

Hultman elements for the hyperoctahedral group

Alexander Woo (U. Idaho)

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Pattern avoidance

For this talk, permutations will usually be written in one line notation; we write $w \in S_n$ as the string $w(1)w(2)\cdots w(n)$.

Given $v \in S_m$ and $w \in S_n$, we say w **contains** v if there exist indices $1 \leq a_1 < \cdots < a_m \leq n$ such that $fl_{(a_1, \dots, a_m)}^n(w) = v$. If w does not contain v , we say w **avoids** v .

For example, **594827631** contains 35142, but 24975138 avoids 35142.

The set of permutations that avoid a set S of permutations is written $Av(S)$.

An interesting permutation class

The permutations in $Av(4231, 35142, 42513, 351624)$ have shown up in several places:

- ▶ Permutations w for which the number of chambers of the inversion hyperplane arrangement \mathcal{B}_w is the same as the number of elements in the Bruhat interval $[\text{id}, w]$
- ▶ Permutations w for which every permutation fitting in the right hull of w is in the Bruhat interval $[\text{id}, w]$
- ▶ Permutations w indexing Schubert varieties defined by inclusions.
- ▶ Permutations w for which the number of matrices over \mathbb{F}_q with support in the complement of the diagram $D(w)$ is equal to the Poincaré polynomial $P(q^{-1})$ for the Bruhat interval $[\text{id}, w]$.

Bruhat order

Given a permutation w , define

$$r_{ij}(w) = \#(\{1, \dots, i\} \cap \{w(1), \dots, w(j)\}).$$

Given permutations $v, w \in S_n$, define $v \leq w$ if, for all i, j with $1 \leq i, j \leq n$, $r_{ij}(v) \geq r_{ij}(w)$. This gives a partial order known as **Bruhat order**. (The definition I just gave, one of many possible, is the **tableau criterion**.)

Given any fixed n , the smallest permutation is the identity id (with $r_{ij}(\text{id}) = \min(i, j)$) and the largest is $w_0 = n(n-1) \cdots 21$ (with $r_{ij}(w_0) = i + j - n$).

Bruhat order by transpositions

Alternatively, define the **length** of a permutation w , denoted $\ell(w)$ as the smallest number of adjacent transpositions $s_i = (ii + 1)$ such that $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ for some i_1, \dots, i_ℓ .

Given a permutation $w \in S_n$, a pair (i, j) is an **inversion** of w if $i < j$ and $w(i) > w(j)$. Length is equal to the number of inversions.

Bruhat order is the transitive closure of the relation defined by $v \prec w$ if there exists a transposition (ij) such that $w = (ij)v$ and $\ell(w) > \ell(v)$.

Bruhat order example

We have $13254 < 35142$:

The rank matrices (r_{ij} appropriately arranged) are:

$$\begin{array}{ccccc}
 1 & 1 & 1 & 1 & 1 & & 0 & 0 & 1 & 1 & 1 \\
 1 & 1 & 2 & 2 & 2 & & 0 & 0 & 1 & 1 & 2 \\
 1 & 2 & 3 & 3 & 3 & \text{and} & 1 & 1 & 2 & 2 & 3 \\
 1 & 2 & 3 & 3 & 4 & & 1 & 1 & 2 & 3 & 4 \\
 1 & 2 & 3 & 4 & 5 & & 1 & 2 & 3 & 4 & 5
 \end{array}$$

One chain of cover relations is

$$13254 < 31254 < 32154 < 35124 < 35142$$

Inversion arrangements

Given a permutation $w \in S_n$, define the **inversion arrangement** \mathcal{B}_w to be the set of hyperplanes $\{x_i - x_j = 0\} \in \mathbb{R}^n$ for which (i, j) is an inversion.

A hyperplane arrangement cuts \mathbb{R}^n into **chambers**.

The HLSS theorem

Theorem (Hultman–Linusson–Shareshian–Sjöstrand)

The number of chambers of \mathcal{B}_w is less than or equal to the number of permutations $v \leq w$ in Bruhat order. Equality occurs if and only if w avoids 4231, 35142, 42513, and 351624.

The proof of inequality is via an injection, but *not* from the set of chambers. Rather, the injection is the **broken circuits** of the inversion arrangement, which are known to be equinumerous with the chambers by deletion-contraction.

The proof of equality relies on a complicated recursive decomposition of the class of permutations avoiding 4231, 35142, 42513, 351624.

