TROPICALIZATION, SYMMETRIC POLYNOMIALS, AND COMPLEXITY

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ABSTRACT. D. Grigoriev-G. Koshevoy recently proved that tropical Schur polynomials have (at worst) polynomial tropical semiring complexity. They also conjectured tropical skew Schur polynomials have at least exponential complexity; we establish a polynomial complexity upper bound. Our proof uses results about (stable) Schubert polynomials, due to R. P. Stanley and S. Billey-W. Jockusch-R. P. Stanley, together with a sufficient condition for polynomial complexity that is connected to the saturated Newton polytope property.

1. INTRODUCTION

The **tropicalization** of a polynomial

$$f = \sum_{(i_1,i_2,\ldots,i_n)\in\mathbb{Z}^n_{\geq 0}} c_{i_1,\ldots,i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \in \mathbb{C}[x_1, x_2, \ldots, x_n]$$

(with respect to the trivial valuation $\mathsf{val}(a) = 0$ for all $a \in \mathbb{C}^*$) is defined to be

$$\mathsf{Trop}(f) := \max_{i_1,\ldots,i_n} \{i_1 x_1 + \cdots + i_n x_n\}. \tag{1}$$

This is a polynomial over the tropical semiring $(\mathbb{R}, \oplus, \odot)$, where

$$a \oplus b = \max(a, b) \text{ and } a \odot b = a + b$$

respectively denote tropical addition and multiplication, respectively. We refer to the books [ItMiSh09, MaSt15] for more about tropical mathematics.

Let Sym_n denote the ring of symmetric polynomials in $x_1, \ldots, x_n$. A linear basis of Sym_n is given by the *Schur polynomials*. These polynomials are indexed by partitions λ (identified with their Ferrers/Young diagrams). They are generating series over semistandard Young tableaux T of shape λ with entries from $[n] := \{1, 2, \ldots, n\}$:

$$s_\lambda(x_1, \ldots, x_n) := \sum_T x^T \quad \text{where} \quad x^T := \prod_i x_i^{\#i\text{'s in } T}.$$

The importance of this basis stems from its applications to, for example, enumerative and algebraic combinatorics, the representation theory of symmetric groups and general linear groups, and Schubert calculus on Grassmannians; see, for example, [Fu97, St99].

D. Grigoriev and G. Koshevoy [GrKo16] studied the complexity of the tropical polynomial $\mathsf{Trop}(s_\lambda)$ over $(\mathbb{R}, \oplus, \odot)$. An *arithmetic circuit* is a circuit where inputs are each labelled by a single variable x_i or a fixed constant, each gate performs a single $\oplus$ or $\odot$ operation, and there is one output. An arithmetic circuit C naturally gives an expression $\mathsf{res}(C)$, the tropical polynomial in the variables $x_1, \ldots, x_n$ that it computes. The circuit C *evaluates* f if $\mathsf{res}(C) = f$ as tropical polynomials, meaning that one can show $\mathsf{res}(C) = f$ using the *tropical semiring axioms*, by which we mean the semiring axioms along with the idempotence

Date: March 26, 2019.

property $a \oplus a = a$. The *tropical semiring complexity* of f is the smallest number of gates in a circuit C evaluating f; see [JeSn82, Section 2].

The following is [GrKo16, Theorem 2.1]:

Theorem 1.1 (D. Grigoriev-G. Koshevoy)**.** *The tropical semiring complexity of* $\mathsf{Trop}(s_\lambda)$ *is at most* $O(n^2 \cdot \lambda_1)$.

