

PRESENTING THE COHOMOLOGY OF A SCHUBERT VARIETY

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ABSTRACT. We extend the short presentation due to [Borel '53] of the cohomology ring of a generalized flag manifold to a relatively short presentation of the cohomology of any of its Schubert varieties. Our result is stated in a root-system uniform manner by introducing the *essential set of a Coxeter group element*, generalizing and giving a new characterization of [Fulton '92]'s definition for permutations. Further refinements are obtained in type A.

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1. INTRODUCTION

The cohomology of the generalized flag manifold G/B , for any complex semisimple algebraic group G and Borel subgroup B , has a classical presentation due to Borel [B53]. Pick a maximal torus $T \subset B \subset G$, choose a field \mathbb{k} of characteristic zero, and let $V = \mathbb{k} \otimes_{\mathbb{Z}} X(T)$, where $X(T)$ is the coweight lattice of T . Let $\mathbb{k}[V] := \text{Sym}(V^*)$ be the symmetric algebra for V^* ; it naturally carries an action of the Weyl group $W := N_G(T)/T$. Furthermore let $\mathbb{k}[V]^W$ be the ring of W -invariants of $\mathbb{k}[V]$, and $\mathbb{k}[V]_+^W$ the ideal of $\mathbb{k}[V]$ generated by the W -invariants of positive degree. A classical theorem of Borel [B53] states that

$$(1.1) \quad H^*(G/B, \mathbb{k}) \cong \mathbb{k}[V]/(\mathbb{k}[V]_+^W).$$

A desirable feature of Borel's presentation is its shortness: since W acts on V^* as a finite reflection group, by Chevalley's Theorem, the W -invariants $\mathbb{k}[V]^W$ are a polynomial algebra $\mathbb{k}[f_1, \dots, f_n]$, where $n := \dim_{\mathbb{k}} V$ is the rank of G [H90, Section 3.5]. Hence Borel's presentation shows that $H^*(G/B)$ is a complete intersection as it has n generators and n relations.

There is a second way to describe $H^*(G/B)$ arising from a cell decomposition of G/B . One has

$$\begin{aligned} G &= \bigsqcup_{w \in W} BwB && \text{(Bruhat decomposition)} \\ G/B &= \bigsqcup_{w \in W} X_w^\circ && \text{(Schubert cell decomposition), with} \\ X_w^\circ &:= BwB/B \cong \mathbb{C}^{\ell(w)} && \text{(open Schubert cell).} \end{aligned}$$

Here $\ell(w)$ denotes the Coxeter group length of w under the Coxeter system (W, S) , where S are the reflections in the simple roots Π among the positive roots Φ^+ within the natural root system Φ associated to our Lie pinning. This gives a CW-decomposition for G/B in which all cells have even real dimension. Hence the fundamental homology classes $[X_w]$ of their closures $X_w := \overline{X_w^\circ}$, the **Schubert varieties**, form a \mathbb{Z} -basis for the integral homology $H_*(G/B)$, and their (Kronecker) duals $\sigma_w := [X_w]^*$ form the dual \mathbb{Z} -basis for the integral cohomology $H^*(G/B)$.

The above facts lead to an *a priori* presentation for $H^*(X_w)$; see also [C92, Cor. 4.4] and [GR02, Prop. 2.1]: The Schubert variety X_w is the union of the Schubert cells X_u° for which $u \leq w$ in the (*strong*) Bruhat order; hence it inherits a cell decomposition from the flag variety. Consequently, the map on cohomology

$$H^*(G/B) \rightarrow H^*(X_w)$$

induced by including $X_w \hookrightarrow G/B$ is surjective with kernel

$$(1.2) \quad I_w := \mathbb{k}\text{-span of } \{\sigma_u : u \not\leq w\}.$$

This gives rise to presentations

$$(1.3) \quad \begin{aligned} H^*(X_w) &\cong H^*(G/B)/I_w && \text{working over } \mathbb{Z} \\ &\cong \mathbb{k}[V]/(I_w + \mathbb{k}[V]_+^W) && \text{working over } \mathbb{k}. \end{aligned}$$

This presentation (1.3) involves a generating set (1.2) for I_w with at most $|W|$ generators. However, this generating set for I_w is wasteful in that it not only generates I_w as an ideal but actually spans it \mathbb{k} -linearly within $H^*(G/B)$.

Therefore, a basic question is to request more efficient generating sets for I_w . For type A_{n-1} , earlier work [GR02] reduced the $|W| = n!$ upper bound on the number of generators

for I_w to a polynomial bound of n^2 for the class of Schubert varieties X_w defined by inclusions; this class includes all smooth X_w . Moreover, for a certain subclass of smooth Schubert varieties X_w considered originally by Ding [D97, D01], they gave a smaller generating set for these I_w having only n generators. This latter result was applied in [DMR07] to classify these varieties up to isomorphism and homeomorphism (the generating set (1.2) having proved too unwieldy). One motivation for this work arises from the desire to extend this classification to general Schubert varieties of type A. Experience suggests that presentations for I_w that are as simple as possible are best for this purpose.

The question of finding simple presentations of I_w for other root systems appears to have been less studied.

The main goal of this paper is to give a concise and root-system uniform extension of Borel's presentation that produces for arbitrary $w \in W$ an abbreviated list of generators for I_w . Our first main result, Theorem 1.1, achieves this via a strong restriction on the **descent set**

$$\text{Des}(\mathbf{u}) := \{s \in S : \ell(\mathbf{u}s) < \ell(\mathbf{u})\}$$

of the elements \mathbf{u} in W that index elements in our list of ideal generators $\sigma_{\mathbf{u}}$ for I_w . We will need the following definitions, the first two of which are standard:

- An element $\mathbf{u} \in W$ is **grassmannian** if $|\text{Des}(\mathbf{u})| \leq 1$.
- An element $\mathbf{v} \in W$ is **bigrassmannian** if both \mathbf{v} and \mathbf{v}^{-1} are grassmannian.
- Given $w \in W$, the **essential set for w** , denoted $\mathcal{E}(w)$, is the set of $\mathbf{u} \in W$ which are minimal in the Bruhat order among those *not* below w .

The nomenclature “essential set” for $\mathcal{E}(w)$ is justified in Proposition 4.6, where we give a new characterization of Fulton's essential set [F92] for the case of the symmetric group $W = S_n$. Indeed, $\mathcal{E}(w)$ has been previously studied from a different point of view: a result of Lascoux and Schützenberger [LS96, Théorème 3.6], implicit in our first main result below, is that the elements in $\mathcal{E}(w)$ are bigrassmannian for any w in W .¹

Theorem 1.1. *For any $w \in W$, working in a field \mathbb{k} of characteristic zero, the ideal I_w defining $H^*(X_w)$ as a quotient of $H^*(G/B)$ is generated by the cohomology classes $\sigma_{\mathbf{u}}$ where $\mathbf{u} \not\leq w$ and \mathbf{u} is grassmannian.*

More precisely, I_w is generated by the classes $\sigma_{\mathbf{u}}$ indexed by those grassmannian \mathbf{u} for which there exist some bigrassmannian \mathbf{v} in $\mathcal{E}(w)$ satisfying both $\mathbf{u} \geq \mathbf{v}$ and $\text{Des}(\mathbf{u}) = \text{Des}(\mathbf{v})$.

In type A, a similar result was obtained by Akyildiz, Lascoux, and Pragacz [ALP92, Theorem 2.2]. Specifically, they prove the first sentence of Theorem 1.1, though they do not address the strengthening given by the second sentence. Their methods are mainly geometric, as opposed to our essentially combinatorial arguments. Their work provides, to our knowledge, the first inroads towards an abbreviated generating set for I_w .

Theorem 1.1 replaces the general upper bound of $|W|$ on the number of generators needed for I_w with the bound

$$(1.4) \quad \sum_{s \in S} [W : W_{S \setminus \{s\}}],$$

¹For Lascoux and Schützenberger this arises in their work (see also Geck and Kim [GK97] and Reading [R02]) seeking efficient *encodings* of the strong Bruhat order; it is perhaps not surprising that it should arise in our search for efficient cohomology presentations as well.

where, for any subset $J \subset S$, W_J denotes the parabolic subgroup of W generated by J . Our theorem is deduced in Section 2 from a more general result (Theorem 2.4) that applies to Hiller's extension [H82, Chapter IV] of Schubert calculus as introduced by Bernstein, Gelfand, and Gelfand [BGG73] and Demazure [D73] to the *coinvariant algebras* of finite reflection groups W .

Section 2 explains and proves Theorem 1.1. In Section 3, we exploit the particular form of our generators to derive a straightforward extension to Schubert varieties in any partial flag manifold G/P associated to a parabolic subgroup P of G .

Section 4 examines more closely Theorem 1.1 in type A_{n-1} . Here one can take $G = \mathrm{GL}_n(\mathbb{C})$, with B the subgroup of invertible upper triangular matrices, T the invertible diagonal matrices, and $W = S_n$ the permutation matrices. The bound on generators for I_w in (1.4) becomes 2^n , at least a practical improvement on $|W| = n!$. More importantly, one can be even more explicit and efficient in the generating sets for I_w .

