \geq i$. This entry will never be moved by the jeu de taquin during the evacuation procedure, so, eventually, we will have an element $(k, j+1)$. But, since $j+1 < k$, the $j+1$-st anti-diagonal does not meet the k-th row, so we could not have started with an rc-graph for w_n in the first place. □

Now we can define a bijection from rc-graphs for w_n to partitions fitting inside δ_n by letting the partition associated to an rc-graph be the boxes whose label matches the row number in the Q-tableau corresponding to the rc-graph. This is an injection since the Edelman-Greene correspondence is injective, and this is therefore a bijection due to proposition 1.

Proposition 4. *The bijection given in Section 3 is the same as the one given by the Edelman-Greene correspondence.*

Proof. For an rc-graph with i cross pieces in the k-th row, the Edelman-Greene corrspondence produces a tableau with i occurrences of the letter k, or, equivalently, a partition whose conjugate has exactly $n-k+1-i$ parts of size $k-1$. An rc-graph for any permutation of S_{n+1} has $n+1-k$ places, and therefore $n+1-k-i$ elbow joints, in the k-th row, so the two bijections are the same. □

Since the proof of proposition 2 gives a surjection and the Edelman-Greene correspondence is known to be an injection, a purely bijective proof omitting the counting lemma is possible.

5. The Transposition Involution

Since w_n is its own inverse in S_n, transposing an rc-graph for w_n gives another rc-graph for w_n. It is a natural question to ask what this involution translates to on partitions. Unfortunately, the description of this involution on partitions is not immediately evident. However, described on bracketings of a string of length $n+1$ subject to a binary nonassociative operation, it turns out to be simply reversing the string along with the brackets. Equivalently, under the obvious bijection to binary trees, this corresponds to flipping the tree along its vertical axis.

We describe the map from rc-graphs for w_n to bracketings as follows. For convenience, let the "letters" of the string be the numbers from 1 to $n+1$. Now, let D be an rc-graph for w_n, and for each $(i,j) \in E(D)$ (the set of locations of (nontrivial) elbow joints in D), place a left bracket before the letter j and a right bracket after the letter $n+2-i$. It is clear such a map is injective and sends the transposition involution on rc-graphs to reversal of order on parenthesizations. What remains to be shown is that this actually gives a proper full bracketing for a binary associative operation. Actually, more than this is true; the pair of brackets aassociated with each elbow joint is in fact a matching pair.

We prove this by induction on the generalized inverse chute moves in the proof of proposition 2. D_{bot} corresponds to the bracketing $(1(2(\cdots(nn+1)\cdots)))$, and each elbow joint clearly corresponds to a matching pair of brackets. Now suppose there is valid generalized inverse chute move moving an elbow joint at (i,j) to (i',j'). The elbow joint at (i,j) corresponds to a matching pair of brackets with the left bracket before the letter j and the right bracket after the letter $n+2-i$. The second condition for a valid generalized inverse chute move forces the next right bracket to also occur between the letters $n+2-i$ and $n+3-i$; the first condition forces us to have another left bracket to the left of the letter j matching a right bracket after the letter $n+2-i'$, although this could be an imaginary pair of brackets around the letter j (corresponding to a trivial required elbow joint at $(j, n+2-j)$). The remaining conditions merely state that the original pair of brackets is a matching pair, which for us is true by induction. We can draw the situation as follows:

$$(1\cdots(j'((j\cdots n+2-i')\cdots n+2-i))\cdots n+1)$$

The generalized inverse chute move shifts the parentheses to the following configuration:

$$(1\cdots((j'(j\cdots n+2-i'))\cdots n+2-i)\cdots n+1)$$

Clearly, the new pair of brackets is a matching pair whenever we start with a proper full bracketing. Translated into the language of binary trees, this operation is (left) rotation, an operation used in many schemes for keeping binary search trees balanced.

6. Multiplicity on $X_{w'_n}$

A *(complete) flag* $\mathcal{F}$ in $\mathbb{C}^n$ is a sequence of subspaces $\{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n$ such that the subspace F_i has dimension i. Fixing a basis for $\mathbb{C}^n$, we can represent $\mathcal{F}$ non-uniquely by an invertible matrix $m \in GL(n)$, where the first i columns of m form a basis for F_i. Two matrices m and m' represent the same flag precisely when $m' = m \cdot b$ for some $b \in B$, the group of upper-triangular matrices; as a result the *flag variety* which parameterizes set of all flags is the quotient $GL(n)/B$. Note that $GL(n)$ and its subgroups B and B_-, the lower triangular matrices, act on $GL(n)/B$ on the left. Given our choice of basis, $GL(n)/B$ has a distinguished flag E called the *standard flag* represented by the identity matrix.

For each permutation $w \in S_n$ there is a subvariety X_w of $GL(n)/B$ known as the *Schubert variety*; X_w is the closure of the left B-orbit of the flag $F = w(E)$. F is the flag whose i-th vector space F_i is spanned by the vectors $e_w(1), \ldots, e_w(i)$, or, alternatively, F is the flag represented by w as a permutation matrix. Given two permutations v and w, $X_v \subseteq X_w$ iff $v < w$ in the Bruhat order on S_n; in particular, for e the identity permutation, $X_e \in X_w$ for every $w \in S_n$. Note that X_e consists of a single point, namely the flag E. A dense open neighborhood of E is given by Ω_e°, the orbit of E under the left action of B_-. A detailed treatment of flag and Schubert varieties can be found in, for example, [8].

The *multiplicity* of a variety X at a point p is the degree of the projective tangent cone $\mathrm{Proj}(\mathrm{gr}_{\mathfrak{m}_p} \mathcal{O}_{X,p})$, considered as a subvariety of the projective tangent space $\mathrm{Proj}(\mathrm{Sym}^* \mathfrak{m}_p/\mathfrak{m}_p^2)$. The multiplicity is one measure of "how singular" X is at p; in particular it is always 1 if X is smooth at p. General semi-continuity theorems imply that the multiplicity of X_w at X_e is at least the multiplicity of X_w at any other point.

The multiplicity of X_w at X_e can be calculated using local equations for X_w on Ω_e°. In general, these equations will be a specialization of the equations for the matrix Schubert varieties given in [7]. However, if w satisfies the condition that, for every (i, j) with $i + j > n$, either $(w_0 w)^{-1}(i) \leq j$ or $w_0 w(j) \leq i$, the local equations are exactly the equations for the matrix Schubert varieties. Therefore, in this special case, the multiplicity of X_w at X_e is exactly the degree of the matrix Schubert variety.

Let w'_n be the permutation $(n+2)23\cdots(n+1)1 \in S_{n+2}$. These permutations satisfy the condition stated above, so the multiplicity of $X_{w'_n}$ at X_e is given by the degree of the matrix Schubert variety. Knutson and Miller [9] relate rc-graphs to degenerations of matrix Schubert varieties so that, in particular, the degree of a matrix Schubert variety is given by $\mathfrak{S}_{w_0 w}(1, \ldots, 1)$.

Note that $w_0 w'_n = 1(n+1)\cdots 2(n+2)$, and Schubert polynomials are unchanged under the inclusion of S_k into $S_k + 1$ fixing the last element, as can easily be seen by adding an anti-diagonal of elbow joints to every rc-graph. Therefore, the multiplicity of $X_{w'_n}$ at X_e is given by $\mathfrak{S}_{w_n}(1, \cdots, 1) = C_n$.

7. Acknowledgements

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