APOLARITY AND REFLECTION GROUPS

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Abstract. We determine the Waring rank of the fundamental skew invariant of any complex reflection group whose highest degree is a regular number. This includes all real reflection groups.

Given a homogeneous polynomial f of degree d, the **Waring rank** of f, denoted $r(f)$, is the smallest positive integer r such that there exist linear forms $\ell_1, \dots, \ell_r$ with $f = \ell_1^d + \cdots + \ell_r^d$. For example, $xy = \frac{1}{4}((x+y)^2 - (x-y)^2)$, so $r(xy) \le 2$, and

$$xyz = \frac{1}{24}\Big((x+y+z)^3 - (x+y-z)^3 - (x-y+z)^3 + (x-y-z)^3\Big),$$

so $r(xyz) \le 4$. Waring ranks are notoriously difficult to determine. Even over 160 years after Sylvester's work [Syl51] studying this question, Waring ranks are known only for a few families and particular cases of examples. For a quadratic form, the rank is equal to the rank of the associated symmetric matrix. Ranks of binary forms have been known since the 19th century, see [Syl51, CS11]. Ranks of plane cubics such as xyz have been determined by different methods several times, such as by [Yer32] in the 1930s, [CM96], and [LT10]. From these results, one can see that that in fact $r(xy) = 2$ and $r(xyz) = 4$.

A theorem of Alexander and Hirschowitz [AH95] gives the rank of a general polynomial of degree d in n variables. However, it is not known how to determine whether a given form is indeed general, nor is it known in general how to determine the rank of a given form with any reasonable efficiency. Hence there has been a body of recent work determining the rank of certain polynomials of special interest. Some specific ranks were determined in [LT10], such as $r(x(y_1^2 + \cdots + y_m^2)) = r(x(y_1^2 + \cdots + y_m^2 + x^2)) = 2m$, $r(x^2yz) = 6$, and $r(xyzw) = 8$. A result of Ranestad and Schreyer [RS11] gives a lower bound for rank which turns out to be tight for monomials of the form $(x_1 \cdots x_n)^d$; they have rank $(d+1)^{n-1}$. More generally, the Waring ranks of arbitrary monomials were found by Carlini, Catalisano, and Geramita [CCG12]: $r(x_1^{a_1} \cdots x_n^{a_n}) = (a_2+1)\cdots(a_n+1)$ when $a_1 \le \cdots \le a_n$. They also found the ranks of sums of pairwise coprime monomials. Among these examples are a few sporadic cases of polynomials having greater rank than that of a general polynomial with the same number of variables and degree; see, for example, [CCG12, Section 4.1], [LT10, Remark 7.3].

One class of polynomials of great interest, generalizing the case of monomials, is the class of products of linear forms, possibly with multiplicities. Geometrically, the vanishing loci of these polynomials are hyperplane (multi)-arrangements. Even in this case, the only products of linear forms for which the Waring rank was previously known are monomials and binary forms.

In this paper, we treat the case where our hyperplane arrangement is the multi-arrangement associated to a complex reflection group satisfying the technical hypothesis that the highest

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degree is a regular number (see section 1.3). In particular, this includes all real reflection arrangements as well as monomials of the form $(x_1 \cdots x_n)^d$. As we recall below, the defining equation of the arrangement is the fundamental skew invariant of the reflection group. For example, the symmetric group S_n acts on $\mathbb{C}^n$ as a reflection group with the transposition $t_{i,j}$ acting by reflecting across the hyperplane $H_{i,j}$ defined by $x_i - x_j = 0$. The defining equation of the union of these hyperplanes is $V_n = \prod_{i<j}(x_i - x_j)$, the classical Vandermonde determinant. This is a form of degree $\binom{n}{2}$ in n variables. We show that $r(V_n) = (n-1)!$.