Identify points of G/B with complete flags

$$\langle 0 \rangle \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n.$$

Under this identification, each Schubert variety X_w is the set of flags satisfying certain specific conditions derived from w of the form $\dim_{\mathbb{C}}(V_r \cap \mathbb{C}^s) \geq t$. The bigrassmannians v comprising $\mathcal{E}(w)$ correspond to Fulton's *essential Schubert conditions*, a minimal list of such conditions defining the Schubert variety X_w . Our second main result (Theorem 4.8) provides for each bigrassmannian v in $W = S_n$ a generating set for the ideal

$$(1.5) \quad J_v := \mathbb{k}\text{-span of } \{\sigma_u : u \geq v\}$$

in type A that is smaller than the one used as a general step (Theorem 2.3) in the proof of Theorem 1.1 for arbitrary finite Coxeter groups W . Our proof of Theorem 4.8 is based on symmetric function identities that we devise for this purpose.

Therefore, concatenating these generating sets for J_v gives a generating set for

$$(1.6) \quad I_w = \sum_{v \in \mathcal{E}(w)} J_v$$

that is smaller than the one provided by Theorem 1.1. We remark that this result subsumes (and slightly improves upon; see Example 4.9) the generating set of size n^2 given by [GR02] in the case of Schubert varieties defined by inclusions.

Actually, we conjecture that this smaller generating set for J_v in type A is minimal (although the generating set (1.6) for I_w obtained by concatenation is not always minimal; see Example 4.12.) The significance of this minimality conjecture, as explained in Section 4.5, is that it implies an exponential *lower* bound of at least

$$\binom{n/2}{n/4} \sim \frac{\sqrt{2}^{n+2}}{\sqrt{\pi n}}$$

on the number of generators needed for I_w , accompanying our exponential upper bound of 2^n . Thus one would not be able to expect short presentations for $H^*(X_w)$ in general, at least in type A .

2. PROOF OF THEOREM 1.1

As in Section 1, let \mathbb{k} be a field of characteristic zero. For v, w in W , define two \mathbb{k} -linear subspaces J_v, I_w of the cohomology $H^*(G/B)$ with \mathbb{k} coefficients:

$$(2.1) \quad \begin{aligned} J_v &:= \mathbb{k}\text{-span of } \{\sigma_u : u \geq v\} \\ I_w &:= \mathbb{k}\text{-span of } \{\sigma_u : u \not\leq w\} \left(= \sum_{v \in \mathcal{E}(w)} J_v \right). \end{aligned}$$

Recall the essential set $\mathcal{E}(w)$ is the set of all Bruhat-minimal elements of $\{v \in W : v \not\leq w\}$.

The Schubert cell decomposition of G/B shows that both of these \mathbb{k} -subspaces are actually ideals in the cohomology ring $H^*(G/B)$: one has that I_w (respectively J_v) is the kernel of the surjection $H^*(G/B) \rightarrow H^*(X)$ induced by the inclusion of the B -stable subvariety $X \subseteq G/B$, where $X = X_w$ (respectively $X = \bigcup_{u \not\geq v} X_u$).

Also recall from Section 1 the following important property of bigrassmannians in the Bruhat order on Coxeter groups, originally due to Lascoux and Schützenberger [LS96, Théorème 3.6] and Geck and Kim [GK97, Lemma 2.3 and Theorem 2.5].

Lemma 2.1. *For any Coxeter system (W, S) and w in W , every element of $\mathcal{E}(w)$ is bigrassmannian.*

See Section 4.2 below for a further interpretation of $\mathcal{E}(w)$ when W is a Weyl group of type A_{n-1} . As a consequence of Lemma 2.1 and (2.1), finding generators of J_v for bigrassmannian v automatically gives generators for the ideals I_w .

We will actually work at the level of generality of irreducible Coxeter systems (W, S) with W finite, using Hiller's version [H82] of the Schubert calculus [BGG73, D73] for coinvariant algebras. This emphasizes that the arguments of this section and the next only depend on Coxeter combinatorics and formal properties of divided difference operators and coinvariant algebras. We review here the relevant facts from [H82, Chapter IV].

Let W be a finite and irreducible Coxeter group, and V its reflection representation. One then picks a (possibly non-crystallographic) root system Φ for W as follows: $\Phi \subset V^*$ is any W -stable choice² of a set of a linear functionals α and $-\alpha$ such that the perpendicular spaces α^\perp in V run through the reflecting hyperplanes of W . These reflecting hyperplanes divide V into chambers; we pick one which we call the dominant chamber \mathcal{C} . One declares the positive roots $\Phi^+ \subseteq \Phi$ to be the roots which are positive on this chamber. The Coxeter generators S of W are declared to be the reflections across the hyperplanes which are facets of \mathcal{C} . Among the positive roots are the simple roots $\Pi = \{\alpha_s\}_{s \in S}$, where α_s is the positive root vanishing on the reflecting hyperplane of s .

With these choices, one defines for each $w \in W$ the **Hiller Schubert polynomial**³ S_w in $\mathbb{k}[V] = \text{Sym}(V^*)$ by

$$S_w := \partial_{w^{-1}w_0} S_{w_0},$$

²Note that this may require coefficients in a subfield \mathbb{k} of \mathbb{R} strictly larger than \mathbb{Q} when W is not crystallographic.

³"BGG/Demazure Schubert polynomial" would be as fair, but for us the main point is to distinguish these polynomials from the type A Lascoux-Schützenberger version forthcoming in Section 4, which are not the same as elements of $\mathbb{k}[V]$, although equivalent mod $\mathbb{k}[V]_+^W$.

where w_0 is the unique longest element in W , for which one declares that

$$S_{w_0} := \frac{1}{|W|} \prod_{\alpha \in \Phi^+} \alpha.$$

The images of the polynomials S_w as w ranges over all of W will form a basis for the **coinvariant algebra** $\mathbb{k}[V]/(\mathbb{k}[V]_+^W)$. Here the **divided difference** operators ∂_u on $\mathbb{k}[V]$ are defined first for $s \in S$ by

$$\partial_s(f) := \frac{f - s(f)}{\alpha_s}$$

and then for any $u \in W$ of Coxeter length $\ell = \ell(u)$ by

$$\partial_u := \partial_{s_{i_1}} \cdots \partial_{s_{i_\ell}},$$

where $u = s_{i_1} \cdots s_{i_\ell}$ is any choice of a (reduced) decomposition expressing u in terms of the generators $s \in S$.

The relation to a generalized flag manifold G/B is that G , B , and T come equipped with a (crystallographic) root system and Weyl group W . Let $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ be the Lie algebras associated to $T \subset B \subset G$. Taking the reflection representation V to be $V = \mathfrak{h}$ (or more generally $V = \mathbb{k} \otimes_{\mathbb{Z}} X(T)$) and the root system Φ to be the set of weights of the adjoint representation of \mathfrak{g} acting on itself, with Φ^+ the weights of the action of \mathfrak{g} on \mathfrak{b} . In this case, it was proven in [BGG73, D73] that the element $S_w \in \mathbb{k}[\mathfrak{h}]$ is a lift under the surjection

$$\mathbb{k}[\mathfrak{h}] \rightarrow \mathbb{k}[\mathfrak{h}]/(\mathbb{k}[\mathfrak{h}]_+^W) \quad (\cong H^*(G/B))$$

of the Schubert cohomology class σ_w in $H^{\ell(w)}(G/B)$ which is *Kronecker dual* to the fundamental homology class $[X_w]$ of X_w in $H_{\ell(w)}(G/B)$ and *Poincaré dual* to the fundamental homology class $[X_{w_0 w}]$ in $H_{\ell(w_0) - \ell(w)}(G/B)$.

We collect basic properties of the divided difference operators ∂_s and Schubert polynomials S_w that we need below.

- (a) **Leibniz rule:** $\partial_s(fg) = \partial_s(f) \cdot g + s(f)\partial_s(g)$.
- (b) $\partial_s(f) = 0$ if and only if $s(f) = f$. In the case where $f = S_w$ for some w , then $\partial_s(S_w) = 0$ if and only if $\ell(ws) > \ell(w)$, or equivalently if and only if $s \notin \text{Des}(w)$.
- (c) Consequently, ∂_s preserves the ideal $I = \mathbb{k}[V]_+^W$ generated by the W -invariants of positive degree in $\mathbb{k}[V]$.
- (d)

$$\partial_u \partial_v = \begin{cases} \partial_{uv} & \text{if } \ell(u) + \ell(v) = \ell(uv) \\ 0 & \text{otherwise.} \end{cases}$$

- (e) If $\ell(w) = \ell(w')$ then $\partial_{w'} S_w = \delta_{w, w'}$. (By definition, $\delta_{a,b} = 1$ if $a = b$ and $\delta_{a,b} = 0$ otherwise.)
- (f) Consequently, the **Schubert structure constants** $c_{u,v}^w$ which uniquely express

$$S_u S_v = \sum_{\substack{w \in W: \\ \ell(w) = \ell(u) + \ell(v)}} c_{u,v}^w S_w \quad \text{mod } \mathbb{k}[V]_+^W$$

can be computed by the formula $c_{u,v}^w = \partial_w(S_u S_v)$.

- (g) These structure constants also satisfy this sparsity rule:

$$c_{u,v}^w = 0 \text{ unless } w \geq u, v \text{ in Bruhat order.}$$

This is a consequence of the Pieri formula [H82, §IV.3] for multiplying the S_u by any of the degree one elements that generate $\mathbb{k}[V]/(\mathbb{k}[V]_+^W)$.

The key to the proof of Theorem 1.1 turns out to be the following lemma (perhaps of independent interest) about some further sparsity of the Schubert structure constants $c_{u,v}^w$ appearing in property (f) above. Given $J \subset S$, recall that every w in W has a unique length-additive **parabolic factorization**

$$w = u \cdot x$$

where x lies in the parabolic subgroup W_J generated by J , and u lies in the set W^J of minimum-length coset representatives for W/W_J , characterized by the property that $\text{Des}(u) \cap J = \emptyset$.