The *skew-Schur polynomial* $s_{\lambda/\mu}(x_1, \ldots, x_n) = \sum_T x^T$ is the generating series for semistandard tableau of skew shape λ/μ with entries from $[n]$. When $\mu = \emptyset$ then $s_{\lambda/\emptyset} = s_\lambda$; hence skew-Schur polynomials generalize Schur polynomials. Also,

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu,\nu} s_\nu, \tag{2}$$

where $c^{\nu}_{\lambda,\mu} \in \mathbb{Z}_{\geq 0}$ is the *Littlewood-Richardson coefficient*. The next statement is from [GrKo16, Section 5]:

Conjecture 1.2 (D. Grigoriev-G. Koshevoy)**.** *The tropical semiring complexity of* $\mathsf{Trop}(s_{\lambda/\mu})$ *is at least exponential.*

We will show the following:

Theorem 1.3. *There is an explicitly described partition β, depending on λ/μ, with $\beta_1 = \lambda_1$, such that*

$$\mathsf{Trop}(s_{\lambda/\mu}(x_1, \ldots, x_n)) = \mathsf{Trop}(s_\beta(x_1, \ldots, x_n))$$

over the tropical semiring $(\mathbb{R}, \oplus, \odot)$.

Example 1.4. Let $\lambda = (2, 1)$ and $\mu = (1)$. Then the tableaux contibuting to $s_{\lambda/\mu}$ are:

$$\begin{array}{cc} & \boxed{1} \\ \boxed{1} & \end{array}, \quad \begin{array}{cc} & \boxed{1} \\ \boxed{2} & \end{array}, \quad \begin{array}{cc} & \boxed{2} \\ \boxed{1} & \end{array}, \quad \begin{array}{cc} & \boxed{2} \\ \boxed{2} & \end{array}.$$

Hence $s_{\lambda/\mu} = x_1^2 + 2x_1x_2 + x_2^2$. On the other hand, (2) in this case is:

$$s_{\lambda/\mu} = s_{1,1} + s_2 = (x_1x_2) + (x_1x_2 + x_1^2 + x_2^2).$$

By definition,

$$\begin{aligned} \mathsf{Trop}(s_{\lambda/\mu}) &= \max\{x_1 + x_2, x_1 + x_2, 2x_1, 2x_2\} \\ &= x_1 \odot x_2 \oplus x_1 \odot x_2 \oplus x_1^{\odot 2} \oplus x_2^{\odot 2} \\ &= x_1 \odot x_2 \oplus x_1^{\odot 2} \oplus x_2^{\odot 2} \quad \text{(idempotence)} \\ &= \mathsf{Trop}(s_2), \end{aligned}$$

in agreement with Theorem 1.3. □

The following addresses Conjecture 1.2:

Corollary 1.5 (of Theorems 1.1 and 1.3)**.** $\mathsf{Trop}(s_{\lambda/\mu})$ *has at most* $O(n^2 \cdot \lambda_1)$ *tropical semiring complexity.*

In Section 2, we describe a sufficient condition for a tropicalized symmetric polynomial to have polynomial tropical semiring complexity. This condition uses the notion of a polynomial having a *saturated Newton polytope* [MoToYo17]. Section 3 applies this condition to *Stanley symmetric polynomials* [St84]. Since skew-Schur polynomials are a special case of Stanley symmetric polynomials, we thereby deduce Theorem 1.3. In Section 4, we remark on how this condition applies to other families of symmetric polynomials.

2. Dominance order, Newton polytopes and saturation

Let $\mathsf{Par}(d) = \{\lambda : \lambda \vdash d\}$ be the set of partitions of size d. *Dominance order* $\leq_D$ on $\mathsf{Par}(d)$ is defined by

$$\mu \leq_D \lambda \quad \text{if} \quad \sum_{i=1}^{k} \mu_i \leq \sum_{i=1}^{k} \lambda_i \quad \text{for all } k \geq 1. \tag{3}$$

Definition 2.1. Suppose $f \in \mathsf{Sym}_n$ is homogeneous of degree d such that

$$f = \sum_{\mu \in \mathsf{Par}(d)} c_\mu s_\mu$$

with $c_\mu \geq 0$ for all μ. Moreover, assume there exists λ with $c_\lambda \neq 0$ such that $c_\mu \neq 0$ only if $\mu \leq_D \lambda$. Then we say f is *dominated* by s_λ. □

The *Newton polytope* of a polynomial f is the convex hull of its exponent vectors, so

$$\mathsf{Newton}(f) = \mathsf{conv}((i_1, i_2, \ldots, i_n) : c_{i_1, i_2, \ldots, i_n} \neq 0\}) \subseteq \mathbb{R}^n.$$

C. Monical, N. Tokcan and the second author [MoToYo17] define f to have *saturated Newton polytope* (SNP) if $c_{i_1,\ldots,i_n} \neq 0$ whenever $(i_1, \ldots, i_n) \in \mathsf{Newton}(f)$.