We briefly sketch our proof here for the case of the Vandermonde determinant. To obtain an expression for V_n as a sum of powers of linear forms, one may start naively with a general linear form ℓ and skew-symmetrize the power $\ell^{\binom{n}{2}}$ to give an expression of V_n as a signed sum of powers involving $|S_n| = n!$ terms. We improve this by choosing ℓ to be a regular eigenvector of a Coxeter element of S_n, namely a linear form with the n-th roots of unity as its coefficients. With this choice, each term in the expression of V_n as a sum of $n!$ powers turns out to be repeated n times. This reduces the number of distinct terms to $(n-1)!$, so $r(V_n) \leq (n-1)!$. A lower bound for $r(V_n)$ is given by combining the Ranestad–Schreyer result with well-known facts about the invariant theory of S_n, yielding $r(V_n) \geq (n-1)!$.

The same strategy works for other real reflection groups and for complex reflection groups where the largest degree is a regular number. Our main theorem is the following.

Theorem 1. *Let W be a finite complex reflection group acting on $\mathbb{C}^n$ with degrees $d_1 \leq \cdots \leq d_n$. Suppose that the largest degree d_n of W is a regular number. (In particular, this holds for any W that is a real reflection group.) Let f be the defining equation of the reflection multi-arrangement for W. Then $r(f) = d_1 \cdots d_{n-1} = |W|/d_n$.*

In general, when our hypothesis that d_n is a regular number fails, a weaker result holds; for any finite complex reflection group W with degrees $d_1 \leq \cdots \leq d_n$ and with greatest regular number D, $d_1 \cdots d_{n-1} = |W|/d_n \leq r(f) \leq |W|/D$.

Other notions of rank are of interest, including cactus rank, smoothable rank, and border rank, see [BBM12]. Since the lower bound of Ranestad and Schreyer also applies to cactus rank and smoothable rank, our theorem determines these values as well and shows that they are equal to the rank. We are not, however, able to determine the border ranks of the forms that we consider.

We discuss background both on the aforementioned theorem of Ranestad and Schreyer and on complex reflection groups and their skew invariants in Section 1. The proof of our theorem is given in Section 2, and Section 3 discusses specific examples of reflection groups.

1. Background

1.1. **The Ranestad–Schreyer lower bound.** Now we recall the theorem of Ranestad and Schreyer giving a lower bound for the rank of a homogeneous polynomial f in n variables.

Consider the polynomial ring $S = \mathbb{C}[x_1, \ldots, x_n]$. The ring $T = \mathbb{C}[\partial_1, \ldots, \partial_n]$ acts on S by differentiation; explicitly, given nonnegative integer vectors $\alpha, \beta \in \mathbb{N}^n$,

$$\partial^\alpha(\mathbf{x}^\beta) = \left(\prod_{i=1}^{n} \alpha_i! \binom{\beta_i}{\alpha_i}\right) \mathbf{x}^{\beta-\alpha}$$

if $\beta - \alpha$ is a nonnegative vector, and $\partial^\alpha(\mathbf{x}^\beta) = 0$ otherwise, with the action extending $\mathbb{C}$-linearly. This gives rise to the **apolar pairing** between S and T. Letting S_k, T_k denote the homogeneous pieces of degree k, this gives a perfect pairing between S_k and T_k for each

k, and a map $S_d \otimes T_a \to S_{d-a}$ whenever $d \geq a$, or for all d, a with the understanding that $S_k = T_k = 0$ when $k < 0$. This makes S a T-module; it is a graded T-module when the grading is reversed (that is, when each S_k is given degree $-k$ rather than k).

Given $f \in S$, define

$$f^\perp = \{D \in T \mid D(f) = 0\}.$$

Then $f^\perp$ is an ideal of T, called the **apolar ideal** of f. The ring $A^f := T/f^\perp$ is a zero-dimensional Gorenstein ring, which is known as the **apolar Artinian Gorenstein ring** for f. If f is a homogeneous polynomial, the ideal $f^\perp$ is homogeneous and A^f is graded. As a T-module, A^f is isomorphic to the T-submodule of S generated by f (but note that the grading is reversed).

A homogeneous ideal I is said to be **generated in degree (at most)** δ if there exist polynomials $g_1, \ldots, g_k$, all of degree at most δ, such that $I = \langle g_1, \ldots, g_k \rangle$. The theorem of Ranestad and Schreyer is as follows.