HLSS theorem examples

If $w = 4321$, then \mathcal{B}_w is the arrangement of all hyperplanes $x_i - x_j = 0$ with $1 \leq i < j \leq 4$, also known as the **braid arrangement**. This divides \mathbb{R}^n into 24 chambers, one for each permutation (indicating the relative order on the coordinates), and all 24 permutations in S_4 are ≤ 4321

If $w = 4231$, then \mathcal{B}_w is missing the hyperplane $x_3 - x_2 = 0$. This missing hyperplane would cut through 6 chambers of \mathcal{B}_w (where $x_2, x_3 > x_1 > x_4$, et c.), so \mathcal{B}_w has 18 chambers. On the other hand, the only permutations not smaller than 4231 are 4321, 3421, 4312, and 3412, so there are 20 permutations in $[\text{id}, 4231]$.

If $w = 3412$, then \mathcal{B}_w has 4 hyperplanes $x_3 - x_1 = x_3 - x_2 = x_4 - x_1 = x_4 - x_2 = 0$. The only impossible combinations are $x_3 > x_1 > x_4 > x_2 > x_3$ and its reverse, so there are 14 chambers. There are 14 permutations in $[\text{id}, 3412]$.

The right hull property

Given a permutation w , the **right hull** of w , denoted $H(w)$ is the set of points $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ satisfying the property that there exist $i, j \in \mathbb{Z}$ with $i \leq x \leq j$ and $w(i) \geq y \geq w(j)$.

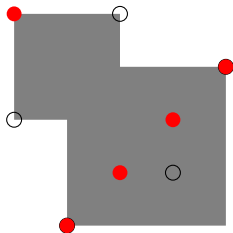
Any permutation $v \leq w$ must be contained in $H(w)$ (meaning $(i, v(i)) \in H(w)$ for all i). A permutation satisfies the **right hull property** if the reverse implication holds: $v \in H(w)$ implies $v \leq w$.

Theorem (Sjöstrand)

The permutation w satisfies the right hull property if and only if w avoids 4231, 35142, 42513, and 351624.

Right hull example

For example, 35142 does *not* have the right hull property since 15432 fits in the right hull but $15432 \not\leq 35142$



Essential sets

Fix a single $w \in S_n$ and consider the conditions on $r_{ij}(v)$ that determine whether a given $v \leq w$ in Bruhat order. By the definition, we need $r_{ij}(v) \leq r_{ij}(w)$ for all i and j , but many of these conditions will be redundant.

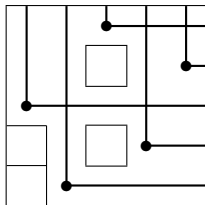
Lemma (Fulton)

Given any fixed $w \in S_n$, there is a unique minimal set of indices $\{(i_1, j_1), \dots, (i_m, j_m)\}$ such that $v \leq w$ if and only if $r_{i_k, j_k}(v) \leq r_{i_k, j_k}(w)$ for all k , $1 \leq k \leq m$.

This set is called the **essential set** of w , written $\mathcal{E}(w)$.

Calculating the essential set

We can calculate the essential set of a permutation w by a picture (here for $w = 35142$, where the essential set is $\{(3, 1), (1, 1), (3, 3)\}$):



The rank matrix is

$$\begin{array}{ccccc} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 2 & 3 & 4 \end{array}$$

Permutations defined by inclusions

A permutation w is **defined by inclusions** if $r_{ij}(w) = \min(i, j)$ for all $(i, j) \in \mathcal{E}(w)$.

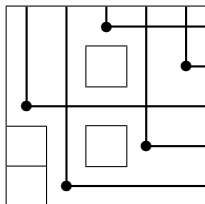
This means having $v \leq w$ is defined by conditions of the form $\{1, \dots, i_k\} \subseteq \{v(1), \dots, v(j_k)\}$ or $\{v(1), \dots, v(j_k)\} \subseteq \{1, \dots, i_k\}$.

The GR theorem

Theorem (Gasharov–Reiner)

A permutation w is defined by inclusions if and only if w avoids 4231, 35142, 42513, and 351624.

The proof is as follows. If $r_{ij} \neq \min(i, j)$ for some $(i, j) \in \mathcal{E}(w)$, then there are 1s in the permutation matrix both strictly SW and strictly NE of (i, j) . This happens if and only if w contains one of the 4 patterns.



Schubert varieties

The **flag variety** GL_n/B is the moduli space of **flags**: configurations of subspaces

$$\{0\} \subset F_1 \subset \cdots \subset F_{n-1} \subset \mathbb{C}^n$$

with $\dim F_j = j$ for all j .

