Electric Power and Controls

Copper pipe

Fan

Sensor

Heater
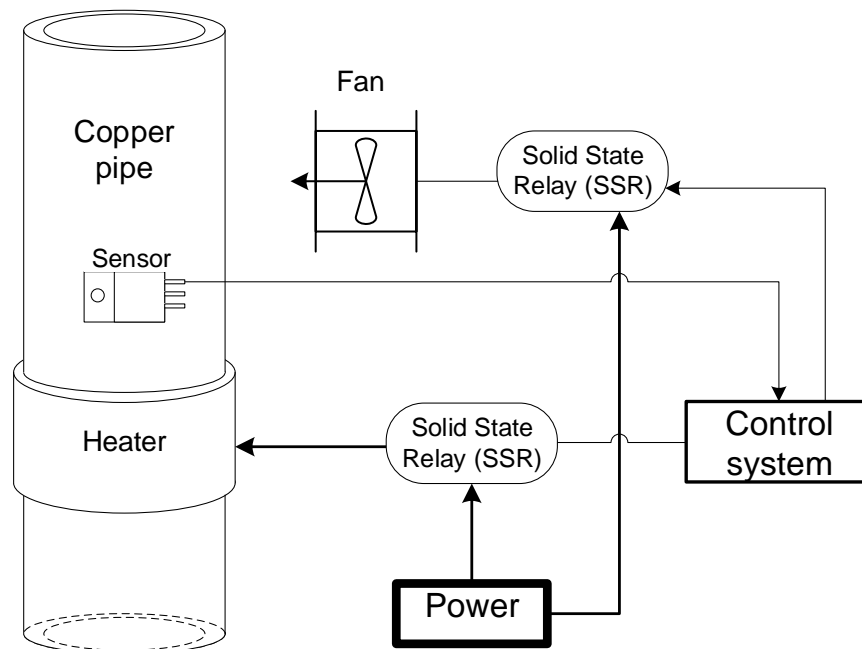
Solid State Relay (SSR)

Solid State Relay (SSR)

Control system

Power

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Electric Power and Controls

This Electric Power and Controls handout will give students an understanding of electric power and control systems applied to biological engineering. The course covers electrical system design and implementation following the National Electrical Code, understanding of electric motors (DC, induction, stepper, and servo), motor controls systems, and programmable logic controllers (PLCs). This course will give students a solid understanding of controls theory and be able to use industrial-grade PLC in their subject areas.

This handout is divided into three distinct sections. After the first section, the students will understand the household and industrial power supply and be able to design a branch circuit with various electrical component following the National Electric Code. The students will gain hands-on experience in the lab to design and put a branch circuit with both 120 and 240V household power supply.

The second portion of this handout is devoted to the electric motors. The students will learn how an electric motor runs, types of motors, advantages, and limitations of using each type of motor and proper motor selection.

The third portion of the course is devoted to controls. Students will learn what does a control system do, how to model a system, and how to tune a PID controller to achieve the desired performance.

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Chapter 1 Electricity Basics

Current

Current is the rate of flow of electrical charges.

$$I = \frac{dQ}{dt}$$

Q is electrical charge in Coulombs[1] and t is the time is seconds. The current is in Amperes (1A=1Coulomb/s).

Example 1.1

If 0.2 Coulombs of charge has moved from a cross-section of wire in 0.1 seconds, what is the average current flow during that period?

Solution: Given: dQ = 0.2 C. dt = 0.1 s, I = dQ/dt = 2 A.

Example 1.2

One mole of a substance has $6.02\text{x}10^{23}$ molecules. If a mammalian cell is 2000 μm³ in volume with four mM (milli Molarity[2]) of sodium (Na+) concentration. Find electrical charges in Coulombs inside the cell from sodium.

Solution:

Cell Volume in L =

$$2000 \times 10^{-18} \times 1000L = 2 \times 10^{-12}L$$

Number of moles = Cell Volume in L *
Concentration = $2 \times 10^{-12} \times 4 \times 10^{-3}$

$$= 8 \times 10^{-15}M$$

Charge = $\frac{Number\ of\ moles\ *\ 6.02x10^{23}}{6.25 \times 10^{18}}$

$$= 7.7 \times 10^{-10}C$$

Note that the charge is positive because the sodium ions are positive. The flow of either positive or negative charge produces electric current except for their conventional direction of flow would be opposite.

Measuring Current

Current is measured as shown in *Figure 1.1*. There are typically two types of multimeters, In-line type, and clamp type. For the in-line type, the circuit must be broken, and the meter is placed in-line to bridge the gap to measure current. In-line type multimeter measures current by measuring the voltage across a small internal resistor. Be careful with this kind of multimeter, not to connect the meter directly to the supply voltage, as a large current will flow through a small resistor, blowing the multimeter fuse.

A clamp-type ammeter uses a current transformer (frequently called a CT), meaning, convert the current signal into a voltage signal using a clamping coil. This type of ammeter doesn't need breaking of the circuit to hook up the meter, does not blow a fuse and usually

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[1] 1 Coulomb = Combined charges in $6.25 \times 10^{18} electrons$ or protons

[2] Molarity (M) is the concentration of a solution expressed as the number of moles of solute per liter of solution.

makes the current measurement easier. However, the sensitivity of a clamp type ammeter is usually lower and does not work well for small currents.
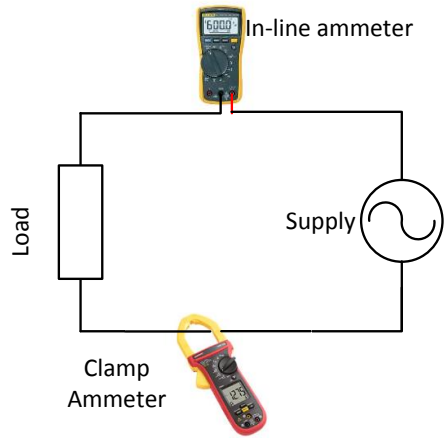
Figure 1.1: Measuring current at a point

Example 1.3

For the following sinusoidal current, find the net flow of charge over one complete cycle.
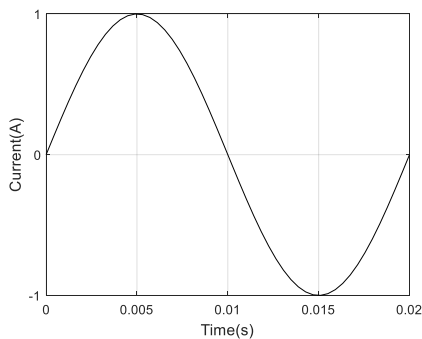
Solution:

$$Q = \int I\, dt$$

The area under the sinusoidal curve gives the total flow of charge. The area under the sinusoid for a complete cycle = 0. Hence net flow of charge = 0.

Voltage

Voltage is the unit to measure the electrical potential difference between two points that forces electrons to move. If one *Joule* of work is required to move one *coulomb* of charge from point A to point B, then the electrical potential at point B is one volt higher than at point A (*Figure 1.2*).
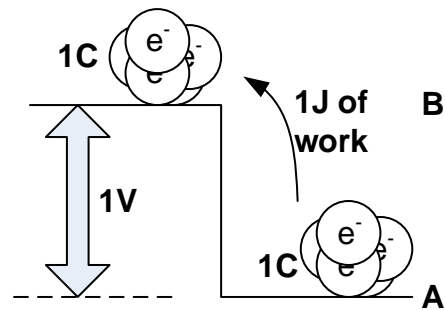
Figure 1.2: One Joule of work is required to move one-coulomb charge against one volt.

Conversely, if electrical charges flow from point B to A, then it will release one Joule of energy.

The equation for power comes directly from the definition of the voltage. Unlike current, the voltage is always measured across two points as shown in *Figure 1.3*.
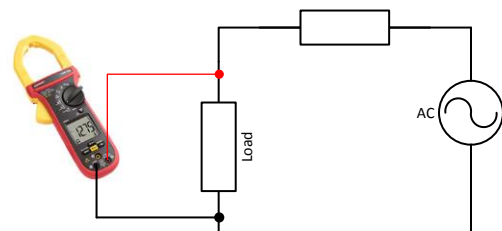
Figure 1.3: Voltage is always measured across two points

Sometimes, the voltage is referenced to a common ground such as earth. So, when you hear 120 V is the household power supply voltage, the voltage is being referenced to earth.

If 'E' joules of work is done moving 'Q' Coulombs over potential difference of 'V' volt then,

$$E = V.Q$$

$$\frac{dE}{dt} = V.\frac{dQ}{dt}$$

$$P = V.I$$

P is power in Watt (W) and I in ampere (A).

Since voltage and current can continuously vary over time, instantaneous power delivered from flow of current from point A to point B is given by

$$P(t) = V(t).I(t)$$

and total energy delivered over a period T

$$E = \int_T P(t).dt$$

Example 1.4

For a sinusoidal voltage V(t) = 3 Sin(120πt) and I(t) = 2 sin(120πt), plot the voltage, current and power for one complete cycle of sinusoid.