Theorem 2 ([RS11])**.** *Let $f \in S$ be a homogeneous polynomial, and suppose $f^\perp$ is generated in degree δ. Then*

$$r(f) \geq \frac{\dim_{\mathbb{C}} A^f}{\delta}.$$

The proof of this theorem relies on the well-known Apolarity Lemma [IK99, Lemma 1.31] which states that f can be written as a sum of s powers of linear forms if and only if there is an ideal $I \subseteq f^\perp$ which is the homogeneous defining ideal of a set of s distinct points in projective space.

1.2. **Reflection groups and their invariant theory.** We now give a brief overview of the invariant theory of complex reflection groups. Two recent books on this subject are those of Lehrer and Taylor [LT09] and Kane [Kan01]. A classic reference, addressing only the case of real reflection groups, is that of Humphreys [Hum90]. The statements of this section are originally due to Steinberg [Ste64] and Chevalley [Che55].

Let V be a $\mathbb{C}$-vector space of dimension n. A non-identity element $t \in \mathrm{GL}(V)$ is called a **pseudo-reflection** if it fixes a hyperplane $H_t \in V$ (so it has the eigenvalue 1 with geometric multiplicity $n-1$) and has one eigenvalue which is not equal to 1. This exceptional eigenvalue must be a k_t-th root of unity where k_t is the order of t. If $k_t = 2$, then t is a **(real) reflection**. A **complex reflection group** W is a finite group generated by pseudo-reflections. If we have only real reflections, then W is a **real reflection group**. We can now change viewpoints and consider W as an abstract group equipped with a distinguished represention V; from this viewpoint V is known as the **reflection representation**.

We identify $\mathrm{Sym}(V)$ with the polynomial ring $S = \mathbb{C}[x_1, \ldots, x_n]$ by choosing a basis $x_1, \ldots, x_n$ of V. (Note this is an unusual choice of convention; S is usually identified with $\mathrm{Sym}(V^*)$, but we find it more convenient for our purposes to identify T with $\mathrm{Sym}(V^*)$ and hence S with $\mathrm{Sym}(V)$. In the case of a real reflection group, $V \cong V^*$, so this choice of convention is irrelevant.) For each reflection $t \in W$, let $L_t \in V$ be an exceptional eigenvector of t, an eigenvector with eigenvalue $\neq 1$. Now let $f_W = \prod L_t$, where the product is over all reflections in W. (Note that this will usually include more reflections than are necessary to generate W.) Let $H_t \subset V^*$ be the reflecting hyperplane of t in the adjoint action of W on V^*. Note that $H_t \subset V^*$ is defined by the linear equation L_t. Two reflections s, t satisfy $H_s = H_t$ if and only if $L_s = L_t$ up to a scalar multiple, which occurs if and only if s and t lie in a cyclic subgroup generated by a reflection. Therefore, up to a scalar multiple,

$f_W = \prod L_H^{k_H - 1}$, where the product is over the hyperplanes H fixed by reflections in W, L_H is the complementary exceptional eigenvector of any reflection $t \in W$ fixing H, and k_H is the order of the cyclic subgroup of reflections fixing H. Thus f_W is the defining equation of the reflection multi-arrangement of W in V^*. (Of course, f_W is only defined up to multiplication by a nonzero constant, but this will be irrelevant for our purposes.)

We say that a polynomial $p \in \mathrm{Sym}(V)$ is **skew invariant** (for W) if $g \cdot p = (\det g)^{-1} p$ for every $g \in W$. (We take the determinant of g by considering it as an element of $GL(V)$.) The polynomial f_W divides every skew invariant polynomial in $S = \mathrm{Sym}(V)$, and we call f_W the **fundamental skew invariant** [Kan01, Prop. 20-1B] [LT09, Lemma 9.10]. Let $D = \deg f = \sum k_H - 1$ where the sum is over all reflecting hyperplanes H; this is the number of reflections in W. As f_W divides every skew invariant polynomial, the only skew invariant polynomial of degree less than D is 0, and every skew invariant polynomial of degree D must be a constant multiple of f_W. (Note that most sources consider instead elements of $\mathrm{Sym}(V^*)$ on which W acts by det, so they regard f_W as the element of $\mathrm{Sym}(V^*)$ defining the reflection multi-arrangment in V.)