Given a permutation w , the **Schubert variety** $X_w \subseteq GL_n/B$ are the points corresponding to flags with $\dim(E_i \cap F_j) \geq r_{ij}(w)$, where $E_i = \langle e_1, \dots, e_i \rangle$ and e_k is the k -th standard basis vector.

Context for Gasharov–Reiner

If w is defined by inclusions, then the Schubert variety X_w is defined by conditions of the form $F_{j_k} \subseteq E_{i_k}$ or $E_{i_k} \subseteq F_{j_k}$.

Gasharov and Reiner were studying the cohomology rings of smooth Schubert varieties, and found their theorem naturally extends to all Schubert varieties defined by inclusions.

The Schubert varieties defined by inclusions turn out to be an important subclass of the local complete intersection Schubert varieties.

Diagram varieties

Given a permutation w , let O_w be the variety of invertible matrices whose support (i.e. nonzero entries) is contained in the *complement* of $D(w)$.

Theorem (Morales–Lewis)

The number of \mathbb{F}_q points of O_w is $(q-1)^n q^{\binom{n}{2}} \sum_{v \leq w} q^{\ell(w) - \ell(v)}$ if and only if w avoids 4231, 35142, 42513, and 351624.

The proof of this depends on the HLSS theorem but is *not* positive; it relies on a recursion that involves negative signs.

They have conjectured that one always has $(q-1)^n q^{\binom{n}{2}} P(q)$ for some polynomial P with positive integer coefficients. This suggests O_w has a cell decomposition, but finding one in general seems difficult.

Finite reflection groups

A **finite reflection group** is finite subgroup of $SO(V)$ generated by reflections. The vector space V is known as the reflection representation.

The symmetric group S_n is a finite reflection group with each permutation being identified with its permutation matrix.

The reflections give us a hyperplane arrangement known as the **generalized reflection arrangement**. Pick a **fundamental chamber**; the reflections across the hyperplanes bordering the fundamental chamber are the **simple reflections**. Denote the set of simple reflections by S and the set of all reflections by T .

Bruhat order and inversion arrangements

All Coxeter groups have a notion of **length** and **Bruhat order**.

The **length** of an element w is the minimum number of elements of S required to write w .

Bruhat order is the partial order generated by the relation where $v \prec w$ if there exists $t \in T$ such that $w = tv$ and $\ell(w) > \ell(v)$.

Any element $w \in W$ has an inversion arrangement \mathcal{B}_w consisting of hyperplanes corresponding to reflections $t \in T$ with $\ell(wt) < \ell(w)$.

Bruhat graphs and absolute length

Given a Coxeter group (W, S) , the **Bruhat graph** of W is the graph whose vertices are the elements of W , with a directed edge $v \rightarrow w$ if $w = tv$ for some reflection t and $\ell(v) < \ell(w)$.

The **absolute length** $\ell_T(v, w)$ is the *undirected* distance (length of the shortest path) from v to w , and the **directed absolute length** $\ell_D(v, w)$ is the *directed* distance from v to w . By definition, $\ell_T(v, w) \leq \ell_D(v, w)$ for all $v, w \in W$.

Hultman's theorem

Theorem (Hultman)

Let (W, S) be a finite Coxeter group, and $w \in W$. The number of chambers of \mathcal{B}_w is at most the number of elements less than w in Bruhat order, with equality if and only if $\ell_T(v, w) = \ell_D(v, w)$ for all $v \leq w$.

The proof is bijective on broken circuits, but, when the condition fails, the v not satisfying the condition do not correspond to “missing” broken circuits.

The hyperoctahedral group

Let B_n denote the subgroup of S_{2n} defined by

$$B_n = \{w \in S_{2n} \mid w(i) + w(2n + 1 - i) = 2n + 1 \forall i, 1 \leq i \leq n\}.$$

This is the symmetry group of the n -dimensional cube. Label opposite facets with labels i and $2n + 1 - i$; this group writes a symmetry of the cube by its action on the facets. Embedding the cube in \mathbb{R}^n gives the reflection representation of B_n .

B_n as a finite reflection group

Picking the fundamental chamber with $0 < x_1 < \cdots < x_n$, we get simple reflections $s_0 = (n(n+1))$ and $s_i = ((n+i)(n+i+1))((n-i)(n-i-1))$ for $1 \leq i \leq n-1$. The representation of B_n as a subgroup of B_{2n} respects Bruhat order (but not length).