Lemma 2.2. *Let $J \subset S$. Suppose w, w' in W lie in the same coset $wW_J = w'W_J$, so that $w = u \cdot x$ and $w' = u \cdot x'$ for some u in W^J and x, x' in W_J . Then $c_{u,x}^{w'} = \delta_{w',w} = \delta_{x',x}$.*

Proof. Using property (f) of divided differences above, one can rephrase the lemma as saying that, for any $w' \in W$ with $w' = u \cdot x'$ where $x' \in W_J$ and $\ell(x') = \ell(x)$ (so that $\ell(w') = \ell(w)$), one has

$$(2.2) \quad \partial_{w'}(S_u S_x) = \delta_{x,x'}.$$

We prove (2.2) by induction on $\ell(x')$. In the base case where $\ell(x') = 0$, one has $x' = x = 1$ and $w' = w = u$, so the assertion (2.2) follows from property (e) above.

In the inductive step, let $\ell(x') > 0$. Thus, there exists $s \in J$ with $\ell(x's) < \ell(x')$. Consequently,

$$\ell(w') = \ell(u) + \ell(x') = \ell(u) + \ell(x's) + \ell(s).$$

Hence by properties (d), (a) and (b), one has, respectively,

$$(2.3) \quad \begin{aligned} \partial_{w'}(S_u S_x) &= \partial_u \partial_{x's} \partial_s(S_u S_x) \\ &= \partial_u \partial_{x's} (\partial_s(S_u) \cdot S_x + s(S_u) \partial_s(S_x)) \\ &= \partial_u \partial_{x's} (S_u \partial_s(S_x)). \end{aligned}$$

Now consider two cases.

Case 1. $\ell(xs) > \ell(x)$. Then property (b) says that $\partial_s(S_x) = 0$. Using this in the last line of (2.3), one concludes that

$$\partial_{w'}(S_u S_x) = 0 = \delta_{x,x'}$$

since $\ell(x's) < \ell(x')$ implies $x \neq x'$.

Case 2. $\ell(xs) < \ell(x)$. Then $\partial_s(S_x) = S_{xs}$, and

$$\begin{aligned} \partial_{w'}(S_u S_x) &= \partial_u \partial_{x's} (S_u \partial_s(S_x)) \\ &= \partial_u \partial_{x's} (S_u S_{xs}) \\ &= \delta_{xs,x's} \\ &= \delta_{x,x'}, \end{aligned}$$

where the second-to-last equality applied the inductive hypothesis to $x's$ and xs . \square

We now use this lemma to find a smaller generating set for the ideal J_v as defined in (2.1) based on the descent set $\text{Des}(v)$. Working more generally, for any finite Coxeter group W and a choice of root data for the Hiller Schubert calculus, define the following two \mathbb{k} -subspaces within the coinvariant algebra $\mathbb{k}[V]/(\mathbb{k}[V]_+^W)$:

$$(2.4) \quad \begin{aligned} J_v &:= \mathbb{k}\text{-span of } \{S_u : u \geq v\} \\ I_w &:= \mathbb{k}\text{-span of } \{S_u : u \not\leq w\} = \sum_{v \in \mathcal{E}(w)} J_v. \end{aligned}$$

Note that in this context we can appeal to property (g) to see that J_v and I_w are actually *ideals* within the coinvariant algebra $\mathbb{k}[V]/(\mathbb{k}[V]_+^W)$.

Theorem 2.3. *Let v be an element of a finite Coxeter group W and J a subset of S . Assume v lies in W^J , or, equivalently, $\text{Des}(v) \cap J = \emptyset$.*

Then J_v is generated as an ideal within the coinvariant algebra $\mathbb{k}[V]/(\mathbb{k}[V]_+^W)$ by the set

$$\{S_u : u \in W^J, u \geq v\}.$$

Proof. For any $t \in W^J$, let J'_t denote the ideal

$$\{S_u : u \in W^J, u \geq t\}.$$

We need to show $J'_v = J_v$. Certainly $J'_v \subseteq J_v$ by definition, so it remains to show the reverse inclusion.

Proceed by induction on the **colength** $\ell(w_0) - \ell(v)$ of v . In the base case where v has colength 0, $v = w_0$; therefore the assumption $v \in W^J$ implies $J = \emptyset$, so $W^J = W$, and there is nothing to prove.

In the inductive step, given $w \geq v$, one must show that S_w lies in J'_v . Factor $w = u \cdot x$ uniquely with $u \in W^J$ and $x \in W_J$. We will use repeatedly the fact (see [BB05, §2.5]) that the map

$$\begin{aligned} W &\xrightarrow{p^J} W^J \\ w &\longmapsto u \end{aligned}$$

is order-preserving for the Bruhat order. In particular, since it was assumed that $w \geq v$ above, one has $u \geq v$ here.

By Lemma 2.2, one has

$$\begin{aligned} S_u S_x &= S_w + \sum_{w'} c_{u,x}^{w'} S_{w'}, \quad \text{so that} \\ S_w &= S_u S_x - \sum_{w'} c_{u,x}^{w'} S_{w'}. \end{aligned}$$

Here each w' appearing in the sums satisfies $w' \geq u$ by property (g), and hence if one factors $w' = u' \cdot x'$ with $u' \in W^J$ and $x' \in W_J$, then $w' \geq u$ implies $u' \geq u$. But then Lemma 2.2 also says that $c_{u,x}^{w'} = 0$ unless $u' > u$ ($\geq v$); hence, by induction, for any w' with $S_{w'}$ appearing with nonzero coefficient in the right hand sum, $J_{u'} = J'_u$ holds for the corresponding u' in the factorization $w' = u' \cdot x'$. By definition, $J_{u'} \subset J'_v$, and since S_u also lies in J'_v as $u \in W^J$, one concludes that S_w lies in J'_v as desired. \square

Theorem 1.1 is then a special case of the following result, which is immediate from Lemma 2.1 and Theorem 2.3:

Theorem 2.4. *Let W be a Coxeter system (W, S) with W finite, and let w be an element of W . Then the ideal I_w of the coinvariant algebra $\mathbb{k}[V]/(\mathbb{k}[V]_+^W)$, defined as in (2.4), is generated by the Schubert polynomials*

$$\{S_u : u \not\leq w, \text{ and } u \text{ is grassmannian}\}.$$

More precisely, I_w is generated by the set $\{S_u\}$ for those u for which there exist some (bigrassmannian) v in $\mathcal{E}(w)$ satisfying both $u \geq v$ and $\text{Des}(u) = \text{Des}(v)$.

3. REDUCING PRESENTATIONS IN $H^*(G/P)$ TO THOSE IN $H^*(G/B)$

We explain in this section how Theorem 2.4 leads to a shorter presentation more generally for the cohomology of Schubert varieties in any partial flag manifold.

Given $J \subset S$, one has the parabolic subgroup W_J of W generated by J . One also has a corresponding parabolic subgroup P_J of G , generated by the Borel subgroup B together with representatives within G that lift the elements $J \subset W = N_G(T)/T$. The Borel picture identifies the cohomology ring of the **(generalized) partial flag manifold** G/P_J as the subring of W_J -invariants inside the cohomology of G/B . In other words, the quotient map

$$G/B \xrightarrow{\pi} G/P_J$$

induces the inclusion

$$(3.1) \quad H^*(G/P_J) \cong H^*(G/B)^{W_J} \xhookrightarrow{i} H^*(G/B).$$

Recall that W^J denotes the set of minimum-length coset representatives for W/W_J . Inside $H^*(G/B)$, the cohomology classes $\{\sigma_w : w \in W^J\}$ lie in this W_J -invariant subalgebra $H^*(G/B)^{W_J}$ and form a \mathbb{k} -basis identified with the \mathbb{k} -basis of Schubert cohomology classes σ_{wW_J} for $H^*(G/P_J)$. One also has that the pre-image of Schubert varieties in G/P_J are certain Schubert varieties of G/B : specifically,

$$\pi^{-1}(X_{wW_J}) = X_{w_{\max}},$$

where w_{\max} is the unique *maximum*-length coset representative in wW_J .

Working more generally with the Hiller Schubert calculus for any finite Coxeter system (W, S) , the inclusion (3.1) generalizes to the inclusion

$$(3.2) \quad \mathbb{k}[V]^{W_J}/\mathbb{k}[V]_+^W \mathbb{k}[V]^{W_J} \cong (\mathbb{k}[V]/(\mathbb{k}[V]_+^W))^{W_J} \xhookrightarrow{i} \mathbb{k}[V]/(\mathbb{k}[V]_+^W).$$

The first isomorphism shown in (3.2) is a consequence of the fact that one has an *averaging* map

$$(3.3) \quad \begin{array}{ccc} \mathbb{k}[V] & \xrightarrow{\rho} & \mathbb{k}[V]^{W_J} \\ f & \longmapsto & \frac{1}{|W_J|} \sum_{w \in W_J} w(f) \end{array}$$

which provides a $\mathbb{k}[V]^{W_J}$ -linear (and hence also $\mathbb{k}[V]^W$ -linear) retraction map for i , meaning that $\rho \circ i = 1_{\mathbb{k}[V]^{W_J}}$. Inside $\mathbb{k}[V]/(\mathbb{k}[V]_+^W)$, the Hiller Schubert polynomials $\{S_w : w \in W^J\}$ lie in this W_J -invariant subalgebra $(\mathbb{k}[V]/(\mathbb{k}[V]_+^W))^{W_J}$ and provide a \mathbb{k} -basis for it.