Solution: For one complete cycle, period T = 1/60s. The following Matlab code

```
t = linspace(0,1/60,100)';
vt = 3*sin(120*pi*t);
it = 2*sin(120*pi*t);
pt = vt.*it;
plot(t,vt,t,it,'--',t,pt,'-.k')
xlabel('Time (s)')
legend({'V(t)','I(t)','P(t)'})
```

Will produce
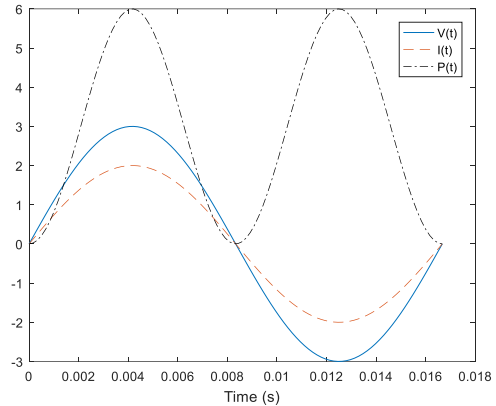
Note that the power delivered by a single phase supply varies from 0 to 6W. This may not be a problem in applications such as lightbulb as even the light bulb goes off 60 times a second our eye can't detect it. However, it could be a problem such as driving a large motor that needs to run smoothly under load. A single phase motor may not produce smooth rotations as the power delivered to the motor is not constant. Uneven power supply usually translates into higher motor vibration. This is one of the drawbacks of a single phase power to supply large load. As we will see later, the problem is solved by using a three-phase power supply.

Resistance

If it takes work to move electrons through an element, then the element is said to have electrical resistance or simply resistance. The resistance of a wire is given by:

$$R = \rho\frac{L}{A}$$

For this equation to be unit compatible, the resistivity has a unit of Ω-m in SI units.

The resistivity ρ is a temperature dependent property of the material. Resistance (R) of metal

wire increases with temperature (T). Increase in resistance per unit resistance per unit temperature increase is called "*temperature coefficient*" (α).

$$\alpha = \frac{dR/R}{dT}$$

The resistivity of some common electrical conductors at 20°C and their temperature coefficient are:

Material	Resistivity($\mu\Omega$-cm)	α (1/°C)
Silver	1.59	3.8×10^{-3}
Copper	1.72	3.9×10^{-3}
Aluminum	2.62	4.3×10^{-3}

Conductor Size

The conductor wire size is usually specified by American Wire Gauge (AWG). By definition, No. 36 AWG is 5 Mil[3] in diameter, and No. 0000 (or commonly written as 4/0) is 0.46 inches in diameter. There are 40 gauge sizes from No. 36 to No. 4/0. Each successive gauge number increases cross-sectional area by a constant multiple.

Example 1.5

Each step decrease in AWG size corresponds to what percent increase in the cross-sectional area?

Solution: Since there are 39 steps between wire gauge 4/0 and 36,

$$A_{4/0} = A_{36} r^{39}$$

where, r is the common ratio.

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[3] 1 Mil = 1/1000 inch

$$r = \left(\frac{A_{4/0}}{A_{36}}\right)^{\frac{1}{39}}$$

And since, Area is proportional to square of the diameter,

$$r = \left(\frac{d_{4/0}}{d_{36}}\right)^{\frac{2}{39}} = \left(\frac{0.46}{0.005}\right)^{\frac{2}{39}} \approx 1.26$$

Percent increase in area = 26%

Example 1.6

Derive the approximate equation for wire cross section to AWG.

Solution: Cross-sectional areas of AWG 36 wire is $A_{36} = \frac{\pi d_{36}^2}{4} = 19.6 \ \mu \ in^2$

$A_n = 19.6 * 1.26^{(36-n)} \mu \ in^2$
Use n = −1 for 2/0, −2 for 3/0 and−3 for 4/0 AWG.

For a quick reference, the cross-sectional areas of commonly used electrical cables are:

AWG	A (in^2)	A (mm^2)
12	5.13×10^{-3}	3.31
14	3.23×10^{-3}	2.08
16	2.03×10^{-3}	1.31

Example 1.7

Calculate the resistance of 100' (30.5m) 14 AWG copper conductor at 20°C.

Solution: For copper, $\rho = 1.72 \ \mu\Omega - cm$

$A_{14} = 19.6 * 1.26^{36-14} = 3165 \, \mu \, in^2$

Since 1 in = 2.54 cm and 1 ft. = 12 in,

$R = \rho \frac{L}{A}$

$R = \frac{\left(1.72\times10^{-6}\Omega \, cm \times \frac{1 \, in}{2.54 \, cm} \times 100 \, ft \times \frac{12 \, in}{1 \, ft}\right)}{3165\times10^{-6} in^2}$

$R = 0.26\Omega$

Basic Laws

Ohm's law:

For a linear circuit element,

$$I = \frac{V}{R} \qquad V = I \cdot R$$

Using Ohm's law, we can express power as:

$$P = V.I = I^2 R = \frac{V^2}{R}$$

Example 1.8

Calculate the resistance of the circuit in example 4.

Solution: R = V(t)/I(t) = 3 Sin(120πt)/2
sin(120πt) = 1.5 Ω

Kirchhoff's Voltage Law (KVL):

The algebraic sum of the voltage drops along any closed path in a circuit and the electromotive forces in that path is equal to zero. i.e. $\Sigma \, \Delta V_i = 0$.
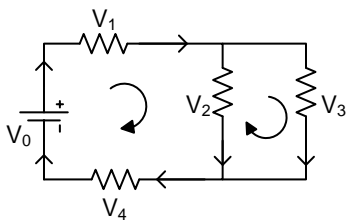
Figure 1.4: The algebraic sum of the voltage drops along any closed path is zero.

$V_0 - V_1 - V_2 - V_4 = 0$

$V_0 - V_1 - V_3 - V_4 = 0$

$V_2 - V_3 = 0$

Kirchhoff's Current Law (KCL):

The algebraic sum of electric currents at a node (wire junction) is 0, i.e. $\Sigma \, I_i = 0$
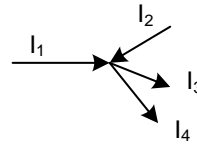
Figure 1.5: Algebraic sum of the currents at a node is zero

$I_1 + I_2 - I_3 - I_4 = 0$

DC Vs. AC Circuit

In direct current (DC), the electric charge (current) only flows in one direction, although the magnitude of the current may vary over time. Electric charge in alternating current (AC), on the other hand, changes direction periodically.

Average current and Voltage of alternating current is 0. The AC voltage is measured by its

Root Mean Squared (RMS) value. By definition, the RMS voltage of an AC power system is the equivalent DC voltage that would have produced the same amount of average power over a cycle.

$$P = V_{rms}^2 R$$

Average power of AC is

$$P = \frac{1}{T}\int_T V(t)^2 R dt$$

from the above two equations,

$$V_{rms} = \sqrt{\frac{1}{T}\int_T V(t)^2 dt}$$

If AC is sinusoidal, which applies to all commercial power supply in the United States, the general equation is,

$$V(t) = V_m \sin(\omega t + \theta)$$

where θ is the *phase* angle or simply 'phase'.

For the sinusoidal power supply of any frequency, it can be shown that:

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$
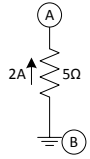
Example 1.9

Calculate the Peak voltage for 120V household power supply

Solution: $V_m = \sqrt{2} * 120 = 169.7\ V$

Exercise

1. Write down the equation of the sinusoid current curve shown in Example 1.3. What is the frequency of this sinusoid?
2. For the question in Example 1.3, derive the equation and plot the charge versus time for t = 0 to 0.02s. What is the maximum charge that would ever flow in one direction? At what time the charge Q is maximum?
3. How many electrons are passed every 1/120 s from a cross-section of wire if the wire was carrying 1A of DC current?
4. How many electrons are passed every half of cycle of 60 Hz sinusoidal current from a cross-section of wire if the wire was taking 1A of peak current?

5. How much energy will an electron dissipate moving down 1V? Compare this energy to the definition of an electron–volt (Google electron–volt)
6. Calculate the total energy consumed in Example 3 during each voltage cycle.
7. Calculate V_A if V_B=0V.

8. Calculate the resistance of 100' (30.5m) 14 AWG aluminum conductor at 20°C.
9. Calculate the resistance of 100' (30.5m) 14 AWG copper conductor at 40°C.
10. What is the resistance of the 1500 Watt electric heater designed to operate at 120V AC?
11. If the household voltage can fluctuate 5% of its nominal value of 120V, what would be the corresponding average power fluctuation in a resistive load in terms of percentage of nominal power?
12. A 100 ft #16 AWG (X–sectional area = 1.3 mm²) copper cable is being used to supply power to an electric heater 1,500W rated at 120V at an ambient temperature of 32 °F (0 °C). The source voltage is 120V.

a) Find the voltage dropped in the cable and the heater?
b) How much power is lost in the cable?
c) What is the power transmission efficiency of this system? Transmission efficiency is the ratio of useful power used by the heater to total power used.

d) Calculate the power loss if #14 AWG was used instead of #16 AWG cable.

True or False?

13. If 10C of charge is moved across a voltage difference of 120V, the work done is 1440 Joule.
14. If you double the electric cable diameter, the resistance of the cable will reduce by a half.
15. For a given supply voltage, higher the wattage rating of an electric load, higher will be the resistance of the load.
16. A good electrical conductor such as copper has no resistance to the flow of electricity.
17. A 1000W electric heater will have less resistance than 1500W electric heater rated for the same voltage.
18. RMS voltage is same as average voltage for DC
19. RMS voltage is the same as the average of the sinusoidal wave.
20. Peak voltage is same as RMS Voltage for AC.
21. The resistance of a given wire does not depend on wire length.
22. The resistivity of a given wire does not depend on the cross-sectional area of the wire.
23. RMS voltage of a sinusoidal wave is independent of frequency.
24. RMS voltage of any kind of power source (like DC or Sine wave) is Peak voltage divided by $\sqrt{2}$.