A classical theorem, proved on a case by case basis by Shephard and Todd [ST54] and uniformly by Chevalley [Che55], states that the subring $S^W \subset S$ of W-invariant polynomials is itself a polynomial ring, generated by a set of homogeneous and algebraically independent invariants. These invariants are not uniquely determined, but their degrees are. These degrees will be denoted $d_1, \ldots, d_n$, with $d_1 \leq \cdots \leq d_n$; they are known as the **degrees** of the complex reflection group W. The product of the degrees turns out to be the order of the group W; in notation, $\prod_{i=1}^n d_i = |W|$. The degrees satisfy $\sum_{i=1}^n (d_i - 1) = D$.

A **coinvariant** of W is a homogeneous element of T which is invariant under the action of W (induced by the adjoint action on V^*). Let J_W be the ideal generated by all positive-degree coinvariants in T. Then J_W is generated as an ideal by the generators of the coinvariant ring, and these have the same degrees $d_1, \ldots, d_n$ as the basic invariants in S. A theorem of Steinberg [Ste64] states that $J_W = f_W^\perp$ (see also [Kan01, Chapter 26] or [LT09, Lemma 9.36]) so the apolar ring A^{f_W} for the fundamental skew invariant f_W is isomorphic to the **covariant ring** T/J_W both as a graded ring and as a representation of W. Since J_W is a complete intersection ideal generated in degrees $d_1, \ldots, d_n$, $\dim_{\mathbb{C}} T/J_W = \prod_{i=1}^n d_i = |W|$. Thus we have the following corollary to Theorem 2.

Corollary 3. *Let W be a complex reflection group, f_W its fundamental skew invariant, and $d_1 \leq \cdots \leq d_n$ its degrees. Then*

$$r(f_W) \geq \frac{d_1 \cdots d_n}{d_n} = d_1 \cdots d_{n-1} = \frac{|W|}{d_n}.$$

1.3. **Regular numbers.** A vector $v \in V$, where V is the reflection representation, is called a **regular vector** if v does not lie on any reflecting hyperplane. A group element $g \in W$ is called a **regular element** if g has a regular eigenvector. A positive integer d is called a **regular number** if there exists a regular element having a regular eigenvector of eigenvalue $e^{2\pi i/d}$. These definitions are originally due to Springer [Spr74].

The remainder of this section is not necessary for our theorem or its proof but elaborates on our technical hypothesis requiring that the highest degree is a regular number. In particular, we describe some cases in which it holds.

For a real reflection group, it was classically known (even before Springer defined the concept) that d_n is a regular number. Indeed, pick any set of n reflections $t_1, \ldots, t_n$ such that the linear forms $L_{t_1}, \dots, L_{t_n}$ span V. Then the element $t_1 t_2 \cdots t_n \in W$ is known as a **Coxeter element** (and all Coxeter elements are conjugate). The Coxeter element always has order d_n, and there is a real plane in V, usually known as the **Coxeter plane**, on which it acts by rotation by $2\pi/d_n$; hence it has an eigenvector of eigenvalue $e^{2\pi i/d_n}$ in the complexification of the Coxeter plane. This eigenvector does not lie on a reflecting hyperplane, because the intersection of any reflecting hyperplane with the real Coxeter plane is a real line, so the reflecting hyperplane cannot contain the complex line containing the complex eigenvector. See [Hum90, Chap. 3] for details.

For an *irreducible* complex reflection group, the largest degree is a regular number whenever the group is **well-generated**, which means there are some $n = \dim V$ complex reflections $t_1, \dots, t_n$ which generate the group. See section 2.2 of [BR11] for more details.