The retraction in (3.3) also provides the relation between the cohomology presentations for the Schubert varieties X_{wW_J} and $X_{w_{\max}}$. Recall that when one has an inclusion of rings $R \xhookrightarrow{i} \hat{R}$, one can relate ideals of R and \hat{R} by the operations of *extension* and *contraction*: given an ideal I in R , its extension $\hat{R}I$ to \hat{R} is the ideal it generates in \hat{R} , and given an ideal

\hat{I} of \hat{R} , its contraction to R is the intersection $\hat{I} \cap R$. Say that the inclusion $R \xrightarrow{i} \hat{R}$ is a *split inclusion* if it has an R -linear *retraction* $\hat{R} \xrightarrow{\rho} R$, meaning that $\rho \circ i = 1_R$. The following proposition about this situation is straightforward and well-known.

Proposition 3.1. *Assume $R \xrightarrow{i} \hat{R}$ is a split inclusion, and \hat{I} is an ideal of \hat{R} which is generated by its contraction $I := \hat{I} \cap R$ to R .*

Then a set of elements $\{g_\alpha\}$ lying in R generate \hat{I} as an ideal of \hat{R} if and only if the same elements $\{g_\alpha\}$ generate $I = \hat{I} \cap R$ as an ideal of R .

We will apply this proposition to the split inclusion (3.2) and these ideals

$$(3.4) \quad \begin{aligned} I = I_{wW_J} &:= \mathbb{k}\text{-span of } \{S_u : u \in W^J, uW_J \not\subseteq wW_J\} \subset (\mathbb{k}[V]/(\mathbb{k}[V]_+^W))^{W_J} = R \\ \hat{I} = I_{w_{\max}} &:= \mathbb{k}\text{-span of } \{S_u : u \in W, u \not\subseteq w_{\max}\} \subset \mathbb{k}[V]/(\mathbb{k}[V]_+^W) = \hat{R} \end{aligned}$$

which in the case where (W, S) comes from an algebraic group G have the following interpretations as kernels:

$$(3.5) \quad \begin{aligned} I_{wW_J} &= \ker \left(H^*(G/P_J) \xrightarrow{i^*} H^*(X_{wW_J}) \right) \\ I_{w_{\max}} &= \ker \left(H^*(G/B) \xrightarrow{i^*} H^*(X_{w_{\max}}) \right). \end{aligned}$$

Borel's picture already gives a very short presentation for $H^*(G/P_J)$ or more generally $(\mathbb{k}[V]/(\mathbb{k}[V]_+^W))^{W_J}$, as we now explain. The isomorphism in (3.2) says that a presentation of $(\mathbb{k}[V]/(\mathbb{k}[V]_+^W))^{W_J}$ is equivalent to a presentation of the quotient $\mathbb{k}[V]^{W_J}/\mathbb{k}[V]_+^W \mathbb{k}[V]^{W_J}$. Since both W_J and W are finite reflection groups acting on V , their invariant rings are both polynomial algebras

$$\begin{aligned} \mathbb{k}[V]^{W_J} &= \mathbb{k}[g_1, \dots, g_n] \\ \mathbb{k}[V]^W &= \mathbb{k}[f_1, \dots, f_n], \end{aligned}$$

and hence the quotient can be presented as a graded complete intersection ring:

$$\begin{aligned} (\mathbb{k}[V]/(\mathbb{k}[V]_+^W))^{W_J} &\cong \mathbb{k}[V]^{W_J}/\mathbb{k}[V]_+^W \mathbb{k}[V]^{W_J} \\ &\cong \mathbb{k}[g_1, \dots, g_n]/(f_1, \dots, f_n). \end{aligned}$$

Thus we only need to provide generators for the ideal I_{wW_J} .

Theorem 3.2. *Let (W, S) be a finite Coxeter system, with $J \subseteq S$ and w in W , and w_{\max} the unique maximum-length representative of wW_J . Consider the inclusion*

$$R := (\mathbb{k}[V]/(\mathbb{k}[V]_+^W))^{W_J} \xrightarrow{i} \hat{R} := \mathbb{k}[V]/(\mathbb{k}[V]_+^W).$$

Then the following hold.

- (i) *The essential set $\mathcal{E}(w_{\max})$ lies entirely in W^J .*
- (ii) *The ideal $I_{w_{\max}}$ of \hat{R} is generated by its contraction $I_{w_{\max}} \cap R$.*
- (iii) *This contraction is the same as I_{wW_J} .*
- (iv) *The set*

$$\bigcup_{v \in \mathcal{E}(w)} \{S_u : u \geq v, \text{Des}(u) = \text{Des}(v)\}$$

both generates $I_{w_{\max}}$ as an ideal of \hat{R} and also generates the contraction I_{wW_J} as an ideal of R .

Proof. For assertion (i), assume for the sake of contradiction that $v \in \mathcal{E}(w_{\max})$ and that $vs < v$ for some s in J . Since v is Bruhat-minimal among the elements *not* below w_{\max} , this implies $vs \leq w_{\max}$. However $w_{\max}s < w_{\max}$ by maximality of w_{\max} within wW_J , so the *lifting property* [BB05, Prop. 2.2.7, Cor. 2.2.8] of Bruhat order implies that $v \leq w_{\max}$, a contradiction.

For assertion (ii), apply Theorem 2.4 to w_{\max} to conclude that I_w is generated by the set $\{S_u\}$ for those u for which there exist $v \in \mathcal{E}(w_{\max})$ satisfying both $u \geq v$ and $\text{Des}(u) \subseteq \text{Des}(v)$. By (i), this forces u to lie in W^J , so that S_u is W^J -invariant and therefore lies in R .

For assertion (iii), from the definition (3.4) and the fact that R has a \mathbb{k} -basis given by $\{S_u : u \in W^J\}$, it suffices to show that for any u in W^J , one has $uW_J \leq wW_J$ if and only if $u \leq w_{\max}$. By definition, $uW_J \leq wW_J$ if and only if $u \leq w_{\min}$, where w_{\min} is the unique representative of wW_J lying in W^J . The usual parabolic factorization $W = W^J \cdot W_J$ allows one to write down a reduced word ω for w_{\max} in the concatenated form

$$\omega = \omega_1 \cdot \omega_2$$

where the prefix ω_1 factors w_{\min} , and the suffix ω_2 contains only generators in J . The *subword characterization* of Bruhat order [BB05, Cor. 2.2.3] shows that $u \leq w_{\max}$ if and only if u is factored by a reduced subword of this word ω ; this subword must necessarily use no generators from J (since u is in W^J) and hence must actually be a subword of ω_1 . Thus $u \leq w_{\max}$ if and only if $u \leq w_{\min}$, as desired.

Assertion (iv) then follows from assertions (ii), (iii) and Proposition 3.1. \square

4. REFINEMENTS IN TYPE A

We investigate further the situation when W is a Weyl group of type A_{n-1} , which exhibits extra features; one can:

- be more explicit about bigrassmannians and their essential sets $\mathcal{E}(w)$,
- produce even smaller generating sets for the ideals I_w and J_w , which are conjecturally minimal in the case of J_w , and
- work with \mathbb{Z} coefficients rather than over a field \mathbb{k} of characteristic zero.

4.1. Schubert conditions and bigrassmannians. In type A_{n-1} , points in the variety G/B are identified with complete flags of subspaces

$$\langle 0 \rangle \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n$$

having $\dim_{\mathbb{C}} V_i = i$. Pick as our particular base flag $\{\mathbb{C}^i\}_{i=0}^n$, where \mathbb{C}^i is spanned by the first i standard basis elements; this flag is fixed by the Borel subgroup B consisting of the invertible upper-triangular matrices within $G := GL_n(\mathbb{C})$. Picking the maximal torus T of invertible diagonal matrices, one identifies the Weyl group $W = N_G(T)/T$ with the symmetric group S_n . The Coxeter generators S for $W = S_n$ associated to our Borel subgroup B is the set of adjacent transpositions $S = \{(1 \leftrightarrow 2), (2 \leftrightarrow 3), \dots, (n-1 \leftrightarrow n)\}$.

The Schubert variety X_w corresponding to a permutation $w = w_1 w_2 \cdots w_n \in S_n$ (written in one-line notation) can be defined as the subvariety of flags satisfying the conjunction of the **Schubert conditions**

$$(4.1) \quad \dim_{\mathbb{C}}(V_r \cap \mathbb{C}^s) \geq t$$

where

$$(4.2) \quad t = t_{r,s}(w) := |\{w_1, w_2, \dots, w_r\} \cap \{1, 2, \dots, s\}|$$

for all $r, s = 1, 2, \dots, n-1$ is the **rank function** associated to w . Denote the condition (4.1) by $C_{r,s,t}$ (for arbitrary t , not necessarily of the form (4.2)). Note that $C_{r,s,t}$ is vacuous unless $t > r + s - n$.

The following explicit identification of bigrassmannian permutations is well-known and straightforward.

Lemma 4.1. *The bigrassmannian permutations (other than the identity) are parameterized by r, s, t with $1 \leq t \leq r, s \leq n$ and $t > r + s - n$. Let $v_{r,s,t,n}$ denote the unique bigrassmannian permutation $v_1 \dots v_n \in S_n$ such that*

- $\text{Des}(v) = \{(r \leftrightarrow r+1)\}$
- $\text{Des}(v^{-1}) = \{(s \leftrightarrow s+1)\}$, and
- $v_t = s+1$.