Chapter 2 Complex Numbers, Phasors, and Sinusoids

Phasor representation of a Sinusoid

AC power supply voltage is represented by equation $V(t) = V_m \sin(2\pi f t + \theta)$, with a constant frequency f. The only two variables in this equation as V_m and θ. Also, for a sinusoid, we have seen that V_m is equal to $\sqrt{2}V_{rms}$. If one is known the other is known too. A pair of number V_{rms} or simply V and phase angle θ can be represented by a vector. Since the second number of this vector is a phase angle, the vector representing a sinusoid is commonly referred to as a *phasor*, the short form of **phase**-**vector**. The advantage of using phasor is that the sinusoidal equations becomes a simple algebraic operation.
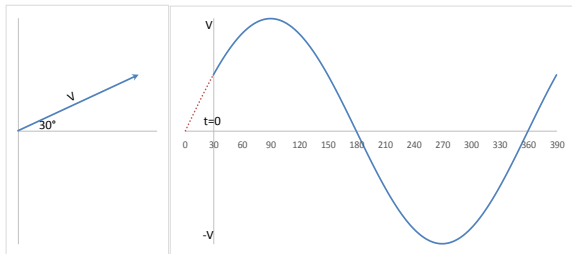
Figure 2.1: Phasor representation of a sinusoid

Example 2.1

Draw a phasor diagram of $170 \sin(377t)$,

$170 \cos(377t)$ and $170 \sin(377t) + 170 \sin(377t)$.

Solution: This first expression has a phase angle of 0 and magnitude of 170. The second expression can be written as $170 \cos(377t) = 170 \sin(377t + 90)$. This expression has

magnitude of 170 and phase angle of 90°. The third expression is the algebraic sum of first two, so we can simply make a phasor addition of above two as shown below. Note that each phasor represents a single sinusoidal wave with certain magnitude and phase. In this case, it is easy to see that cosine wave is a sine wave with phase angle of 90° and if you add a sine and a cosine the resultant is another sine wave (time shifted) with magnitude of 240 ($=\sqrt{170^2 + 170^2}$ and phase of 45° ($= \tan^{-1}\frac{170}{170}$).

Once, the magnitude and the phase is known, you can use Excel, Matlab or any other engineering software to plot it. The following Matlab code will produce the plot

```
p0 = [0 0]; % Origin
[x1,y1]= pol2cart(0,170); % Polar
to Cartesian
[x2,y2]= pol2cart(pi/2,170);
[x3,y3]= pol2cart(pi/4,240);
min = -10;% For axis minimum
max = 180;% For axis maximum


% Plot
quiver(p0(1),p0(2),x1,y1,0)
% see quiver help
text(50,5,'170 sin(377t)')
hold on
quiver(p0(1),p0(2),x2,y2,0)
h=text(5,50,'170 cos(377t)');
set(h,'Rotation',90)
quiver(p0(1),p0(2),x3,y3,0)
h=text(55,50,'170 sin (377t)+170
cos(377t)');
set(h,'Rotation',45)
axis([min  max  min  max])
set(gca,'xtick',[0 170])
```

```
set(gca,'ytick',[0 170])

title('Phasor plot')
axis square
grid on
```

will produce the phasor plot
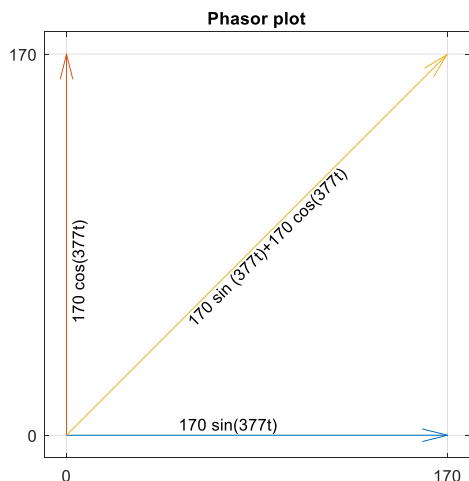
Phasor rotation and complex number

As we have seen in the above example, that sine and cosine can be represented by a phasor diagram and that they are merely rotated by 90°. In electrical analysis, phase rotation is a common task and we would want to have a simple operator that could rotate a phasor by 90°. We know that if a phasor is multiplied by negative 1 (–1), this would be equivalent to rotating a phasor by 180°. We are looking for a similar operator that would rotate a phasor by just 90°, let's call it operator 'j' when a phasor is multiplied by j, it would roatate that phasor by 90° counterclockwise. So, if the phasor 170 (pointing towards x-axis) represents $170 \sin(377t)$. The phasor $j170$ would represent $170 \cos(377t)$.

If we are to rotate $j170$ by another 90°, it will be $j^2\,170$. But since the phasor is now rotated

by 180°, it would be the same as phasor –170. This alludes to: $j^2 = -1$ or $j = \sqrt{-1}$. Which is commonly known as an imaginary number, but for us it is a simple operator that would rotate any phasor by 90° in counterclockwise direction.

Since $\sqrt{-1}$ can not be simply added with a real number, and that it has a magnitude of 1, $\sqrt{-1}$ can also serves as a unit vector pointing in y direction. Therefore, instead of traditional $\vec{\imath}$ and $\vec{\jmath}$ unit vectors pointing in x and y axis, we can use, the real number 1 pointing in x direction and imaginary number $\sqrt{-1}$ (o r j) pointing in y direction. A vector $x\vec{\imath} + y\vec{\jmath}$ will be written as x+yj.
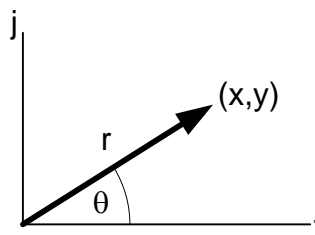
The phasor can also be written in polar form as r∠θ; where, r is the length of the phasor and represents the magnitude of the sine wave. When two phasors $r_1\angle\theta_1$ and $r_2\angle\theta_2$ are multiplied, the product is $r_1.r_2\angle(\theta_1 + \theta_2)$. This is the magnitude gets multiplied and angle gets added.

This property alludes us to a fact that if we multiplied a phasor by another phasor with unit magnitude and phase angle θ, this is equivalet to rotating the original phasor by angle θ. So $1\angle\boldsymbol{\theta}$ is an operator in polar form

that rotates a phasor by angle θ. In cartesian form, $1\angle\alpha = cos\theta + j\ sin\theta$. But using Euler's formula $e^{j\theta} = cos\theta + j\ sin\theta$ when θ is expressed in radians. Therefore, $e^{j\theta}$ can be viewed as an operator that rotates, a phasor counterclockwise by angle θ radians.

Word of Caution Using Phasors

We saw how easy additions of sine and cosine waves gets using phasors. Keep in mind that each phasor in a phasor diagram represents a sine wave at a specific frequency. You cannot add two phasors with different frequencies.

Also, be careful interpreting phasor multiplication. The phasor multiplication does not represent the resultant of two sine wave multiplied. For instance, $sin(\omega t)$ is represented by a phasor 1. If we multiply $sin(\omega t)$ with $sin(\omega t)$ the resultant phasor is still 1. Which obviously does not represent $sin^2(\omega t)$. Also, since $sin^2(\omega t)$ can be written as $\frac{1-cos(2\omega t)}{2}$, this term has a sinusoidal curve with twice the frequency, and a constant number which can be considered as a sine wave of 0 frequency. Therefore, these two phasors cannot be added or with the phasors with different frequencies for that matter using the phasor diagram.

Example 2.2

Show that $e^{j\pi} = e^{-j\pi} = -1$.

Solution:

$e^{j\pi}$ or $e^{-j\pi}$ are operators that rotates a phasor by π radians or 180°, clockwise or counterclockwise they both represent the same reversal in direction. This is equivalent to

multiplying a phasor by −1. So, they all must be equal. You can verify this using Euler's formula.

Remember: The product of two phasors do NOT represent the product of two sine waves

Example 2.3

Find an operator that would rotate a phasor by −90°.

Solution: Since rotating −90° is equivalent to first rotating by 90° (operator j) and then by 180° (operator −1), the combined operator would be −j. Also, logically, if multiplying by j rotates a phasor by 90°, dividing a phasor by j should restore its original direction, that is should rotate the phasor by −90°. In fact, either multiplying by −j or dividing by j rotates a phasor by −90°. Since they are mathematically equivalent, $-j = \frac{1}{j}$ or $j^2 = -1$ which, we have seen is true.

Example 2.4

What do you get when you add $Sin(\omega t)$ and $Cos(\omega t)$?

Solution: Although you could solve this trigonometrically, we will use phasor to solve this. In phasor, $Sin(\omega t)$ can be writtin as 1, and $Cos(\omega t)$ can be writtne as j. Therefore, the addition of two phasors is 1+j. The magnitude of which is $\sqrt{2}$ and phase angle is 45°. This phasor represents$\sqrt{2}\ Sin(\omega t + 45°)$. Therefore, $Sin(\omega t) + Cos(\omega t) = \sqrt{2}\ Sin(\omega t + 45°)$.

Reactive Circuit Components

Circuit components can be divided into two categories, *resistive* and *reactive*. Resistive

component to a circuit dissipates electrical energy as heat or mechanical work. Resistive part of the circuit does not store any energy.

On the other hand, the energy delivered to a reactive component (a capacitor or an inductor) is stored in one-quarter of the sinusoidal cycle, and the same amount of energy is released back to the circuit on the other quarter of the cycle. Reactive components serve as energy storing devices.