We now describe a more general criterion due to Lehrer and Springer [LS99] (later given a case-free proof by Lehrer and Michel [LM03]) for a number to be regular, see also [LT09, Chapter 11.4.2]. Consider the action of W on the covariant ring $\mathrm{Sym}(V^*)/J_W$. Let A_i be the degree i graded piece of $\mathrm{Sym}(V^*)/J_W$. Given any representation U of W, we define the **fake degree** of U to be the generating function

$$g_U(t) = \sum_{i \geq 0} (\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}W}(A_i, U)) t^i.$$

We can write $g_U(t)$ in the form

$$g_U(t) = t^{e_1(U)} + \cdots + t^{e_r(U)},$$

with $e_1(U) \leq e_2(U) \leq \cdots \leq e_r(U)$. The numbers $e_1(U), \dots, e_r(U)$ are known as the **U-exponents**. In the case where $U = V$ is the reflection representation, these are known simply as the **exponents**, and it is classically known that $e_i = d_i - 1$ and hence that $\sum_{i=1}^n e_n$ is the degree of f_W. In the case $U = V^*$, the V^*-exponents are known as the **coexponents**, and the numbers $d_i^* = e_i(V^*) - 1$ are the **codegrees**.

Now the criterion of Lehrer and Springer is as follows.

Theorem 4 ([LS99])**.** *A number d is regular if and only if the number of degrees that d divides is the same as the number of codegrees d divides, or, in notation,*

$$\#\{i : d | d_i\} = \#\{i : d | d_i^*\}.$$

Note that $d_1^* = 0$, since A_1, the degree 1 piece of $\mathrm{Sym}(V^*)/J_W$, is V^*. Hence, every number divides at least one codegree, so every regular number divides at least one degree, and in particular no regular number exceeds the largest degree d_n.

2. Proof of Main Theorem

Now we give upper bounds for $r(f_W)$ by giving expressions for f_W as sums of powers of linear forms.

For a polynomial $p \in S = \mathrm{Sym}(V)$ the **skew-symmetrization** of p is

$$\mathrm{alt}(p) := \frac{1}{|W|} \sum_{w \in W} (\det w)^{-1} (w \cdot p). \tag{1}$$

Note that for $w \in W$, $w \cdot \mathrm{alt}(p) = (\det w)^{-1} \mathrm{alt}(p)$, so $\mathrm{alt}(p)$ is skew invariant for any p.

Recall that if $p \in S$ is a skew invariant polynomial, then f_W divides p. In particular, if $D = \sum_{i=1}^n (d_i - 1)$ is the degree of f_W, and p has degree D, then p must be a (possibly zero) multiple of f_W. Therefore, given any linear form $L \in V$, we see that $\mathrm{alt}(L^D)$ is a (possibly zero) scalar multiple of f_W.

We now show the following lemma.

Lemma 5. With D as above, the polynomial $\mathrm{alt}(L^D)$ is zero if and only if L lies on a reflecting hyperplane in V.

Proof. Let $\mathbb{C}_{\det^{-1}}$ be the rank one representation of W on which $w \in W$ acts as $(\det w)^{-1}$, where the determinant is given by the determinant of w in its action on the reflection representation V. Then the skew-symmetrization operator alt is a $\mathbb{C}W$-linear map $\mathrm{alt} : \mathrm{Sym}^D(V) \to \mathbb{C}_{\det^{-1}}$, where $\mathbb{C}_{\det^{-1}}$ is identified with the span of f_W. We can tensor this map by the determinant representation $\mathbb{C}_{\det}$ to get a map $\mathrm{alt} \otimes \mathbb{C}_{\det} : \mathrm{Sym}^D(V) \otimes \mathbb{C}_{\det} \to \mathbb{C}$, which we can identify with an invariant element of $\mathrm{Sym}^D(V^*) \otimes \mathbb{C}_{\det^{-1}}$, or equivalently an element of $\mathrm{Sym}^D(V^*)$ on which W acts by det.

Just as f_W divides every element of $\mathrm{Sym}(V)$ on which W acts by $\det^{-1}$, we have a corresponding element $f_W^* \in \mathrm{Sym}(V^*)$ (defined only up to a nonzero scalar multiple), vanishing precisely on the reflecting hyperplanes $H_t^* \subset V$, each to order $k_t - 1$, that divides every element of $\mathrm{Sym}(V^*)$ on which W acts by det.