Then explicitly we have:

$$(4.3) \quad v_{r,s,t,n} := (1, 2, \dots, t-1, \\ s+1, s+2, \dots, s+r-t+1, \\ t, t+1, t+2, \dots, s, \\ s+r-t+2, s+r-t+3, \dots, n).$$

There is a simple relation between these Schubert conditions $C_{r,s,t}$ and the bigrassmannian permutations in $W = S_n$:

Proposition 4.2. *Let $w \in S_n$. Then the Schubert condition $C_{r,s,t}$ is satisfied by all flags in X_w if and only if $v_{r,s,t,n} \not\leq w$.*

Proof. Note $C_{r,s,t}$ is a Schubert condition on X_w if and only if $t_{r,s}(w) \geq t$. It is then straightforward to check that the latter is equivalent to $v_{r,s,t,n} \not\leq w$ using the *tableau criterion* [BB05, Theorem 2.6.3] for comparing elements in the Bruhat ordering. \square

Note that imposing an arbitrary conjunction of Schubert conditions on complete flags cuts out a B -stable subvariety of G/B , but this subvariety may be a *reducible* union of Schubert varieties rather than a single Schubert variety X_w . However, in type A , when one imposes a *single* Schubert condition, the result is always a (single) Schubert variety. This fact can be traced to special properties of the Bruhat order in type A , first identified by Lascoux and Schützenberger [LS96], and exploited further by Geck and Kim [GK97] and Reading [R02]. To explain this, we first recall some terminology.

Definition 4.3. In a poset P , say that an element v is a *dissector* of P if there exists a (necessarily unique) element w in P for which P decomposes as the disjoint union of the principal order filter above v and the principal order ideal below w :

$$P = \{u \in P : u \geq v\} \sqcup \{u \in P : u \leq w\}.$$

Say that an element a in a poset P (which need not be a lattice) is *join-irreducible* if there does not exist a subset $X \subset P$ with $a \notin X$ such that a is the least element among all upper bounds for X in P .

There are two subtle issues to point out in this definition of join-irreducibles. Firstly, when the finite poset is a *lattice*, an element is join-irreducible if and only if it covers a unique element. However, for non-lattices, one can have join-irreducibles that cover more than one element. For example, the strong Bruhat order in type A_2 has four non-minimal, non-maximal elements, each of which is join-irreducible, but two of them cover two elements. Secondly, all of the posets that we will consider have a unique least element, (e.g. in Bruhat order on W , the least element is the identity of W), and this least element is *not* considered join-irreducible because it is the least element among all upper bounds for the empty set $X = \emptyset$.

Theorem 4.4. [LS96, GK97, R02]

- (i) *In any finite poset, every dissector is join-irreducible. When the poset is the Bruhat order for a Coxeter system (W, S) of type A, B, H_3, H_4 or $I_2(m)$, the converse holds: the join-irreducible elements are exactly the dissectors.*
- (ii) *In the Bruhat order for any finite Coxeter system (W, S) , every join-irreducible element is bigrassmannian. In type A , the converse holds: the (non-identity) bigrassmannian elements are exactly the join-irreducibles.*

In particular, in type A , for every bigrassmannian v in W , there exists a (necessarily unique) element w in W for which $\mathcal{E}(w) = \{v\}$.

Corollary 4.5. *In type A_{n-1} , a single Schubert condition $C_{r,s,t}$ on the flags in G/B cuts out the Schubert variety $X_{w_{r,s,t,n}}$ in G/B , where $w_{r,s,t,n}$ is the unique element with $\mathcal{E}(w_{r,s,t,n}) = \{v_{r,s,t,n}\}$ as in Theorem 4.4.*

Thus in type A_{n-1} , for any bigrassmannian $v_{r,s,t,n}$, one has equality of the two ideals $J_{v_{r,s,t,n}} = I_{w_{r,s,t,n}}$ within the coinvariant algebra $\mathbb{k}[V]/(\mathbb{k}[V]_+^W)$.

We remark that, as with $v_{r,s,t,n}$, one knows $w_{r,s,t,n}$ explicitly (see [R02, §8]):

$$w_{r,s,t,n} = (n, n-1, \dots, (n-r+t+1), \\ s, s-1, \dots, s-t+1, \\ n-r+t, n-r+t-3, \dots, s+1, \\ s-t, s-t-1, \dots, 1).$$

4.2. Bigrassmannians and essential Schubert conditions. Next, we explain the relation between what we have called the essential set $\mathcal{E}(w)$ for w and Fulton's essential set of Schubert conditions for X_w .

Note that there are implications among the various Schubert conditions $C_{r,s,t}$. Fulton introduced the **essential set** of a permutation, a set of coordinates $\{(r_i, s_i)\} \subset n \times n$ which give an inclusion-minimal subset of Schubert conditions $C_{r_i, s_i, t}$ with $t = t_{r_i, s_i}(w)$ that suffice to define X_w as a subset of the flag manifold. (See further remarks in Example 4.7 below.) Correspondingly, we call these Schubert conditions the **essential Schubert conditions** for X_w ; see [F92, §3], [FP98, pp. 20-21], and [EL96, §2].

Proposition 4.6. *The Schubert condition $C_{r,s,t}$ implies the Schubert condition $C_{r',s',t'}$ if and only if $v_{r,s,t,n} \leq v_{r',s',t',n}$ in Bruhat order.*

Therefore in type A_{n-1} , Fulton's essential set of Schubert conditions $C_{r,s,t}$ for X_w correspond bijectively to the elements of the essential set $\mathcal{E}(w)$ for w defined for a general Coxeter group.

Proof. For the first assertion, note that the Schubert cell decomposition for X_w and Theorem 4.4 give the following:

$$\begin{aligned}
C_{r,s,t} \text{ implies } C_{r',s',t'} &\Leftrightarrow X_{w_{r,s,t,n}} \subseteq X_{w_{r',s',t',n}} \\
&\Leftrightarrow \{u \in W : u \leq w_{r,s,t,n}\} \subseteq \{u \in W : u \leq w_{r',s',t',n}\} \\
&\Leftrightarrow \{u \in W : u \geq v_{r,s,t,n}\} \supseteq \{u \in W : u \geq v_{r',s',t',n}\} \\
&\Leftrightarrow v_{r,s,t,n} \leq v_{r',s',t',n}.
\end{aligned}$$

The second assertion follows immediately from the first. \square

Example 4.7. In order to be explicit about the bijection asserted in Proposition 4.6, it will be convenient for us to use a slight adaptation of Fulton’s essential set. This bijection can be inferred from the discussion in [FP98] and [GR02] (our conventions are in line with those the latter text), and one also can thereby also give an explicit bijection between our essential set and Fulton’s essential set as originally defined.

Given $w = w_1 w_2 \cdots w_n \in S_n$, draw an $n \times n$ matrix of “bubbles” \circ . Replace the bubbles in positions (i, w_i) with an \times and erase all bubbles in the “hooks” weakly below and (nonstandardly) to the *left* of each \times . The **diagram** of w , denoted $\mathcal{D}(w)$, consists of all bubbles left, which are those not in any hook. Under this convention $|\mathcal{D}(w)| = \ell(w_0 w)$. This reflects the fact that our diagram is the left \leftrightarrow right mirror image of the *standard* diagram of $w_0 w$; see [FP98, p.11]. Fulton’s essential set is then defined as the subset of $\mathcal{D}(w)$ such that neither $(i + 1, j)$ nor $(i, j - 1)$ is in $\mathcal{D}(w)$. Let us denote Fulton’s essential set by $\mathcal{E}_{\text{Fulton}}(w)$. The desired bijection between bubbles in $\mathcal{E}_{\text{Fulton}}(w)$ and $\mathcal{E}(w)$ sends the essential bubble with (row,column) indices $(r, s + 1)$ to the bigrassmannian $v_{r,s,t,n}$ where t is the number of bubbles weakly above the essential one in the same column.

For example, let $w = 425163$. The figure below shows the positions (i, w_i) marked with an \times , and the bubbles in the diagram $\mathcal{D}(w)$ shown as \bullet or \circ depending upon whether or not they lie in $\mathcal{E}_{\text{Fulton}}(w)$:

| | | | | | | |
|---|----------|----------|-----------|----------|-----------|-----------|
| | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | | | | \times | \circ | \circ |
| 2 | | \times | \bullet | | \bullet | \circ |
| 3 | | | | | \times | \circ |
| 4 | \times | | \bullet | | | \bullet |
| 5 | | | | | | \times |
| 6 | | | \times | | | |

The following table then summarizes the bijection between the bubbles lying in Fulton’s essential set $\mathcal{E}_{\text{Fulton}}(w)$, the essential Schubert conditions defining X_w , and the bigrassmannians that comprise $\mathcal{E}(w)$.

| $(r, s + 1) = (\text{row}, \text{column})$ for bubble in $\mathcal{E}_{\text{Fulton}}(w)$ | Schubert condition $C_{r,s,t}$: $\dim V_r \cap \mathbb{C}^s \geq t$ | bigrassmannian $v = v_{r,s,t,n}$ in $\mathcal{E}(w)$ |
|--|---|---|
| (2, 3) | $\dim V_2 \cap \mathbb{C}^2 \geq 1$ | 341256 |
| (2, 5) | $\dim V_2 \cap \mathbb{C}^4 \geq 2$ (i.e. $V_2 \subset \mathbb{C}^4$) | 152346 |
| (4, 3) | $\dim V_4 \cap \mathbb{C}^2 \geq 2$ (i.e. $\mathbb{C}^2 \subset V_4$) | 134526 |
| (4, 6) | $\dim V_4 \cap \mathbb{C}^5 \geq 4$ (i.e. $V_4 \subset \mathbb{C}^5$) | 123645 |

4.3. Grassmannians and symmetric functions. In looking for generators for the ideals $J_{v_{r,s,t,n}}$, we wish to take advantage of symmetric function identities, so we briefly review here the relation between symmetric functions and the Schubert calculus in type A. We also point out how the calculations may be performed over \mathbb{Z} rather than a coefficient field \mathbb{k} of characteristic zero.