Capacitor:
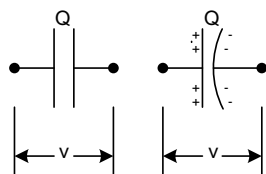
Figure 2.3: Symbol for a Non-polarized and polarized capacitor

A capacitor is an electrical component that stores energy in electrical charges. The fundamental law that governs the relation between stored charge and applied voltage for an ideal capacitor is $Q \propto v \ or \ Q = Cv$. The constant of proportionality C is called the 'capacitance.' The apparent current through a capacitor is given by

$i = \dfrac{dQ}{dt} = C\dfrac{dv}{dt}$ This is the general equation for

a capacitor that relates voltage and current, the same way ohm's law does for a resistor. For a sinusoidal voltage $v = V_m \sin(\omega t)$,

$$i = C\frac{dv}{dt} = CV_m\omega\cos(\omega t) = \omega Cvj$$

Note that for a capacitor, the flow of current is maximum when the slope of voltage curve is maximum and for a sinusoid voltage applied, the current is maximum when the voltage = 0

V. The current phasor has a phase angle of 90° whereas voltage phasor has a phase angle of 0°. The current is said to lead voltage by 90° for a capacitor.

Reactance vs. Resistance

Since differentiation and integration are linear operations, (which simply means doubling voltage also doubles current), the voltage and current calculation can be used in Ohm's law for a resistor using algebraic operations.

Although a reactive component such as a capacitor opposes the flow of electricity, unlike a resistor, it does not dissipate energy. The ratio of voltage to current is called reactance and measured in ohms (same unit as resistance). The reactance of a capacitor (X_c) is given by:

$$X_c = \frac{v}{i} = \frac{v}{\omega Cvj} = \frac{1}{\omega Cj}$$

The instantaneous power to a capacitor is given by $P = v * i = Cv\frac{dv}{dt}$. Since an ideal capacitor does not dissipate energy, the applied power is stored as energy. The stored energy is given by:

$$E = \int Pdt = \int Cvdv$$

If initial and final voltage applied across a capacitor is V_1 and V_2, then net energy to the capacitor is

$$E = \int_{V_1}^{V_2} Cvdv = C\frac{V_2^2 - V_1^2}{2}$$

When a capacitor is charging, the E is positive ($V_2 > V_1$) and when a capacitor is discharging, E is negative. For capacitor in Farad, Voltage in Volts, P has a unit of Watt and E has the unit of Joule.

Example 2.5

If a capacitor known to have a capacitance of 100μF is connected to a 60 Hz source, what is the capacitive reactance?

Solution: $X_c = \frac{1}{2\pi fCj} = \frac{1}{2\pi*60*10^{-4}j} = -26.5j \ \Omega$

Example 2.6

A capacitor connected across 120V was drawing 0.5A of current. What is the reactance and capacitance of that capacitor?

Solution: The magnitude of the reactance, $X_c = \frac{v}{i} = \frac{120}{0.5} = 240\Omega$

Capacitance, $C = \frac{1}{\omega X_c} = \frac{1}{377*240} = 11\mu F$

Example 2.7

Plot stored energy and current flow in a 1000μF capacitor connected across household power supply for a complete voltage cycle.

Solution: The following code in Matlab was written to generate the plot.

```
Frequency = 60;
Th = 0:360;
Om = 2*pi* Frequency;
V_m=120*sqrt(2);
C=1e-3;
vt = V_m*sin(deg2rad(Th));
it = Om*C*V_m*cos(deg2rad(Th));
et = C*vt.^2./2; %in Joules
plot(Th,vt)
ylabel('V(t), I(t)');
hold on
plot(Th,it,'--')
yyaxis right
plot(Th,et,'-.');
ylabel('E(t)');
xlabel('\Theta')
```

```
legend({'V(t)' 'I(t)' 'E(t)'})
grid on
set(gca,'XTick',0:90:360)
```
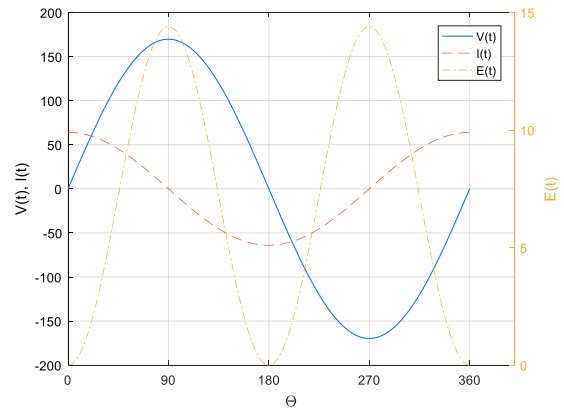

Note that the stored energy in a capacitor is always a positive value. For θ=0 to 90°, the energy is stored in the capacitor, from 90° to 180, the stored energy is released back to the system. When a capacitor is charging, both voltage and current has the same sign, indicating positive power, i.e. work is being done to store the charge. When the capacitor is discharging, they have opposite sign indicating the power is negative, that is capacitor is doing the work.

Inductor

An inductor is an electrical component that stores energy in the magnetic field. The relation between opposing induced voltage across and inductor for an ideal capacitor is given by the fundamental law $v = L\frac{di}{dt}$
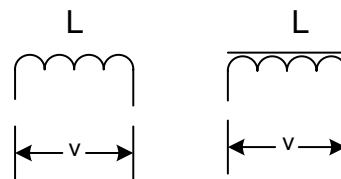
Figure 2.4: Air core and iron core inductor,

for a sinusoidal current, $i = I_m\sin(\omega t)$,

$$v = L\frac{di}{dt} = LI_m\omega\cos(\omega t) = j\omega Li$$

The reactance of an inductor:

$$X_L = \frac{v}{i} = \frac{\omega Lij}{i} = \omega Lj = 2\pi f Lj$$

The instantaneous power to an inductor is given by $P = v * i = Li\frac{di}{dt}$. Since an ideal inductor does not dissipate energy, the applied power is stored as energy. The stored energy is given by:

$$E = \int Pdt = \int Li\,di$$

If initial and final current across an inductor is I_1 and I_2, then net energy to the inductor is can be calculated as

$$E = L(I_2^2 - I_1^2)/2$$

Example 2.8

If an electrical coil known to have an inductance of 0.1 H is connected to a 60 Hz source, what is the inductive reactance?

Solution: $X_L = \omega Lj = 2\pi * 60 * 0.1j = 37.7j\ \Omega$

Example 2.9

Plot stored energy and current flow in a 1mH inductor connected across household power supply for a complete voltage cycle.

Solution: The following code in Matlab was written to generate the plot. The code assumes that the initial current (I_1) was 0.

```
% Th = Theta, Om = Omega
Frequency = 60;
Th = 0:360;
Om = 2*pi*Frequency;
V_m=120*sqrt(2);
```

```
L=10e-3;
vt = V_m*sin(deg2rad(Th));
it = -V_m/(Om*L)*cos(deg2rad(Th));
et = L*it.^2;%in Joules
plot(Th,vt)
hold on
plot(Th,it,'--')
ylabel('V(t), I(t)');
yyaxis right
plot(Th,et,'-.');
ylabel('E(t)');
xlabel('\Theta')
legend({'V(t)' 'I(t)' 'E(t)'})
grid on
set(gca,'XTick',0:90:360);
```
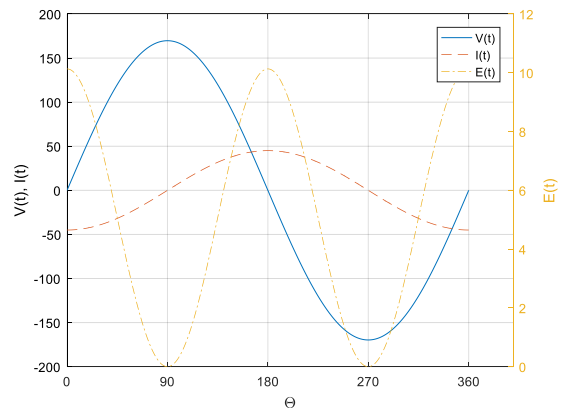

Resistance, Reactance and Impedance

The combined resistance (R) and reactance (X) is called the impedance (Z) of the circuit. The Impedance (Z) is expressed using in phasor as.

$$Z = R + Xj$$

Where R is total resistance and X is algebraic sum of reactance (X_c and X_L). In a circuit consisting of resistance and reactance, if aggregate phase angle of voltage and current is θ, then in-phase product of voltage and current ($I_R * V$)is dissipated as heat (or other forms of energy) and 90° out of phase product ($I_x * V$) of voltage and current is the stored energy.