We can talk about the essential set of an element $w \in B_n$ by considering it as an element of S_{2n} . For $w \in B_n$, we have

$$(i, j) \in \mathcal{E}(w) \Leftrightarrow (2n - i, 2n - j) \in \mathcal{E}(w)$$

and

$$r_{i,j}(w) = \min(i, j) \Leftrightarrow r_{2n-i, 2n-j}(w) = \min(2n - i, 2n - j)$$

for all $(i, j) \in \mathcal{E}(w)$.



The modified right hull condition

Let $w \in B_n \subseteq S_{2n}$. We say that w satisfies the **modified right hull condition** if either

- ▶ w satisfies the right hull condition (as an element of S_{2n}).
- ▶ We have $r_{n,n}(w) = n - 1$, and for any $v \in B_n$, we have $v \leq w$ if and only if $v \subseteq H(w)$ AND $r_{n,n}(v) \geq n - 1$.

This means we allow the lower interval of w to be defined by not just the right hull but also this extra condition.

The Gasharov–Reiner analogue

Given $w \in B_n$, we say w is **defined by pseudo-inclusions** if $r_{i,j} = \min(i,j)$ for all $(i,j) \in \mathcal{E}(w)$ except possibly $r_{n,n} = n - 1$.

The flag variety analogues for B_n are the moduli space of isotropic flags wrt a symmetric form in \mathbb{C}^{2n+1} and the moduli space of isotropic flags wrt a skew-symmetric form in \mathbb{C}^{2n} .

A Schubert variety is defined by pseudo-inclusions if its defining conditions are of the form $E_i \subseteq F_j$, $F_j \subseteq E_i$, or $\dim(F_n \cap E_n) \geq n - 1$.

Billey–Postnikov avoidance for B_n

Therefore, an element $w \in B_n$ **contains**

- ▶ $v \in S_m$ if there exist indices $1 \leq a_1 < \cdots < a_m \leq 2n$, with $a_i + a_j \neq 2n + 1$ for all i, j , such that $w(a_i) < w(a_j) \Leftrightarrow v(i) < v(j)$
- ▶ $v \in B_m$ if there exist indices $1 \leq b_1 < \cdots < b_m < b_{m+1} < \cdots < b_{2m}$, with $b_i + b_{2m+1-i} = 2n + 1$ for all i , such that $w(b_i) < w(b_j) \Leftrightarrow v(i) < v(j)$.

Note a permutation v can be thought of either as living in S_m or in $S_m \subseteq B_m$. This gives rise to different avoidance classes.

Hultman elements for B_n

Theorem

Let $w \in B_n$. Then the following are equivalent:

- ▶ The number of chambers of \mathcal{B}_w is equal to the number of elements $v \in B_m$ with $v \leq w$.
- ▶ For all $v \leq w$, $\ell_T(v, w) = \ell_D(v, w)$.
- ▶ w satisfies the modified right hull condition.
- ▶ w is defined by pseudo-inclusions.
- ▶ w avoids 4231, 35142, 42513, 351624 (thought of as elements of S_m), 10 elements of B_3 , 14 elements of B_4 , and 3 elements of B_5 .

Sketch of proof

We follow the analogous proof of Hultman for S_n .

First we show that, if w satisfies the modified right hull condition, then given any $v < w$, there exists a reflection t such that $v < tv \leq w$, with $\ell_T(tv, w) = \ell_T(v, w) - 1$. We use the fact that any reflection t fixing the fixed subspace of $v^{-1}w$ satisfies $\ell_T(tv, w) = \ell_T(v, w) - 1$, and the modified right hull condition allows us to find such a reflection with $v < tv \leq w$. This shows by induction that $\ell_T(v, w) = \ell_D(v, w)$.

Given w not satisfying the modified right hull condition, we show it is not defined by pseudo-inclusions. We modify the proof of Gasharov–Reiner to show w must contain one of the 31 patterns. None of the 31 patterns satisfy $\ell_T(v, w) = \ell_D(v, w)$ for all $v \leq w$, and this implies w cannot either.

Questions

- ▶ Other finite reflection groups? I have a working hypothesis about the correct abstract analogue of being defined by inclusions.
- ▶ Enumeration? The enumeration for S_n was done by Albert and Brignall and refined several techniques for enumerating pattern classes.
- ▶ Local Schubert classes for Hultman elements? Precisely the ones that are products of roots (possibly with multiplicity)?
- ▶ Presentation of cohomology groups?
- ▶ Rational smoothness = defined by inclusions + ??
- ▶ Analogue for diagram varieties?

Thank you

Thank you for your attention!