Let $J := S \setminus \{s_r\}$ where $s_r = (r \leftrightarrow r + 1)$, so that $W_J = S_r \times S_{n-r}$, and G/P_J is the *Grassmannian* of r -planes in \mathbb{C}^n . The cohomology inclusion (3.1) or (3.2) remains valid working with coefficients in \mathbb{Z} and becomes

$$\begin{aligned} H^*(G/P_J) &\cong H(G/B)^{W_J} && \hookrightarrow H(G/B) \\ \mathbb{Z}[\mathbf{x}]^{W_J} / \mathbb{Z}[\mathbf{x}]_+^{W_J} &\cong (\mathbb{Z}[\mathbf{x}] / (\mathbb{Z}[\mathbf{x}]_+^{W_J}))^{W_J} && \hookrightarrow \mathbb{Z}[\mathbf{x}] / (\mathbb{Z}[\mathbf{x}]_+^{W_J}). \end{aligned}$$

Here $\mathbb{Z}[\mathbf{x}] := \mathbb{Z}[x_1, \dots, x_n]$ is viewed as the symmetric algebra $\mathbb{Z}[V]$, where V is no longer the irreducible reflection representation of dimension $n - 1$ for $W = S_n$ but rather the natural permutation representation of dimension n . In order to work over \mathbb{Z} , one can replace the retraction in (3.3) with the **Demazure operator**

$$\mathbb{Z}[\mathbf{x}] \xrightarrow{\pi_{w_0(J)}} \mathbb{Z}[\mathbf{x}]^{W_J}$$

associated to the longest element $w_0(J)$ in W_J , where

$$\pi_{s_i}(f) := \frac{x_i f - x_{i+1} s_i(f)}{x_i - x_{i+1}}$$

and $\pi_w := \pi_{s_{i_1}} \cdots \pi_{s_{i_\ell}}$ if $w = s_{i_1} \cdots s_{i_\ell}$ is any reduced decomposition for w .

In type A_{n-1} , one can replace the Hiller Schubert polynomial S_w with **Lascoux and Schützenberger's Schubert polynomial** \mathfrak{S}_w (see for example [M91, M01]): one chooses the root linear functionals to be $x_i - x_j$ for $1 \leq i \neq j \leq n$ and replaces the previous choice of $S_{w_0} = \prod_{i < j} (x_i - x_j)$ with an element which is equivalent modulo the ideal $(\mathbb{Z}[\mathbf{x}]_+^W)$, namely

$$\mathfrak{S}_{w_0} := \mathbf{x}^{\delta_n} := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1 x_n^0.$$

Defining $\mathfrak{S}_w := \partial_{w^{-1}w_0} \mathfrak{S}_{w_0}$, property (c) from Section 2 tells us that the images of S_w and \mathfrak{S}_w within $\mathbb{Z}[\mathbf{x}] / (\mathbb{Z}[\mathbf{x}]_+^W)$ are the same for all $w \in W$. These \mathfrak{S}_w :

- lie in $\mathbb{Z}[\mathbf{x}]$ and have nonnegative integer coefficients,

- lift the cohomology classes σ_w in the cohomology with *integer* coefficients

$$H^*(G/B, \mathbb{Z}) \cong \mathbb{Z}[\mathbf{x}] / (\mathbb{Z}[\mathbf{x}]_+^W),$$

and

- give us Schur functions in finite variable sets whenever w is grassmannian: if one has $\text{Des}(w) \subseteq \{(r, r+1)\}$, (in which case we say u is **r-grassmannian**), so that

$$w_1 < w_2 < \cdots < w_r \text{ and } w_{r+1} < w_{r+2} < \cdots < w_n,$$

then

$$\mathfrak{S}_w = s_\lambda(x_1, \dots, x_r),$$

where λ is the partition $\lambda = (w_r - r, \dots, w_2 - 2, w_1 - 1)$. Note that λ has at most r parts, all of size at most $n - r$, so its Young diagram fits inside an $r \times (n - r)$ rectangle.

In order both to suppress the variable set x_1, \dots, x_r from the notation and to make more convenient use of symmetric function identities, we will work within a quotient of the *ring of symmetric functions* with integral coefficients

$$\Lambda = \Lambda_{\mathbb{Z}}(x_1, x_2, \dots);$$

see [M95, Chapter 1], [S99, Chapter 7]. The \mathbb{Z} -basis for Λ given by the Schur functions s_λ has the property that the \mathbb{Z} -submodule $I_{r, n-r}$ spanned by all s_λ with $\lambda \not\subseteq (n-r)^r$ forms an ideal, and the map sending s_λ to $s_\lambda(x_1, \dots, x_r)$ induces an isomorphism

$$\Lambda / I_{r, n-r} \xrightarrow{\sim} \mathbb{Z}[\mathbf{x}]^{W_I} / \mathbb{Z}[\mathbf{x}]_+^W \mathbb{Z}[\mathbf{x}]^{W_I} \quad (\cong H^*(G/P_J, \mathbb{Z}))$$

Thus $H^*(G/P_J, \mathbb{Z})$ has \mathbb{Z} -basis given by

$$\{\sigma_w : w \in W^J\} = \{s_\lambda : \lambda \subseteq (n-r)^r\}.$$

4.4. A shorter presentation in type A. Starting with Theorem 2.3, our goal is to find an even smaller set of generators for the ideal $J_{v_{r,s,t,n}}$ within the coinvariant algebra, so that through (1.6) we obtain an even shorter presentation of I_w in type A.

First note that even though our proof of Theorem 2.3 for all finite Coxeter groups was done in $\mathbb{k}[V]/(\mathbb{k}[V]_+^W)$ where the field \mathbb{k} has characteristic zero, the same proof works in type A more generally for the *integral* coinvariant algebra $H^*(G/B, \mathbb{Z})$. This follows since the Schubert polynomials $\{\mathfrak{S}_w\}$ satisfy the integer coefficient versions of all of the requisite properties (a)-(g) used in Section 2.

The bigrassmannian $v_{r,s,t,n}$ described explicitly in (4.3) is r -grassmannian, and corresponds to the $j \times i$ rectangular partition i^j , where we define

$$\begin{aligned} i &:= s - t + 1, \\ j &:= r - t + 1. \end{aligned}$$

When u and v are r -grassmannian and correspond respectively to partitions λ and μ , the Bruhat order relation $u \geq v$ is equivalent to inclusion $\lambda \supseteq \mu$ of their Young diagrams, meaning that $\lambda_i \geq \mu_i$ for all i . Thus Theorem 2.3 says that $J_{v_{r,s,t,n}}$ is generated as an ideal of $\Lambda/I_{r, n-r}$ by

$$(4.4) \quad \{\mathfrak{S}_u : \text{Des}(u) = \{(r, r+1)\}, u \geq v_{r,s,t,n}\} = \{s_\mu : i^j \subseteq \mu \subseteq (n-r)^r\}.$$

This presentation from Theorem 2.3 can be improved in type A_{n-1} as follows:

Theorem 4.8. Given a bigrassmannian $v = v_{r,s,t,n}$ in type A_{n-1} with $\mathfrak{S}_v = s_{ij}$, let

$$\begin{aligned} a &:= \min(n - r - i, r - j) \\ b &:= \min(i, j). \end{aligned}$$

Then J_v is generated as an ideal of $H^*(G/P_J, \mathbb{Z}) \cong \Lambda/I_{r,n-r}$ by

$$(4.5) \quad \{s_\mu : i^j \subseteq \mu \subseteq ((i+a)^b, i^{j-b})\}.$$

Alternatively, J_v is generated by

$$(4.6) \quad \{s_\mu : i^j \subseteq \mu \subseteq (i^j, b^a)\}.$$

We delay our proof of Theorem 4.8 until Section 4.7.

Note that in both of the asserted generating sets (4.5) and (4.6) for J_v , the shapes μ indexing the generators s_μ run through an interval between the $j \times i$ rectangular shape i^j and the disjoint union of i^j with a smaller rectangle of shape $a \times b$ or $b \times a$. In (4.5) the smaller rectangle is to the right of the rectangle i^j , with both top-justified, while in (4.6) the smaller rectangle is below the rectangle i^j , with both left-justified. Thus in both cases, the generating sets have size $\binom{a+b}{a}$ and consist of generators whose multiset of degrees have generating function

$$\sum_{\mu} q^{|\mu|} = q^{ij} \begin{bmatrix} a+b \\ a \end{bmatrix}_q,$$

where $\begin{bmatrix} a+b \\ a \end{bmatrix}_q$ is a q -binomial coefficient or Gaussian polynomial [M91, I.2 Exer. 3].