Complex Numbers, Phasors, and Sinusoids

Since phase of sine waves are relative, any one of the phasors can be held as 0 phase and draw other phasors relative to that. When drawing the phasor diagram, it is customary to draw supply voltage V at 0 phase.
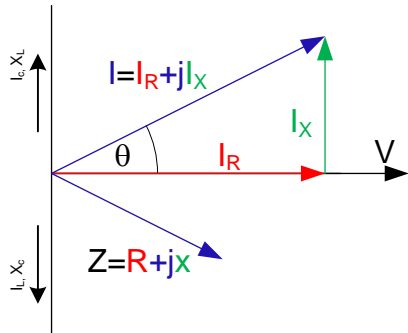
Remember that a circuit with resistance and reactance, the equation for instantaneous power drawn P = V.I still holds. For instance if V(t) = 3 Sin(120πt) and I(t) = 2 sin(120πt+π/3), then the power over a full voltage cycle will be shown as below.
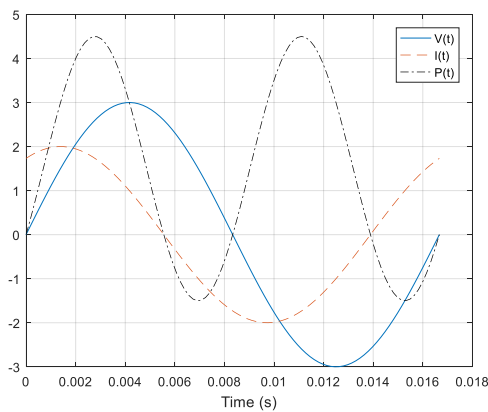
Figure 2.6. Net area under P(t) is the real work done.

We see that the power curve has more positive area than negative area. This power curve is called *Apparent power.*

The average area of the power curve gives the *real power* used by the circuit. If $V(t) = V_m \, Sin(\omega t)$ and $I(t) = I_m \, sin(\omega t + \theta)$

The average area of this power curve is given by:

$$\frac{1}{2\pi} \int_0^{2\pi} V_m \, Sin(\omega t). I_m \, sin(\omega t + \theta) \, d\omega t$$

$$= \frac{1}{2} V_m I_m Cos(\theta)$$

Since $V_{RMS} = \frac{V_m}{\sqrt{2}}$ and $I_{RMS} = \frac{I_m}{\sqrt{2}}$,

$$P_{Average} = V_{RMS} I_{RMS} \, Cos(\theta)$$

In a phasor sense, if phasors are drawn with V_{RMS} and I_{RMS} , the dot product of two phasors gives the average power used by the circuit.

Another way to look at it that that for a power-dissipating component (a resistor), current is always in phase with applied voltage and contributes to the average power used. For an energy storing device (capacitor or inductor), the current and voltage are always 90° out of phase and does not contribute to the average power used. In summary,

Real Power (Power dissipated) = $V. I_R = V. I. Cos\theta$. Where voltage and currents are RMS values. This Power is called 'Real power' and measured in *Watts.*

Reactive Power (Power stored) = $V. I_X = V. I. Sin\theta$.

This power is called 'Reactive power' and measured in VAR (Volt-Ampere Reactive).

The product of Voltage and Current (V. I) is known as 'Apparent power' and measured in (VA).

Apparent power is delivered to the end user by a utility company. The end user consumes the

real power and returns the reactive power in the form of negative power consumption.
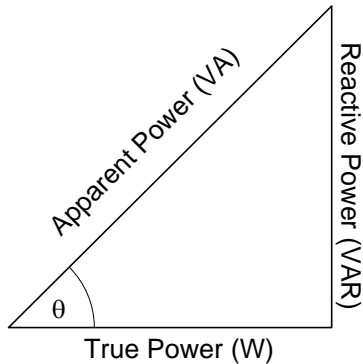
Figure 2.7: Apparent power = V·I, True Power = V·I·Cos (θ); Reactive Power = V·I·Sin (θ) .

The cosine of the phase angle $Cos\theta$, is called 'Power factor' or PF in short. PF = 1indicates the load is purely resistive (i.e. 100% of delivered power is used), a PF = 0 mean the power is purely reactive.

Measuring real power in an AC circuit

For a DC circuit, the power delivered to a load can be calculated by measuring voltage and current and multiplying two. For a DC power supply PF is always 1.0, as for DC, inductive reactance is 0, and no current can pass through a capacitor.

In case of AC, unless the load is entirely resistive, the power factor in addition to voltage and current needs to be measured to calculate the real power. A general-purpose multimeter usually does have a feature to measure the power factor. Therefore, a power meter should be used to measure the AC power. The power meter can measure the real power, supply voltage, and current simultaneously.

Implications of a small power factor system

PF of less than one indicates there is some reactive current I_X that flows back and forth in the transmission line without delivering any power.
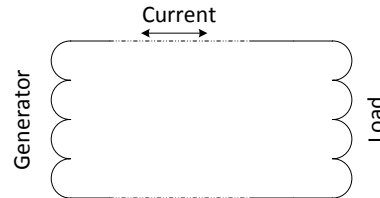
Figure 2.8: PF<1 means reactive currents flows back and forth in transmission line.

However, in reality, utility company having to supply I_X that flows back and forth incurs additional power loss in the transmission line as its resistance is not zero. Also, electrical components such as transformer and wire are rated in VA. Because of I_X, the power company needs to use higher rated equipment, which usually costs more.

Therefore, utility companies usually charge extra to customers with below acceptable value (Usually 0.95) of PF as the 'power factor charge.'

Ohm's law and Kirchhoff's law for AC circuit

Ohm's law and Kirchhoff's law equally applicable to AC circuit provided that resistance is replaced with impedance with phasor notation.

Series and Parallel Circuit

Following the KVL and KCL, the equivalent resistance of series and parallel circuit can be calculated as:
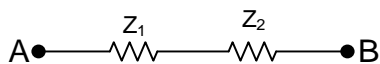
$Z_{AB} = Z_1 + Z_2$ For series connection, and
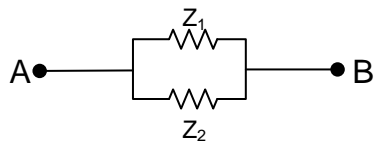
$\frac{1}{Z_{AB}} = \frac{1}{Z_1} + \frac{1}{Z_2}$ or $Z_{AB} = \frac{Z_1.Z_2}{Z_1+Z_2}$ for parallel connection.

If there is no reactance the impedance can be replaced with resistance (R).

Example 2.10

Two capacitors C_1 and C_2 are connected in series, what is their combined capacitance?

Solution: Let the combined capacitance be C_{eq}

Now impedances of C_1, C_2 and C_{eq} are:

$$Z_1 = \frac{1}{j\omega C_1}, Z_2 = \frac{1}{j\omega C_2}, and\, Z_{eq} = \frac{1}{j\omega C_{eq}}$$

Since they are in series, equivalent impedance $Z_{eq} = Z_1 + Z_2$

$$\frac{1}{j\omega C_{eq}} = \frac{1}{j\omega C_1} + \frac{1}{j\omega C_2} \; or, \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

Example 2.11

An electric motor connected across 120V power supply was drawing 8 A at PF = 0.8.
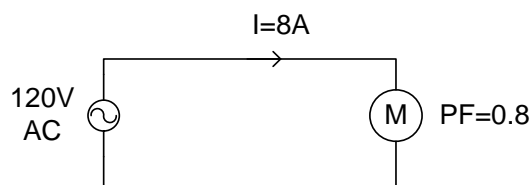
a) Calculate the actual power consumed by the motor.

b) If the cost of electricity was 10 ¢/kWh, how much would it cost to run this motor continuously for a year (8760h)?

c) If the reactance in this circuit is from inductive load, what is the resistance and the inductive reactance of the motor? Assume that they are in series. Plot the phasor of supply voltage V, Line current I and Impedance Z.

d) Calculate the voltage drop across the resistive and inductive component of this circuit.

e) Show that the Kirchhoff's voltage law holds for this circuit.

Solution:

a) Real Power = V·I·PF=120·8·0.8=768W

b) Electricity Cost = 0.768·8760·0.1 = $ 672.77

c) $R = \frac{Real\ Power}{I^2} = \frac{768}{8^2} = 12\Omega$

Since power factor = 0.8

$$|Z| = \frac{R}{PF} = 15\Omega\ ;$$

$|X| = \sqrt{|Z|^2 - R^2} = 9\Omega$. *Therefore, Z =* $12 + 9j = 15\angle 36.9°$.

Since the reactance is inductive, $X_L = j9\ \Omega$. The phase angle of current is $C = \cos^{-1} 0.8 = \pm36.9°$, but since we know that for inductive load, current lags voltage, $I = 8\angle - 36.9°$.
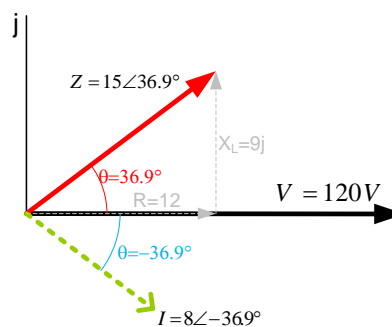
d) For the resistor, voltage and current are in phase, so, V_R points in the same direction as I and $V_R = IR = 8 * 12 = 96\ V$. For the inductor, Voltage leads current by 90° (phase angle = –36.9+90 =53.1°). The magnitude of voltage across conductor $V_L = IX_L = 8 * 9 = 72\ V$

e) The supply voltage is 120V. If you just add V_R and V_L they sum up to 96+72=168V which exceeds supply voltage of 120V and apparently violating the Kirchhoff's voltage law. However, keep in mind that you need to add voltages keeping track of their phase angles.

So, $96\angle - 36.9 + 72\angle 53.1 = 120\angle 0\ V$
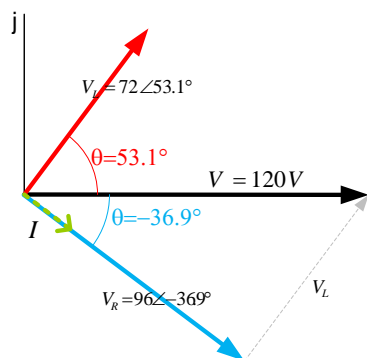
Power Factor Correction

As mentioned earlier, utility company usually charges extra to its clients with a low power factor. Since low power factor is caused mainly from inductive load such as an electric motor, one of the ways power factor can be improved is by using a capacitor. Since X_L and X_C acts in opposite directions, they cancel one another.