The *permutahedron* of $\lambda = (\lambda_1, \ldots, \lambda_n)$, denoted $\mathcal{P}_\lambda$, is the convex hull of the S_n-orbit of λ in $\mathbb{R}^n$. It follows from R. Rado's theorem [Ra52] that

$$\mathsf{Newton}(s_\lambda) = \mathcal{P}_\lambda \quad \text{and} \quad \mathsf{Newton}(s_\mu) \subseteq \mathsf{Newton}(s_\lambda) \text{ if and only if } \mu \leq_D \lambda.$$

A consequence of R. Rado's theorem [MoToYo17, Proposition 2.5] is therefore:

Proposition 2.2. *If $f \in \mathsf{Sym}_n$ is dominated by s_λ, then f is SNP, and*

$$\mathsf{Newton}(f) = \mathsf{Newton}(s_\lambda) = \mathcal{P}_\lambda.$$

We give a technical strengthening of [GrKo16, Theorem 2.5]:

Proposition 2.3 (Sufficient condition for polynomial complexity). *Suppose $f \in \mathsf{Sym}_n$ is dominated by $s_\lambda(x_1, \ldots, x_n)$. Then $\mathsf{Trop}(f) = \mathsf{Trop}(s_\lambda)$ over the tropical semiring $(\mathbb{R}, \oplus, \odot)$. Therefore, f has at most $O(n^2 \cdot \lambda_1)$ tropical semiring complexity.*

Proof. By Proposition 2.2,

$$\mathsf{Newton}(f) = \mathsf{Newton}(s_\lambda), \tag{4}$$

and f is SNP. This proves the first statement.

At this point, we can appeal to Theorem 1.1 to obtain the second claim. However, for convenience, we recall the ideas from [GrKo16, Theorem 2.5], thus indicating the underlying circuit. There it is shown that

$$\mathsf{Newton}(s_\lambda)[\mathbb{Z}] = \sum_{1 \leq k \leq \lambda_1} \mathsf{Newton}(e_{\lambda'_k})[\mathbb{Z}]. \tag{5}$$

In the Minkowski sum of (5),

$$e_k = \sum_{1\leq j_1<j_2<\ldots<j_k\leq n} x_{j_1}\cdots x_{j_k}$$

is the *elementary symmetric polynomial* of degree k. Also, λ' is the conjugate partition of λ, obtained by transposing the Young diagram for λ. Finally, for a polytope $\mathcal{P} \subseteq \mathbb{R}^n$, $\mathcal{P}[\mathbb{Z}]$ denotes the set of integer lattice points of $\mathcal{P}$.

Combining (4) and (5), we see

$$\mathsf{Newton}(f)[\mathbb{Z}] = \sum_{1\leq k\leq \lambda_1} \mathsf{Newton}(e_{\lambda'_k})[\mathbb{Z}]. \tag{6}$$

By Proposition 2.2, f is SNP. This property of f, together with (6), implies

$$\mathsf{Trop}(f) = \bigodot_{1\leq k\leq\lambda_1} \mathsf{Trop}(e_{\lambda'_k}), \tag{7}$$

as tropical polynomials.