Example 4.9. We examine the special case where the bigrassmannian $v := v_{r,s,t,n}$ has t equal to r or s , so that the Schubert condition $C_{r,s,t}$ in 4.1 becomes an inclusion $V_r \subseteq \mathbb{C}^s$ or $V_r \supseteq \mathbb{C}^s$. Schubert varieties X_w in type A for which all Schubert conditions on X_w take one of these two forms were called Schubert varieties *defined by inclusions* in [GR02]. That paper gave a presentation for the cohomology containing

- for each inclusion condition $V_r \supseteq \mathbb{C}^s$, a set of s generators for J_v of the form

$$(4.7) \quad \{e_m(x_1, \dots, x_r)\}_{m=r-s+1, r-s+2, \dots, r},$$

- and for each inclusion condition $V_r \subseteq \mathbb{C}^s$, a set of r generators for J_v of the form

$$(4.8) \quad \{e_m(x_r + 1, \dots, x_n)\}_{m=s-r+1, s-r+2, \dots, n-r}.$$

We compare this with the presentation for J_v in Theorem 4.8, say for the inclusion conditions of the form $V_r \supseteq \mathbb{C}^s$, and using the generators given in (4.6).

Since $t = s$, one has

$$\begin{aligned} i &= s - t + 1 = 1 \\ j &= r - s + 1 \\ b &= \min(i, j) = i = 1 \\ a &= \min(n - r - 1, s - 1) \end{aligned}$$

Hence (4.6) says that J_v is generated by the set of $a + 1$ Schur functions

$$\{s_\mu : 1^{r-s+1} \subseteq \mu \subseteq 1^{a+r-s+1}\} = \{e_m(x_1, \dots, x_r)\}_{m=r-s+1, r-s+2, \dots, a+r-s+1}$$

which is exactly the first $a + 1 = \min(n - r, s)$ out of the s generators listed in (4.7). Hence Theorem 4.8 provides a dramatic reduction in the size of the generating set for J_v whenever $n - r$ is small compared to s .

We remark also that the techniques utilized in [GR02] seem very particular to the case where X_w is defined by inclusions. We do not know how to use them for some alternate approach to the case of general X_w considered in this paper.

4.5. A minimality conjecture. As we shall see in a moment, our generators for I_w are not minimal. However, we believe the following holds:

Conjecture 4.10. *The two generating sets for the ideal $J_{v_{r,s,t,n}}$ given in Theorem 4.8 are both minimal.*

Via computer, we have verified this conjecture for all bigrassmannian permutations where $r \leq 4$ and $n - r \leq 5$.

In fact, Conjecture 4.10 indicates obstructions to short presentations of $H^*(X_w)$ in general. We now give a family of ideals that would require a large number of generators if the conjecture is true.

For a positive integer m , let $n = 4m$, and consider in $W = S_n = S_{4m}$ the bigrassmannian $v_{r,s,t,n}$ that corresponds to $r = n - r = 2m$ and $i = j = m$. Then $a = b = m$, and $J_{v_{r,s,t,n}} (= I_{w_{r,s,t,n}})$ requires

$$(4.9) \quad \binom{2m}{m} \sim \frac{4^m}{\sqrt{\pi m}} = \frac{\sqrt{2}^{n+2}}{\sqrt{\pi n}}$$

generators according to Conjecture 4.10.

The size of any minimal generating set of a homogeneous ideal is well-defined. This is implied by the following well-known fact:

Proposition 4.11. *Let R be a commutative ring, and $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ a graded, connected R -algebra, meaning that $\Lambda_0 = R$ and $\Lambda_i \Lambda_j \subset \Lambda_{i+j}$. Let M be a graded Λ -module, with degrees bounded below, meaning that $M = \bigoplus_{n \geq N} M_n$ for some $N \in \mathbb{Z}$, and $\Lambda_i M_j \subset M_{i+j}$.*

Then a set of homogeneous elements $\{m_i\}_{i=1}^t$ generate M as a Λ -module if and only if their images $\{\bar{m}_i\}_{i=1}^t$ span $M/\Lambda_+ M$ as a R -module. In particular, $\{m_i\}_{i=1}^t$ form a minimal Λ -generating set with respect to inclusion for M if and only if $\{\bar{m}_i\}_{i=1}^t$ form a minimal R -spanning set for $M/\Lambda_+ M$.

In our setting, the well-definedness follows by setting $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ to be the graded ring of symmetric functions with \mathbb{Z} coefficients and setting $M = J_{v_{r,s,t,n}}$, so that the \mathbb{Z} -module $M/\Lambda_+ M$ is a finitely generated abelian group. Thus we conjecture that this abelian group $M/\Lambda_+ M$ requires $\binom{a+b}{a}$ generators, and in fact, we suspect that $M/\Lambda_+ M \cong \mathbb{Z}^{\binom{a+b}{a}}$. So far a proof has eluded us.

Example 4.12. Since $I_w = \sum_{v \in \mathcal{E}(w)} J_v$ and since we have conjectured that the generating sets provided by Theorem 4.8 for J_v are minimal, one might wonder whether their concatenation gives a minimal generating set of I_w . As mentioned above, this turns out to be false in general.

The smallest counterexample is given by $w = 1243$, which has

$$\mathcal{E}(w) = \{v_1 = 2134, v_2 = 1324\}$$

The generating sets given in Theorem 4.8 for the ideals J_{v_1} and J_{v_2} are

$$J_{v_1} = \langle s_{(1)}(x_1) \rangle = \langle x_1 \rangle$$

$$\begin{aligned} J_{v_2} &= \langle s_{(1)}(x_1, x_2), s_{(2)}(x_1, x_2) \rangle \quad \text{or} \quad \langle s_{(1)}(x_1, x_2), s_{(1,1)}(x_1, x_2) \rangle \\ &= \langle x_1 + x_2, x_1^2 + x_1x_2 + x_2^2 \rangle \quad \text{or} \quad \langle x_1 + x_2, x_1x_2 \rangle, \end{aligned}$$

and in each case they minimally generate their ideals J_{v_i} . However, concatenating them gives non-minimal generating sets for I_w , namely

$$\begin{aligned} I_w = J_{v_1} + J_{v_2} &= \langle x_1, x_1 + x_2, x_1^2 + x_1x_2 + x_2^2 \rangle \quad \text{or} \quad \langle x_1, x_1 + x_2, x_1x_2 \rangle \\ &= \langle x_1, x_1 + x_2 \rangle. \end{aligned}$$

Example 4.13. Some readers may find the above earliest example artificial: though $X_w = X_{1243}$ lives inside GL_4/B , it is isomorphic to $X_{21} = GL_2/B \cong \mathbb{P}^2$. However one can easily produce from this more counterexamples with similar properties but no such artificial nature.

For example, take $w = 23541$, which has $\mathcal{E}(w) = \{v_1 = 31245, v_2 = 14235\}$. Then J_{v_1} and J_{v_2} require one and two generators respectively, but the sum $I_w = J_{v_1} + J_{v_2}$ requires only two generators, not three.

4.6. Some symmetric function identities. The proof of Theorem 4.8 on generators for $J_{v_r, s, t, n}$ will use some symmetric function identities which we describe and prove in this section. We will make use of standard terminology, such as in [M95, S01, S99].

In particular, we will use the **Pieri rule** expanding the product of an elementary symmetric function $e_r := s_{1^r}$ with an arbitrary Schur function into Schur functions:

$$(4.10) \quad e_k s_\lambda = \sum_{\mu} s_{\mu},$$

where the sum runs over all partitions μ obtained from λ by adding on a vertical strip of length k . The following easy consequence will be used in the proof of Theorem 4.8 below.

Lemma 4.14. *For any partition ν and nonnegative integer k , one has*

$$(4.11) \quad s_{(\nu, 1^k)} = \sum_{\ell=0}^k (-1)^\ell e_{k-\ell} \sum_{\lambda} s_{\lambda},$$

where the inner sum runs over partitions λ obtained from ν by adding a horizontal strip of length ℓ .

Proof. Using the Pieri rule (4.10) to expand the right side of (4.11), one obtains

$$\sum_{\ell=0}^k (-1)^\ell e_{k-\ell} \sum_{\lambda} s_{\lambda} = \sum_{(\ell, \lambda)} (-1)^\ell s_{\lambda},$$

where the sum runs over pairs (ℓ, λ) in which both $0 \leq \ell \leq k$, and λ is obtained from ν by first adding a horizontal ℓ -strip within the first $\ell(\nu)$ rows then adding an arbitrary vertical $(k - \ell)$ -strip. Cancel all these pairs, except for the one with $\ell = 0$ and $\lambda = (\nu, 1^k)$, via the following sign-reversing involution: if x (respectively y) is the farthest east (respectively, farthest north) box in the horizontal (respectively vertical) strip, then

- when y is to the right of x (or when $\ell = 0$ and $\lambda \neq (\nu, 1^k)$), move y from the vertical to the horizontal strip, and,
- when y is below x , move x from the horizontal strip to the vertical strip.

□

We also need the **Jacobi-Trudi identity**:

$$(4.12) \quad s_\lambda = \det(h_{\lambda_i - i + j})_{i,j=1,2,\dots,\ell(\lambda)}$$

with the usual convention that $h_r := s_{(r)}$ for $r \geq 0$ and $h_r = 0$ for $r < 0$. This has the following consequence, also to be used in the proof of Theorem 4.8 below.

Lemma 4.15. *Let $i < k$, and assume μ is a partition with $\mu_k > i \geq \mu_{k+1}$, so that the $(i + 1)^{\text{st}}$ column of the Young diagram for μ has length $k > i$. Then*

$$s_\mu = \sum_{m=1}^k (-1)^{k-m} h_{\mu_m + k - i - m} s_{\mu^{(m)}},$$

where for $m = 1, 2, \dots, k$ one defines

$$\mu^{(m)} := (\mu_1, \mu_2, \dots, \mu_{m-1}, \widehat{\mu}_m, \mu_{m+1} - 1, \mu_{m+2} - 1, \dots, \mu_k - 1, i, \mu_{k+1}, \mu_{k+2}, \dots, \mu_\ell),$$

where $\ell := \ell(\mu)$ and $\widehat{\mu}_m$ refers to the deletion of the entry μ_m .