The operator alt is clearly nonzero, since $\mathrm{alt}(p) = p$ for the polynomials p on which $w \cdot p = (\det w)^{-1} p$ for all $w \in W$, in particular for $p = f_W$. Hence, regarded as a tensor, alt is a nonzero multiple of f_W^*, so $\mathrm{alt}(L^D)$ is zero if and only if the tensor f_W^* vanishes on L^D, if and only if the polynomial f_W^* vanishes on L, if and only if L lies on a reflecting hyperplane. $\square$

Now suppose L is an eigenvector of $w \in W$ with eigenvalue $e^{\frac{2\pi i}{d}}$. Let $C_w \subset W$ be the subgroup generated by w, and let W/C_w denote the collection of its left cosets. Note $|C_w|$ is some multiple of d and hence $|W/C_w| \leq |W|/d$. Picking any representative σ for each coset in W/C_w, we have

$$\begin{aligned}
\mathrm{alt}(L^D) &= \frac{1}{|W|} \sum_{\sigma \in W/C_w} (\det \sigma)^{-1} \left(\sigma \cdot \sum_{v \in C_w} (\det v)^{-1} (v \cdot L^D) \right) \\
&= \frac{1}{|W|} \sum_{\sigma \in W/C_w} (\det \sigma)^{-1} \left(\sigma \cdot \sum_{j=0}^{|C_w|-1} (\det w)^{-j} e^{2\pi i D j/d} L^D \right) \\
&= \frac{1}{|W|} \left(\sum_{j=0}^{|C_w|-1} (\det w)^{-j} e^{2\pi i D j/d} \right) \sum_{\sigma \in W/C_w} (\det \sigma)^{-1} (\sigma \cdot L^D),
\end{aligned}$$

which is a sum of at most $|W/C_w| \leq |W|/d$ powers of linear forms. Hence we have shown the following proposition.

Proposition 6. *Let W be a complex reflection group, f_W its fundamental skew invariant, and d a regular number for W. Then*

$$r(f_W) \leq \frac{|W|}{d}.$$

Combining Proposition 6 and Corollary 3, we have shown our main theorem:

Theorem 7. *Let W be a finite complex reflection group acting on $\mathbb{C}^n$, $d_1 \leq \cdots \leq d_n$ the degrees of W, and suppose that d_n is a regular number. Then $r(f_W) = |W|/d_n$.*

3. Examples

3.1. **The Vandermonde determinant.** The group $W = S_n$ acts on $\mathbb{C}^n$ by permuting coordinates. It is a real reflection group with reflection hyperplanes defined by the polynomials $x_i - x_j$ for $i \neq j$; hence its fundamental skew invariant is the Vandermonde determinant $f_{S_n} = \prod_{i<j}(x_i - x_j)$, of degree $\binom{n}{2}$. The degrees are $2, 3, \ldots, n$, and since S_n is a real reflection group, the highest degree n is a regular number. Explicitly, the linear form $L = \sum_{j=1}^{n} e^{2\pi i(j-1)/n} x_j$ is a regular element for which the n-cycle $(12\cdots n)$ acts by multiplication by $e^{2\pi i/n}$.

This implies that the rank of the Vandermonde determinant is $(n-1)!$, with an explicit expression given by

$$\begin{aligned}\prod_{i<j}(x_i - x_j) &= \sum_{\sigma \in S_n/C_n} \frac{1}{(n-1)!} \operatorname{sgn}(\sigma)(\sigma \cdot L)^{\binom{n}{2}} \\ &= \frac{1}{(n-1)!} \sum_{\sigma \in S_n/C_n} \operatorname{sgn}(\sigma) \left(\sum_{j=1}^{n} e^{2\pi i(j-1)/n} x_{\sigma(j)} \right)^{\binom{n}{2}},\end{aligned}$$

where C_n is the cyclic subgroup generated by $(12\cdots n)$ and σ denotes a choice of coset representatives in S_n/C_n. For example, the sum can be taken over permutations σ such that $\sigma(1) = 1$.

By the Alexander–Hirschowitz theorem, the rank of a general form of degree $\binom{n}{2}$ in n variables is

$$\left\lceil \frac{1}{n} \binom{\binom{n}{2} + n - 1}{n-1} \right\rceil$$

which is greater than $(n-1)!$ for $n > 1$. Thus the Vandermonde determinant f_{S_n} has less than general rank, and in fact the ratio of $r(f_{S_n})$ to the general rank approaches zero as n goes to infinity.