Therefore, following *loc. cit.*, to calculate $\mathsf{Trop}(f)$ it suffices to compute $\mathsf{Trop}(e_{\lambda'_k})$ for $1 \leq k \leq \lambda_1$. The latter has at worst $O(n^2)$ complexity, using the (tropicalization) of the Pascal-type recurrence

$$e_k(x_1,\ldots,x_n) = e_k(x_1,\ldots,x_{n-1}) + x_n e_{k-1}(x_1,\ldots,x_{n-1}).$$

This proves the second claim. $\square$

Remark 2.4. The assumption in Definition 2.1 that f be Schur-positive ($c_\mu \geq 0$ for each μ) is needed for Proposition 2.2. Consider the *monomial symmetric polynomial* $m_\lambda := \sum_\theta x_1^{\theta_1}\cdots x_n^{\theta_n}$, where the sum is over distinct rearrangements of λ. It is true that $m_\lambda = \sum_{\mu\leq_D\lambda} I_{\lambda,\mu}s_\mu$, where $I_{\lambda,\lambda}=1$. Yet, $\{m_\lambda\}$ has exponential complexity, by [GrKo16]. $\square$

3. Stanley symmetric polynomials and the Proof of Theorem 1.3

For any permutation w, R. P. Stanley [St84] defined the symmetric power series

$$F_w = \sum_{\mathbf{a}\in\mathsf{Red}(w)} \sum_{\mathbf{b}\in C(\mathbf{a})} x_{\mathbf{b}},$$

where the following notation is used. The set $\mathsf{Red}(w)$ consists of the reduced words for w in the simple transpositions $s_i = (i\ i+1)$. This means $\mathbf{a} = (a_1, a_2, \ldots, a_\ell) \in \mathsf{Red}(w)$ if and only if $s_{a_1}s_{a_2}\cdots s_{a_\ell} = w$ and $\ell = \ell(w)$ is the number of inversions of w. Furthermore, by definition, $\mathbf{b} = (b_1,\ldots,b_\ell) \in C(\mathbf{a})$ if and only if

- $1 \leq b_1 \leq b_2 \leq \cdots \leq b_\ell$; and
- $a_i < a_{i+1} \implies b_i < b_{i+1}$.

Finally, set $x_{\mathbf{b}} := x_{b_1}x_{b_2}\cdots x_{b_\ell}$. (This actually defines $F_{w^{-1}}$ in [St84]. Thus we use the results of *loc. cit.* with this swap of convention.)

Remark 3.1. The original motivation for F_w is that $\#\mathsf{Red}(w) = [x_1x_2\cdots x_\ell]F_w$. If we define $a_{w\lambda}$ as the coefficients in

$$F_w = \sum_\lambda a_{w\lambda}s_\lambda, \tag{8}$$

then $a_{w\lambda} \in \mathbb{Z}_{\geq 0}$. This nonnegativity is proved by work of [EdGr87] (see also [LaSc82]). In fact, $a_{w\lambda}$ is a generalization of the Littlewood-Richardson coefficient. A theorem of H. Narayanan [Na06] states that computation of $c_{\lambda,\mu}^{\nu}$ is $\#\mathsf{P}$-complete in L. Valiant's complexity theory for counting problems [Va79]. Hence $a_{w\lambda}$ is a $\#\mathsf{P}$-complete counting problem. In particular, this means that there is no polynomial time algorithm for computing either $c_{\lambda,\mu}^{\nu}$ or $a_{w\lambda}$ unless $\mathsf{P} = \mathsf{NP}$.

Now, $[x_1 \ldots x_\ell]s_\lambda = f^\lambda$ counts standard Young tableaux of shape λ. These numbers are computed by the famous *hook-length formula*. The resulting enumeration $\#\mathsf{Red}(w) = \sum_\lambda a_{w\lambda} f^\lambda$ establishes that $\#\mathsf{Red}(w)$ is a $\#\mathsf{P}$ counting problem. Is it $\#P$-complete? $\square$

Recall that the *Rothe diagram* of w is given by

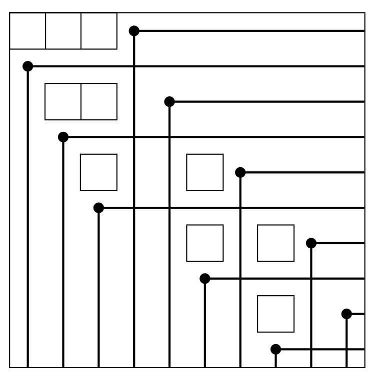
$$D(w) = \{(i,j) : 1 \leq i, j \leq n, j < w(i), i < w^{-1}(j)\}.$$

Pictorially, this is described by placing $\bullet$'s in positions $(i, w(i))$ (in matrix notation), striking out boxes below and to the right of each $\bullet$. Then $D(w)$ consists of the remaining boxes.