A capacitor can be connected in series or parallel to correct for power factor. We will examine the consequence of connecting a capacitor in series or parallel through following two examples.

Example 2.12

A capacitor is connected in series with the motor in above example with $X_C = 9\Omega$. Calculate the new PF and the current in the circuit.
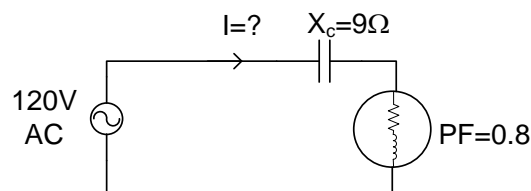
Solution:

Borrowing the results from the previous example, the motor impedance $Z = 12 + 9j$. Since reactance of the capacitor lags 90°, $X_C = -9j$. So, the new circuit impedance $Z_2 = 12 + 9j - 9j = 12\Omega$. Since there is no reactive component in the circuit, the power factor = 1.0 and the new current $I_2 = \frac{120}{12} = 10A$.

The power factor is increased but note that the line current has also increased as impedance has reduced. The increased current will produce higher power in the motor, the new power dissipated by the motor is: $P_2 = I_2^2 R = 1200W$. This is quite a bit more power than original 768W. Usually motors are designed to dissipate heat generated at the rated power. Because of the addition of a capacitor in series, the power delivered to the motor has increased and there is fair chance that the motor will

burn out in short time, unable to dissipate extra heat. Even if the motor did not burn out, probably no cost saving will occur as power use has been increased. So, **connecting a capacitor in series to improve PF from an inductive load is a BAD idea.**

Example 2.13

A capacitor is connected in parallel with the motor in above example with $X_C = 25\Omega$. Calculate the new PF and the current in the circuit, the motor and the capacitor?
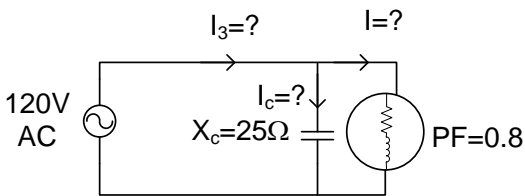
Solution:

Since two branches are in parallel, the equivalent Impedance is

$$Z_3 = -25j \parallel (12 + 9j) = \frac{-25j \times (12 + 9j)}{-25j + 12 + 9j}$$

This expression can be simplified to

$Z_3 = 18.75 + 0j$. Therefore, power factor for the sysem = 1 and new current $I_3 = \frac{120}{18.75} = 6.4\ A$.

Notice that overall current that the power supply is providing has decreased, however, the motor is still connected across the 120V power source, and motor impedance has not changed, so there is no change in the current or the power delivered to the motor. The current through the capacitor is $I_C = \frac{120}{25} = 4.8\ A$. The phasor diagram of the currents are shown below. Notice that the current follow the

Kirchhoff's current law at the node as $I_3 = I_c + I$
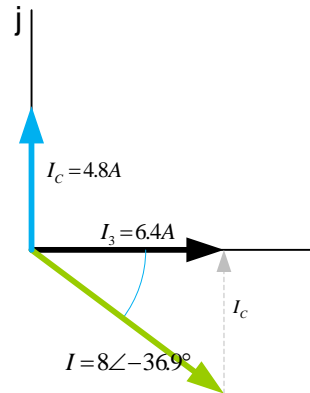
We saw that connecting a capacitor in parallel also increased the power factor without affecting the motor operation in any way. We also saw that higher power factor means low line current from power supply unit.

The heuristic explanation of this phenomenon is that the current that would have been returned to the power line is locally stored in the capacitor because the inductor and the capacitor current are 180° out of phase. When the inductor needed additional current in the following quarter cycle, the capacitor would provide that without having to be drawn from the power supply. So, **connecting a capacitor in parallel to improve PF from an inductive load is a GOOD idea.**

Economic Consideration

The addition of capacitors for power factor improvement is usually based on an economic analysis of the situation. Usually, the penalty is imposed if power factor is below 0.9 (or 0.95 is some cases), so the power factor improvement only up to that is economically practical.

Example 2.14

A capacitor is to be connected in parallel with the motor in above example to improve PF to 0.95. Find the value of X_C for the capacitor?
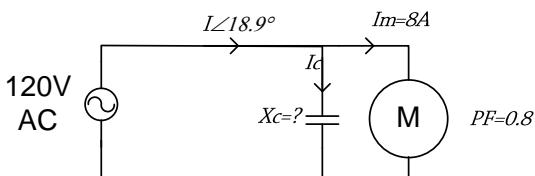
Solution:

For the power factor of 0.95, the phase angle of the supply current $\theta = \cos^{-1} 0.95 = 18.2°$. From the phase diagram below,
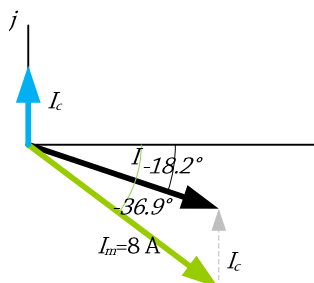
$$I = 8 * \frac{0.8}{0.95} = 6.7A$$

$$I_c = -(I_m. \mathrm{Sin}(-36.9) - I. \mathrm{Sin}(-18.2))$$

$$I_c = 8 * 0.6 - 6.7 * 0.31 = 2.72$$

$$X_c = \frac{120}{I_c} = 44 \ \Omega$$

Capacitor specification

The capacitor is specified either in Farad or in VAR (Volt-Ampere Reactive). Farad rating is an intrinsic property of a capacitor and does not depend on the supply voltage. A capacitor's VAR is related to Reactance as:

$VAR = \frac{V^2}{X_c}$ where V is the voltage across a capacitor.

Example 2.15

Specify the capacitor for $X_c = 25\Omega$ in terms of Farad and VAR, if the capacitor is to be used across 120V, 60 Hz power supply.

Solution:

For $X_c = 25\Omega$,

$$C = \frac{1}{2\pi f X_c} = \frac{1}{377*25} = 106 \ \mu F$$

In terms of VAR, the specification would be:

$$VAR = \frac{V^2}{X_c} = \frac{120^2}{25} = 576$$

Transformer

For an ideal transformer circuit:
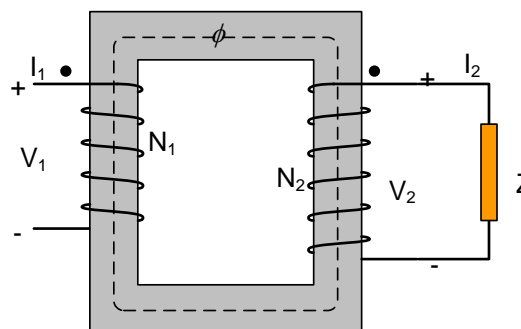
Figure 2.9: Schematics of a transformer

From magnetomotive force balance,

$$N_1. I_1 = N_2. I_2$$

And from the power balance

$$I_1. V_1 = I_2. V_2$$

Combining above two equations,

$$\frac{V_1}{V_2} = \frac{I_2}{I_1} = \frac{N_1}{N_2}$$

The current of the input coil is limited by the reactance of the coil. The reactance decreases at frequency decreases and hence at a lower frequency, more current will flow through the coil. A thicker wire is needed to handle more

current, so, a transformer operating at higher frequency are usually smaller (and cheaper) than the transformer operating at a lower frequency.

Exercise

25. Draw a phasor diagram of $V_1 = 169.7 \sin(2\pi ft + 60°)$ and $V_2 = 169.7 \sin(2\pi ft - 60°)$.

26. Two inductors L_1 and L_2 are connected in parallel. Derive the equation for their combined inductance using KVL and KCL.

27. Prove that $Sin^2(\omega t) + Sin^2\left(\omega t + \frac{2\pi}{3}\right) + Sin^2\left(\omega t - \frac{2\pi}{3}\right) = 1.5$

28. Create a superimposed plot (time is x-axis) for the voltage, the current, and power delivered to the motor with PF = 0.7 over one complete cycle of 120V 60 Hz supply.

29. For the circuit shown, what is the current through and the voltage across each of the three given components?

L=0.3H

AC
240V
60Hz

R=30 Ω

C=50µF
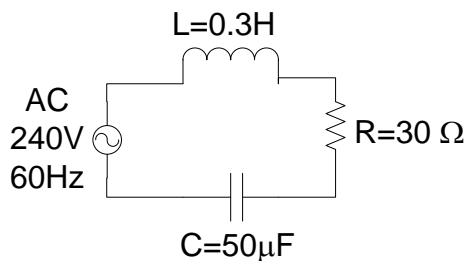
30. For the 120-volt 60 Hz circuit shown, find the reactance of each component and total impedance. Also, calculate the power factor for this circuit.

C = 50 µF

120V
60 Hz

R=30Ω

L=0.1H
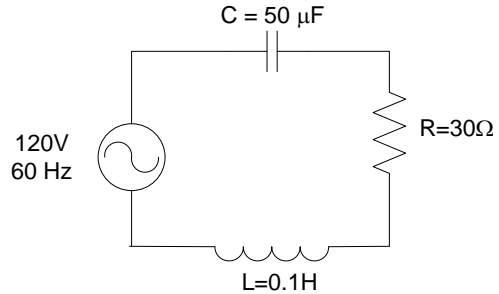
31. Calculate the amount of parallel connected capacitance needed to correct the power factor of the motor circuit shown below to 1.0.