3.2. **Types B and D.** The group B_n of order $2^n n!$ acts on $\mathbb{C}^n$, and its fundamental skew invariant is $f_{B_n} = \prod_{i=1}^{n} x_i \prod_{i<j}(x_i - x_j)(x_i + x_j)$, of degree $n + 2\binom{n}{2} = n^2$. Its highest degree is $2n$. In the realization of B_n as signed permutations, one choice of regular element of order $2n$ is the signed permutation $v \in B_n$ with $v(i) = i + 1$ for $1 \leq i < n$ and $v(n) = \overline{1}$, so in its action on $\mathbb{C}^n$, $v \cdot x_i = x_{i+1}$ for $i < n$ and $v \cdot x_n = -x_1$. The linear form $L = \sum_{j=1}^{n} e^{2\pi i(j-1)/2n} x_j$ is a regular form on which v acts by multiplication by $e^{2\pi i/2n}$.

Thus the rank $r(f_{B_n}) = 2^{n-1}(n-1)!$, and an explicit expression is given by

$$\prod_{i=1}^{n} x_i \prod_{i<j}(x_i - x_j)(x_i + x_j) = \frac{1}{2^{n-1}(n-1)!} \sum_{\sigma \in B_n/C_n} \operatorname{sgn}(\sigma)(\sigma \cdot L)^{n^2},$$

where C_n is the cyclic subgroup generated by v and the σ are coset representatives for B_n/C_n.

The group D_n of order $2^{n-1}n!$ acts on $\mathbb{C}^n$ as a real reflection group with fundamental skew invariant $f_{D_n} = \prod_{i<j}(x_i - x_j)(x_i + x_j)$, of degree $2\binom{n}{2} = n(n-1)$. Its highest degree (assuming $n \geq 3$) is $2n - 2$. Thus $r(f_{D_n}) = 2^{n-2}(n-1)!$. One choice of regular element of order $2n-2$ is the signed permutation $v \in D_n$ with $v(1) = -1$, $v(i) = i+1$ for $2 \leq i \leq n-1$, and $v(n) = -2$, so in its action on $\mathbb{C}^n$, $v \cdot x_1 = -x_1$, $v \cdot x_i = x_{i+1}$ when $2 \leq i \leq n-1$, and $v \cdot x_n = -x_2$. The linear form $L = \sum_{j=2}^{n} e^{2\pi i(j-2)/(2n-2)} x_j$ is a regular form for which v acts by multiplication by $e^{2\pi i/(2n-2)}$.

By the Alexander–Hirschowitz theorem, the rank of a general form of degree n^2, respectively $n(n-1)$, in n variables is

$$\left\lceil \frac{1}{n}\binom{n^2+n-1}{n-1} \right\rceil \qquad \text{respectively} \qquad \left\lceil \frac{1}{n}\binom{n(n-1)+n-1}{n-1} \right\rceil,$$

and these are greater than $2^{n-1}(n-1)!$, respectively $2^{n-2}(n-1)!$. Again f_{B_n} and f_{D_n} have less than general rank, and ratios of their ranks to the general ranks go to zero.

3.3. **Exceptional groups.** We list the fundamental skew invariant, its degree, and its rank for the exceptional real reflection groups E_8, E_7, E_6, and F_4; a similar calculation is possible for H_3 and H_4. These calculations can also be done for the dihedral groups $I_2(m)$ (including $G_2 = I_2(6)$), but these are binary forms and hence their ranks are known due to the work of Sylvester.

First we simply list the fundamental skew invariants of these groups:

W	f_W
E_8	$\displaystyle\prod_{1\le i<j\le 8}(x_i-x_j)(x_i+x_j)\prod_{\substack{\lambda_i=\pm1\\ \prod\lambda_i=1\\ \lambda_1=1}}\sum_{i=1}^{8}\lambda_i x_i$
E_7	$\displaystyle\prod_{2\le i<j\le 7}(x_i-x_j)(x_i+x_j)(x_1-x_8)\prod_{\substack{\lambda_i=\pm1\\ \prod\lambda_i=1\\ \lambda_1=\lambda_8=1}}\sum_{i=1}^{8}\lambda_i x_i$
E_6	$\displaystyle\prod_{3\le i<j\le 7}(x_i-x_j)(x_i+x_j)\prod_{\substack{\lambda_i=\pm1\\ \prod\lambda_i=1\\ \lambda_1=\lambda_2=\lambda_8=1}}\sum_{i=1}^{8}\lambda_i x_i$
F_4	$x_1x_2x_3x_4(x_1+x_2+x_3+x_4)\prod_{1\le i<j\le 4}(x_i-x_j)(x_i+x_j)$