Proof. Start with the $\ell \times \ell$ Jacobi-Trudi matrix for μ . From this create an $(\ell + 1) \times \ell$ matrix by inserting a new row between its row k and row $k + 1$, having entries

$$(h_{i-k+1}, h_{i-k+2}, \dots, h_{i-k+\ell}).$$

Then from this $(\ell + 1) \times \ell$ matrix, create a singular $(\ell + 1) \times (\ell + 1)$ matrix by introducing an $(\ell + 1)^{\text{st}}$ column that duplicates the $(k - i)^{\text{th}}$ column. This last duplicated column is

$$\begin{aligned} & (h_{\mu_1 + k - i - 1}, \dots, h_{\mu_k - i}, h_{i - k + (k - i)}, h_{\mu_{k+1} - i - 1}, \dots, h_{\mu_\ell + k - i - \ell})^T \\ &= (h_{\mu_1 + k - i - 1}, \dots, h_{\mu_k - i}, 1, 0, \dots, 0)^T. \end{aligned}$$

Here we have used the facts that $h_0 = 1$ and that $h_{\mu_m + k - i - m} = 0$ for $m \geq k + 1$ because $\mu_m \leq \mu_{k+1} \leq i$ implies $\mu_m + k - i - m = (\mu_m - i) + (k - m) < 0$. One then checks that cofactor expanding the (zero) determinant of this $(\ell + 1) \times (\ell + 1)$ matrix along this duplicated column gives the asserted identity. □

4.7. Proof of Theorem 4.8. The proof of the second statement will follow from the first, via the well-known ring involution ω on symmetric functions defined by

$$\begin{aligned} \Lambda &\xrightarrow{\omega} \Lambda \\ s_\lambda &\longmapsto s_{\lambda'} \end{aligned}$$

where λ' is the conjugate partition to λ . This means that ω sends the ideal $I_{r,n-r}$ to the ideal $I_{n-r,r}$. Hence the set (4.6) generates J_ν within $\Lambda/I_{r,n-r}$, where ν corresponds to an $i \times j$ rectangle, if and only if the set (4.5) generates the ideal $J_{\nu'}$ within $\Lambda/I_{n-r,r}$, where ν' corresponds to a $j \times i$ rectangle.

The proof for (4.5) is by induction on the degree d , which is the number of boxes in our partition. Our inductive hypothesis is that the portion of J_ν of degree at most d is generated by those elements of (4.5) of degree at most d , or equivalently, that all elements

of (4.4) of degree at most d are writable in terms of elements of (4.5) of degree at most d . The base case, $d = ij$, is clear, since s_{ij} is the only element of degree ij in both sets.

Our proof for the inductive case proceeds in three steps. Start with the generating set for J_ν given in (4.4). We wish to show that, modulo $I_{r,n-r}$, all such s_μ with $|\mu| = d$ lie in the ideal generated by those s_μ with $|\mu| < d$ and those

- Step 1. with μ in the interval $[i^j, (n-r)^j]$, and then furthermore
- Step 2. with μ in the interval $[i^j, (i+a)^j]$, and then finally
- Step 3. with μ in the interval $[i^j, ((i+a)^b, i^{j-b})]$.

Step 1. We will use induction on a certain partial order on partitions which depends on the index j . For a partition λ , define

$$\hat{\lambda} := (\lambda_{j+1}, \lambda_{j+2}, \dots),$$

so that the Young diagram of $\hat{\lambda}$ consists of rows $j+1, j+2, \dots$ from the Young diagram of λ . Then partially order the partitions containing i^j by decreeing $\lambda \prec_j \mu$ if either $|\hat{\lambda}| < |\hat{\mu}|$, or if $|\hat{\lambda}| = |\hat{\mu}|$ but $\hat{\lambda} < \hat{\mu}$ in the *dominance order*, meaning that

$$\lambda_{j+1} + \lambda_{j+2} + \dots + \lambda_k \leq \mu_{j+1} + \mu_{j+2} + \dots + \mu_k$$

for each $k \geq j$.

Now if μ does not already lie in the interval $[i^j, (n-r)^j]$, so that $\ell(\mu) = k + j > j$, let ν be the partition obtained from μ by removing 1 from its last k nonempty parts $\mu_{j+1}, \mu_{j+2}, \dots, \mu_{j+k}$. Then by the Pieri rule (4.10), $e_k s_\nu = s_\mu + \sum_\lambda s_\lambda$, where λ runs through partitions other than μ obtained from ν by adding a vertical strip of size k . One can check that any such λ satisfies $\lambda \prec_j \mu$: either the vertical strip contains some boxes in the first j rows, so that $|\hat{\lambda}| < |\hat{\mu}|$, or if not, the location of the vertical strip forces $\hat{\lambda} < \hat{\mu}$ in dominance. Also, $k \geq 1$, so $|\nu| < |\mu|$. Consequently, by induction on the order \prec , one has an expression for s_μ showing that it is in the ideal generated by s_λ where either $|\lambda| < |\mu|$ or λ is in the interval $[i^j, (n-r)^j]$.

Step 2. We will again use induction, this time on reverse dominance order. We wish to write s_μ where μ is in the interval $[i^j, (n-r)^j]$ in terms of s_λ where $|\lambda| < |\mu|$ or λ lies in the interval $[i^j, (i+a)^j]$. Recall that $a = \min(n-r-i, r-j)$, and if $a = n-r-i$, then $n-r = i+a$ so there is nothing to do after Step 1. Thus we may assume $a = r-j$.

If μ does not already lie in the interval $[i^j, (i+a)^j]$, so that $\mu_1 > i+a$, let $k := \mu_1 - i > a$, and let ν be the partition obtained from μ by removing 1 box from each of the last k nonempty columns in the Young diagram of μ . Note that $\ell(\nu) \geq \ell(\mu)$ since $k < \mu_1$, and $\ell(\mu) \geq j$ since $i^j \subset \mu$. Hence $(\nu, 1^k)$ has length at least $j+k > j+a = r$, so that the partition $(\nu, 1^k) \not\subseteq (n-r)^r$, and hence Lemma 4.14 tells us that

$$\sum_{\ell=0}^k (-1)^\ell e_{k-\ell} \sum_{\lambda} s_\lambda \equiv 0 \pmod{I_{r,n-r}}$$

where in the sum λ runs through partitions having no more than j parts obtained from ν by adding a horizontal strip of length ℓ .

We claim that almost all of the terms in this sum, excepting the single term with $\ell = k$ and $\lambda = \mu$, will have $|\lambda| < |\mu|$ or $\lambda > \mu$. If $\ell < k$, then $|\lambda| < |\mu|$. If $\ell = k$, note the horizontal strip λ/ν cannot have any boxes in the first i columns as those already have length j in ν .

Therefore, the location of the horizontal strip forces $\lambda > \mu$ in dominance. Consequently, by induction, one has an expression for s_μ showing that it is in the ideal generated by s_λ where $|\lambda| < |\mu|$ or λ is in the interval $[i^j, (i+a)^j]$.

Step 3. Now we show that, if μ fits inside $(i+a)^j$ but not inside $((i+a)^b, i^{j-b})$, then s_μ can always be written as a sum of terms of the form h_{r,s_λ} for $r > 0$ and λ containing i^j . Since $r > 0$, $|\lambda| < |\mu|$, this suffices to finish the proof.

Recall that $b = \min(i, j)$, and if $b = j$ then there is nothing to do after Step 2. Thus we may assume $b = i < j$.

Let k be the number of parts of μ which are strictly larger than i , so that k is the size of the $(i+1)^{\text{st}}$ column in the Young diagram of μ . Since μ does not fit inside $((i+a)^i, i^{j-i})$, it must be that $k > i$, and we are in the situation of Lemma 4.15. Hence

$$s_\mu = \sum_{m=1}^k (-1)^{k-m} h_{\mu_m+k-i-m} s_{\mu^{(m)}}$$

where

$$\mu^{(m)} := (\mu_1, \mu_2, \dots, \mu_{m-1}, \mu_{m+1} - 1, \mu_{m+2} - 1, \dots, \mu_k - 1, i, \mu_{k+1}, \mu_{k+2}, \dots, \mu_\ell)$$

and $\ell = \ell(\mu)$. Note that, since μ contains i^j , and hence $k \leq j$, each $\mu^{(m)}$ also contains i^j . Also note that each factor $h_{\mu_m+k-i-m}$ has positive degree: $m \leq k$ implies $\mu_m \geq \mu_k > i$, and hence $\mu_m + k - i - m = (\mu_m - i) + (k - m) > 0$. \square

5. A QUESTION

Question 5.1. Can one find a *minimal* generating set for the ideal I_w in type A_{n-1} ?

Can this at least be done for some of the recently-studied subclasses [GR02, HLSS07, OPY07] where I_w can be generated by n^2 elements, such as

- when X_w is *defined by inclusions*, which occurs when w avoids the patterns $\{4231, 35142, 42513, 351624\}$,
- or more restrictively, when X_w is *smooth*, which occurs when w avoids the patterns $\{3412, 4231\}$?

It was mentioned in the introduction that for a special subclass of smooth Schubert varieties X_w originally considered by Ding [D97, D01], there is a known minimal (in fact, complete intersection) presentation for $H^*(X_w, \mathbb{Z})$ with n relations that was exploited in [DMR07]. Short presentations would be useful to extend that work further.

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