I_M=15A

C=?

230V AC

M

PF=0.8
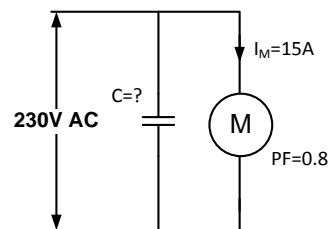
32. Calculate the capacitance (C) to correct the power factor of the motor shown below to 0.9.

120V, 60 Hz AC

C

M   8A, 0.7 PF
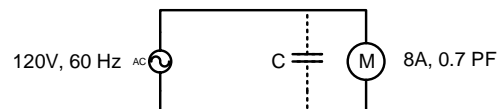
33. A 100 ft #16 AWG (X-sectional area = 1.3 mm^2) copper cable is used to power a 120V, 1500W electric heater rated at ambient temperature of 0°C. One-horsepower motor (with 0.7 PF at full load) is connected in parallel the heater. Assuming the motor is running at full load,

a) Draw a phasor diagram for current in heater, motor and total current.

b) Size the capacitor to increase the motor power factor to 0.95.

c) Draw a schematic of the system showing how one should connect the heater, motor and the capacitor.

1. Calculate the power loss in cable before and after power factor correction.

34. A small bioprocessing plant has determined their load to be 50 kVA with a power factor of 0.76. What size of parallel connected capacitor bank would be required to improve the power factor to

(a) 0.95?

(b) 0.90?

35. In the following transformer circuit, $V_1 =$ 120 V, 60 Hz AC supply. $N_1 = 1000$ turns and $N_2 = 2000$ turns. If $Z = 400$ Ω,

a. Find the current I_1 drawn from the primary supply?
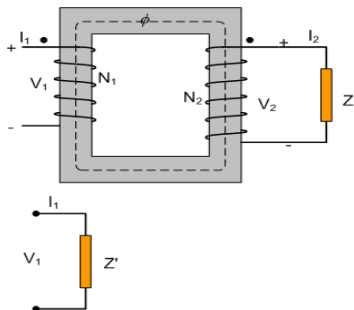
b. What equivalent resistance Z' placed in primary side will draw the same current I_1? Express Z' as a function of N_1, N_2 and Z.

True or False

36. Ohm's law applies only to pure resistors and not to capacitors or inductors.

37. Direct electricity cannot pass through a capacitor.

38. If you add or subtract two sinusoidal waves different in magnitude and phase but at same frequency, the resulting wave will **not** be sinusoidal wave.

39. Reactance of a given inductor coil will be the same regardless of frequency. (Like 60 Hz in US and 50 Hz in Europe).

40. Power factor for a pure inductive load would be 0.

41. As frequency of the supply voltage increases, the reactance of a capacitor will also increase.

42. Reactive components always have voltage and current 90° out of phase.

43. The true power in an AC circuit is always less than or equal to the product of the voltage and the current.

Chapter 3 Three Phase System

As discussed earlier, single phase power supply can't deliver a constant power. A standard solution to supply constant power is providing power in three phases. Each of the phase is mechanically separated 120° apart as shown in figure 1 to generate the voltages that are 120° phase angle apart.
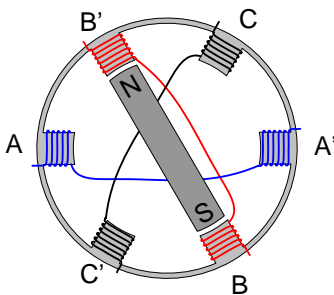
*Figure 3.1:*Three phase consists of three single phases physically separated by 120°.

When the magnetic field is rotated, the coils AA', BB' and CC' produces peak voltages 120° phase angle apart. The magnitude of each single-phase voltages is equal. They only differ in the phase angle. Mathematically, $|V_{A-A'}| = |V_{B-B'}| = |V_{C-C'}|$.

Watch the YouTube video www.youtube.com/watch?v=LisefA_YuVg from 0 to 7:05 minutes to aid your understanding about the AC generation. The windings of three single phases ends can be connected in the Wye (Y) or the Delta (Δ) configuration as shown below to supply power using only 3 or four wires.
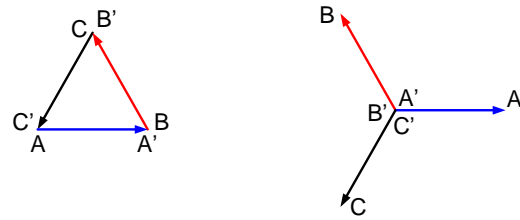
Figure 3.2: single phases shown in Figure 3.1 can be connected as Δ or Y connection to create a 3-phase system

When Δ connection is made, there will be three phases (three live wires) outputted from terminals AC', CB' and BA'. The voltage (magnitude) between any two lines will be equal to the single-phase voltage.

When Y connection is made, there will be three phases outputted from terminal A, B, C and a neutral wire connected to A'B'C'. The voltage (magnitude) between any line to neutral will be equal to the single-phase voltage. The phase to phase voltage will be $\sqrt{3}$ times phase voltage as shown in example below.

Example 16

For a Y-connected system below, if magnitude of A'A = B'B = C'C = 120V, find the magnitude of voltage between terminal A and B.
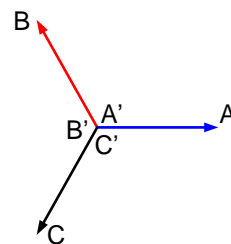
Solution:
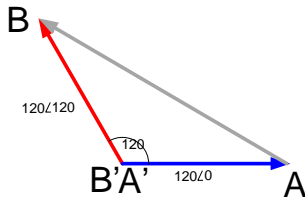
$V_{AB} = V_{B'B} - V_{A'A}$

$V_{AB} = 120\angle120 - 120\angle0$

$V_{AB} = 120\cos(120) + j120\sin(120) - 120$

$|V_{AB}| = 120\sqrt{(\cos120 - 1)^2 + (\sin120)^2}$

$= 120\sqrt{3} \approx 208V$

Example 17

For a three-phase power system as shown below, prove that the total power delivered to a three-phase load is constant at all time.
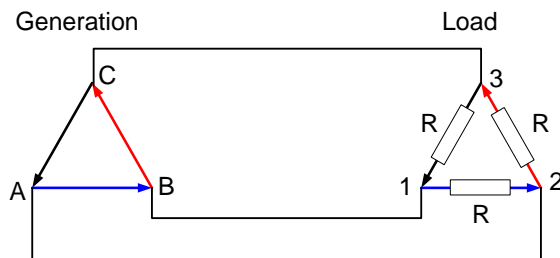
Solution:

Let's hold AB at 0° phase angle. Neglecting line loss, the sinusoidal voltage of each phase can be written as:

$V_{AB} = V_{12} = V_m\sin(\omega t)$

$V_{BC} = C_{31} = V_m\sin(\omega t + 2\pi/3)$

$V_{CA} = V_{23} = V_m\sin(\omega t - 2\pi/3)$

Since power delivered to resistive load = V^2/R, Total power delivered to the power at any given time is

$P = \frac{1}{R}(V_{AB}^2 + V_{BC}^2 + V_{CA}^2)$

But,

$V_{AB}^2 + V_{BC}^2 + V_{CA}^2$

$= V_m^2\sin^2(\omega t) + V_m^2\sin^2\left(\omega t + \frac{2\pi}{3}\right) + V_m^2\sin^2\left(\omega t - \frac{2\pi}{3}\right)$

$= 1.5\,V_m^2$ [The details of this proof are left as an exercise]

Therefore, the total power delivered at any given time = $\frac{1.5\,V_m^2}{R}$. This is a constant for a given power supply and is not a function of time (i.e. there is no 't' in this equation).

Three Phase Power Measurement

The simplest case for a three-phase analysis is when the loads are equal between phases ($Z_{12}=Z_{23}=Z_{31}$). This condition is called balanced loading. For a balanced load, the magnitude of line to line voltage V_L, and line current I_L, and branch currents I_b are equal in all three phases.
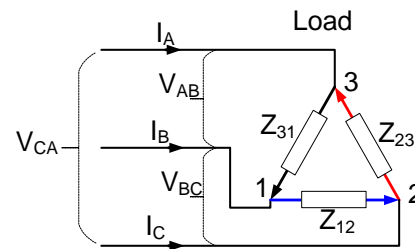
Figure 3.3: Balanced three phase load, $Z_{12}=Z_{23}=Z_{31}$

Figure: For balanced load, $Z_{12}=Z_{23}=Z_{31}$; $|I_{12}| = |I_{23}| = |I_{31}| = i_p$, $|I_A| = |I_B| = |I_C| = I_L$ and $|V_{AB}| = |V_{BC}| = |V_{CA}| = V_L$

The total power delivered to the load is:

$P = |V_{BC}|.|I_{12}|.PF_{12} + |V_{CA}|.|I_{23}|.PF_{23} + |V_{CA}|.|I_{31}|.PF_{31}$

For a balanced load,

$P = 3V_L.I_b.PF$

Practically, it is not always easy to measure I_b for a load as they are usually enclosed, but it is

easy to measure I_L. Applying Kirchhoff's current law at node 1,

$$I_B = I_{12} - I_{31}$$
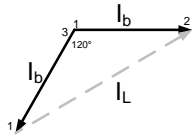
Figure 3.4: $I_L = \sqrt{3}\, I_b$

From the geometry, it can be shown that,

$$I_L = \sqrt{3}I_b$$

Therefore, the real power is expressed as:

$$P = \sqrt{3}\, V_L . I_L . PF$$

Therefore, for a balanced load, just line in case of single phase, V_L, I_L , and PF needs to be measured to determine the real power.