For example, if

(9) $$w = 4\ 1\ 5\ 2\ 7\ 3\ 9\ 6\ 10\ 8 \in S_{10} \quad \text{(in one line notation)},$$

then $D(w)$ is depicted by:



For $w \in S_m$, set q_i to be the number of boxes of $D(w)$ in column i (counting from the left) for $1 \leq i \leq m$. Then $(q_1, q_2, \ldots, q_m)$ is the *code* of w^{-1}. Let $\beta_{\max}(w)$ be the partition obtained by sorting $(q_1, q_2, \ldots, q_m)$ in decreasing order and taking the conjugate shape.

Theorem 3.2 (Complexity of tropical Stanley polynomials). *Let $w \in S_m$. Then*

$$\mathsf{Trop}(F_w(x_1, \ldots, x_n)) = \mathsf{Trop}(s_{\beta_{\max}(w)}(x_1, \ldots, x_n))$$

and hence the tropical semiring complexity of $\mathsf{Trop}(F_w(x_1, \ldots, x_n))$ *is at most* $O(n^2 \cdot \beta_{\max}(w)_1)$.

Proof. By [St84, Theorem 4.1] (up to convention), if $a_{w,\lambda} \neq 0$, then

$$\lambda \leq_D \beta_{\max}(w).$$

Since the $a_{w\lambda}$ in (8) are positive, if $F_w(x_1, \ldots, x_n)$ is nonzero, then $a_{w,\beta_{\max}(w)} \neq 0$ and F_w is dominated by $s_{\beta_{\max}(w)}$. Now use Proposition 2.3. $\square$

Proof of Theorem 1.3 (and Corollary 1.5): We show that $s_{\lambda/\mu}(x_1, \ldots, x_n)$ is dominated by $s_\beta(x_1, \ldots, x_n)$ for some shape β (to be determined) with $\beta_1 = \lambda_1$.

Given λ/μ, construct a permutation $w_{\lambda/\mu}$ by filling all boxes in the same northwest-southeast diagonal with the same entry, starting with 1 on the northeastmost diagonal and increasing consecutively as one moves southwest. Call this filling $T_{\lambda/\mu}$.

For instance, if $\lambda/\mu = (5,4,3,2,1)/(2,2,1,0,0)$ then

$$T_{\lambda/\mu} = \begin{ytableau} \none & \none & 3 & 2 & 1 \\ \none & \none & 4 & 3 \\ \none & 6 & 5 \\ 8 & 7 \\ 9 \end{ytableau}$$

Let $(r_1, r_2, \ldots, r_{|\lambda/\mu|})$ be the left-to-right, top-to-bottom, row reading word of $T_{\lambda/\mu}$. In our example, this is $(3,2,1,4,3,6,5,8,7,9)$.

Define $w_{\lambda/\mu} = s_{r_1} s_{r_2} \cdots s_{r_{|\lambda/\mu|}}$. By [BiJoSt93, Corollary 2.4],

$$F_{w_{\lambda/\mu}}(x_1, \ldots, x_n) = s_{\lambda/\mu}(x_1, \ldots, x_n) \in \mathsf{Sym}_n.$$

By [BiJoSt93, Section 2], λ/μ is obtained by removing empty rows and columns of $D(w_{\lambda/\mu})$ and reflecting across a vertical line. In our example, $w_{\lambda/\mu}$ is the permutation (9). The reader can check from the Rothe diagram that this process gives λ/μ.

By definition, $\beta_{\max}(w_{\lambda/\mu})$ is the conjugate of the decreasing rearrangement of the code of $w_{\lambda/\mu}^{-1}$. Hence, in our example, the code of $w_{\lambda/\mu}^{-1}$ is $(1,2,3,0,0,2,0,2,0,0)$, which rearranges to $(3,2,2,2,1,0,0,0,0,0)$. Therefore,

$$\beta_{\max}(w_{\lambda/\mu}) = \ydiagram{3,2,2,2,1}\,' = \ydiagram{5,4,1}.$$

(Thus, $\beta_{\max}(w_{\lambda/\mu})$ is obtained from λ/μ by first pushing the boxes in each column north and left-justifying the result.)