Now we list, for each group, the number of variables, degree, and rank of its fundamental skew invariant, along with the general rank for forms in that number of variables and of that degree:

W	variables	$\deg(f_W)$	$r(f_W)$	general rank
E_8	8	120	$64\cdot 9!$	11169551972
E_7	8	63	$4\cdot 8!$	149846840
E_6	8	36	$6\cdot 6!$	4028015
F_4	4	24	96	732

Clearly, in the cases listed above, each fundamental skew invariant has rank far smaller than the general rank.

3.4. **Monomials.** The monomial $x_1^{a_1}\cdots x_n^{a_n}$ is the fundamental skew invariant for the reflection group $\mathbb{Z}/(a_1+1)\mathbb{Z}\times\cdots\times\mathbb{Z}/(a_n+1)\mathbb{Z}$, where the generator of the j-th component $\mathbb{Z}/(a_j+1)/\mathbb{Z}$ acts on x_j by multiplication by $e^{-2\pi i/(a_j+1)}$.

The degrees for this group are $d_i = a_i+1$. However, all of the codegrees are 0; since the representation V^* decomposes into n distinct 1-dimensional representations, we have $\dim_{\mathbb{C}}\operatorname{Hom}(V^*,\operatorname{Sym}^1(V^*)) = \dim_{\mathbb{C}}\operatorname{Hom}(V^*,V^*) = n$, so all the coexponents are 1. Therefore, according to Theorem 4, the only case where the largest degree is a regular number is when all the a_i, and hence all the degrees, are equal. In this case we recover the observation of Ranestad and Schreyer that $r((x_1 \cdots x_n)^d) = (d+1)^{n-1}$ [RS11].

For other monomials, the lower bound of Ranestad and Schreyer is not tight, and in most cases our upper bound also is not tight. Indeed, as mentioned in the introduction, Carlini, Catalisano, and Geramita show that $r(x_1^{a_1} \cdots x_n^{a_n}) = (a_2+1)\cdots(a_n+1)$ when $a_1 \leq \cdots \leq a_n$ [CCG12]. The highest degree is $a_n + 1$, so the Ranestad-Schreyer lower bound gives only $r(x_1^{a_1} \cdots x_n^{a_n}) \geq (a_1+1)\cdots(a_{n-1}+1)$. Regarding upper bounds, by Theorem 4, the largest regular number is the $\gcd(a_1+1, \ldots, a_n+1)$, so our construction in Proposition 6 gives a tight upper bound only in the case $a_1 + 1$ divides $a_j + 1$ for all j.

See [CCG12, Section 4.1] for a comparison of the ranks of monomials with general ranks.

It is interesting to note that the explicit expression for $x_1^{a_1} \cdots x_n^{a_n}$ as a sum of powers given in [BBT13] (which is not the only such expression) is exactly the skew-symmetrization of $(x_1 + \cdots + x_n)^D$, $D = a_1 + \cdots + a_n$, over the subgroup $\{1\} \times \mathbb{Z}/(a_2+1)\mathbb{Z} \times \cdots \times \mathbb{Z}/(a_n+1)\mathbb{Z}$ rather than over the whole group. Furthermore $x_1 + \cdots + x_n$ is not necessarily an eigenvector of any element in the group, depending on whether the $a_i + 1$ have a common divisor.

The proof of Theorem 4 by Lehrer and Michel [LM03] and the determination of the ranks of monomials in [CCG12, BBT13] both involve the algebraic geometry of the covariant ring, so we speculate that combining these ideas appropriately could lead to a method to determine the rank of the fundamental skew invariant for an arbitrary complex reflection group even in the case where the largest degree is not a regular number.

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