Since the coefficients $c^{\lambda}_{\mu,\nu}$ in the Schur expansion (2) are positive, $s_{\lambda/\mu}(x_1, \ldots, x_n)$ is dominated by s_β, where $\beta = \beta_{\max}(w_{\lambda/\mu})$. Hence, by Theorem 2.3,

$$\mathsf{Trop}(s_{\lambda/\mu}(x_1, \ldots, x_n)) = \mathsf{Trop}(s_\beta(x_1, \ldots, x_n))$$

over the tropical semiring $(\mathbb{R}, \oplus, \odot)$. By the above process from [BiJoSt93] relating $D(w_{\lambda/\mu})$ and λ/μ,

$$\beta_1 = \beta_{\max}(w_{\lambda/\mu})_1 = \lambda_1,$$

as desired. Hence, Theorem 1.3 holds.

To conclude Corollary 1.5, we may now either apply Theorem 1.1 or Theorem 3.2. $\square$

4. Some other symmetric polynomials

In [MoToYo17, Sections 2 and 3], some symmetric polynomials are observed to be SNP because they are dominated by s_λ (for some λ), or for other reasons. These include:

(1) J. R. Stembridge's polynomial F_M for a totally nonnegative matrix M [St91];

(2) the cycle index polynomial c_G of a subgroup $G \leqslant S_n$ (from Redfield-Pólya theory);
(3) C. Reutenauer's q_λ basis of Sym_n [Re95];
(4) the symmetric Macdonald polynomial (where $(q,t) \in \mathbb{C}^2$ is generic);
(5) the Hall-Littlewood polynomial (for any positive evaluation of t); and
(6) Schur $P-$ and Schur $Q-$ polynomials.

Consequently, by Proposition 2.3, the tropicalizations of these polynomials equal some tropical Schur polynomial. Therefore, as with skew Schur polynomials, one obtains immediate tropical semiring complexity implications:

- The polynomials (1) and (2) are dominated by $s_{(n)}$; see [MoToYo17, Theorem 2.28] and [MoToYo17, Theorem 2.30]. Hence Proposition 2.3 shows their tropicalizations have $O(n^3)$ complexity. In fact, since $\mathsf{Trop}(s_{(n)}) = \mathsf{Trop}((x_1 + \cdots + x_n)^n)$ as tropical polynomials, they have $O(n)$ complexity (see [FoGrNoSc16, Theorem 1.4] for a nontropical version of this statement).
- For (3), if $\lambda = (\lambda_1, \ldots, \lambda_\ell, 1^r)$ where each $\lambda_i \geq 2$, then q_λ is dominated by $s_{a,b}$ where $a = |\lambda| - \ell$ and $b = \ell$ (see [MoToYo17, Theorem 2.3.2] and specifically its proof). Hence Proposition 2.3 asserts $\mathsf{Trop}(q_\lambda)$ has $O(n^2 \cdot (|\lambda| - \ell))$ complexity.
- For a generic choice of $q, t \in \mathbb{C}^2$, it follows from [MoToYo17, Section 3.1] that if $P_\lambda(X; q, t) \in \mathsf{Sym}_n$ is the Macdonald polynomial, then $\mathsf{Trop}(P_\lambda(X; q, t)) = \mathsf{Trop}(s_\lambda)$. Hence $\mathsf{Trop}(P_\lambda(X; q, t))$ has $O(n^2 \cdot \lambda_1)$ complexity.
- (5) and (6) are also indexed by partitions λ and dominated by s_λ. Thus, Proposition 2.3 implies their tropicalizations have $O(n^2 \cdot \lambda_1)$ complexity.

Acknowledgements

AY was supported by an NSF grant. We thank Marc Snir for helpful communications.

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