In practice, a three phase power can be measured using a single phase power meter. The measurement scheme to measure balanced or unbalanced power in 3-wire or 4-wire system is shown in *Figure 3.5*.
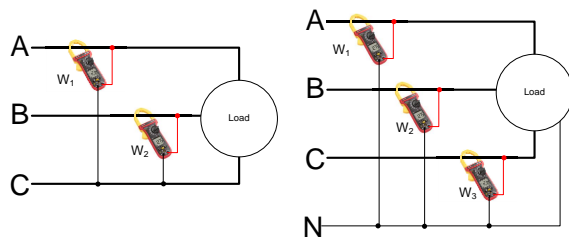
Figure 3.5: Three phase power measurement using a single phase power meter. For a three wire system, total power $P = W_1 + W_2$. For Four wire system, $P = W_1 + W_2 + W_3$. For a balanced load, W_1 should be equal to $W_2 = W_3$.

Note that use of single phase power meter may not work well if the load varies during separate single phase power measurement. If that is a concern then a three phase

Exercise

44. In the following three phase system,

$$V_{1o} = 120\angle 0°$$

$$V_{2o} = 120\angle 120°$$

$$V_{3o} = 120\angle -120°$$

$$Z = 100 + 100j$$

Calculate the line to line voltage, Line current and total power delivered to the delta load.
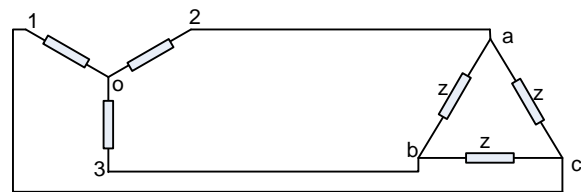
45. For the following balanced three phase Y load, prove that $P = \sqrt{3}\, V_L . I_L . PF$ where, V_L is line to line voltage and I_L is the line current.
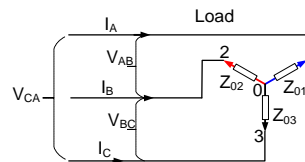
46. Find the line current for a 10 hp 3-phase, 480V motor runnig full load at PF= 0.8.

47. If apparent power consumed by a three-phase electric load is 2 kVA while consuming 1.5 kW of power, what is the reactive power of this system?

48. Given a three-phase wye system with a line voltage of 208V and line current of 100A, determine the true power output of the system, if the power factor was 0.8.

49. Prove that $Sin^2\theta + Sin^2\left(\theta + \frac{2\pi}{3}\right) +$

   $Sin^2\left(\theta - \frac{2\pi}{3}\right) = 1.5$

True or False?

50. Power delivered at any time is constant in balanced three phase system.

51. For three phase Wye connection, line current is $\sqrt{3}$ times the phase current.

Chapter 4 Electrical System Design

Residential Power Supply System

The residential power supply in the US is 120/240V system. The utility company provides power through three wire system to residence. Two live lines and one grounded neutral wire. The two phases are equal in magnitude of 120V with respect to neutral but 180° out of phase. So, the voltage between two phases is 240 V. The grounding is maintained at the primary and secondary side of the service transformer usually seed outside of residence as shown in Figure 1.
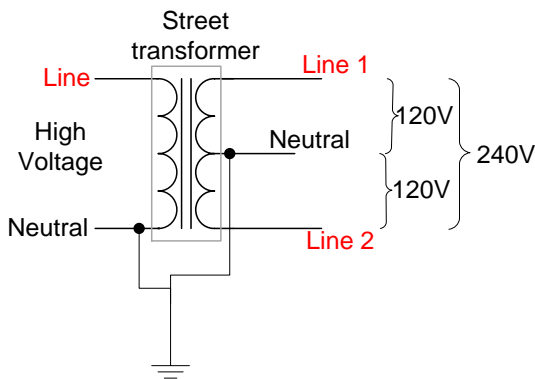
Figure 4.1. A common household power supply system. The Neutral is grounded in both side of the transformer.

The power from a generation station is carried away at very high voltage (Usually Between 70–750kV; 230 kV line from Spokane to Moscow). The voltage is reduced to 120/240V at several steps. City, town, or large electricity users will have set of voltage reducing transformers to reduce voltage from major transmission line called a substation. Since the neutral is grounded, and there is a potential current leak to ground, substations are usually fenced out.

The power from the substation is fed into a city through one or more parallel feeder lines.

Feeder lines are the part of electrical transmission system from a substation to a residential area (generally between 4 – 48kV, but 13 kV for Moscow, ID). For a greater robustness of the system, each residence can receive power from two or more parallel feeder lines. If one feeder line fails, the power is routed through a different feeder line.

If you are designing or installing an electrical system for a residential house. You must design the system downstream of the '**service point**'. Service point is the point where electric utilities provide power to their customers. It could be overhead or underground depending on the distribution system.
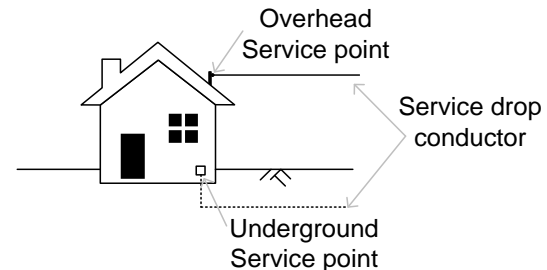
Figure 4.2. When designing a residential electrical system, design your system downstream of the service point. Upstream of service point is the responsibility of Utility.

From the service point, the cable connects down to the meter and then to the main service panel. The main service panel will have two terminals for two lines and a terminal for the ground as shown in the figure below.
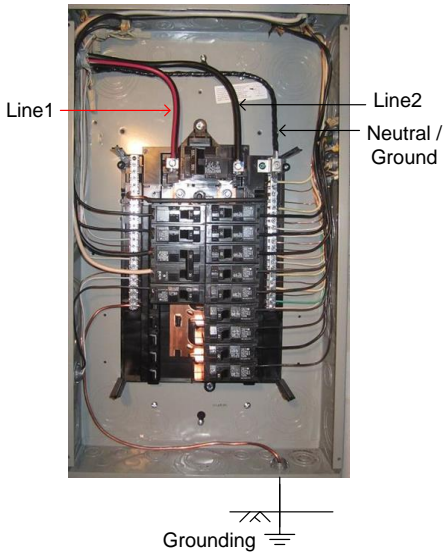
Figure 4.3. Inside of the Main Service Panel. The line to neutral voltage is 120. The line to line voltage is 240.

When designing branch circuits try to balance the electrical loads equally between line 1 and line 2. The phasor sum of current in two lines is carried back by neutral line.

$$I_{Neutral} = I_{Line1} + I_{Line2}$$

The magnitude of the $I_{Neutral}$ is the total current that is measrued in the neutral line. For a perfectly balance load between two lines, neutral carries no current because the line 1 and line 2 are 180° out of phase.
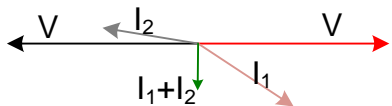
Figure 4.4. Current in neutral is the phasor sum of currents in line1 and line2.

If the loads are resistive, then the current would in phase with voltages.

Example 18

Find the current in Line1, Line2 and neutral for the following wiring condition. Assume that PF of 0.8 is due to an inductive load.
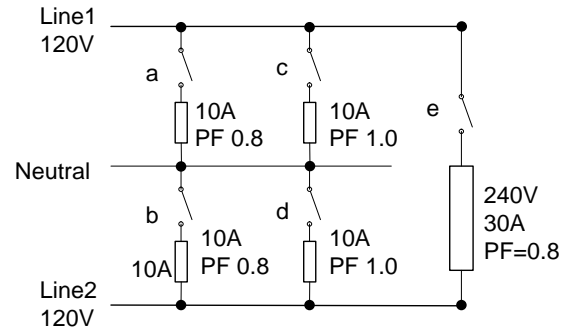
1. Only switch 'a' is closed
2. Only Switch 'a' and 'b' are closed
3. Only switch 'a' and 'c' are closed
4. Only switch 'a' 'b' and 'c' are aclosed
5. All Switches are closed.

Solution:

1. 10 A in both Line1 and neutal, 0A in Line2.
2. Line1 current = 10A. Line2 Current = 10 A. As the line currents are 180° out of phase and has the same power factor, the neutral current = 10−10 = 0A.
3. Line1 Current = Magnitude of the sum of current in 'a' and 'c' branch.
   $$= |10\text{Cos }\theta - 10\text{Sin }\theta\, j + 10 + 0j|$$

   $PF = Cos\theta = 0.8$, Hence, $Sin\theta = \sqrt{1 - Cos^2\theta} = 0.6$

   So, Line1 Current $\approx 19A$
   $I_{Neutral}$ = Line1 current = 19A.
4. $I_{neutral}$ can be calculated by adding currents in branches 'a', 'b', and 'c'. However, since, current in branch 'a' and 'b' are equal and opposire, they

cancel out. Therefore, current in the neutral is 10A.

5. Line1 Current = Sum of (10A at PF=0.8+10A at PF of 1+30A at PF of 0.8)

$$= |10\text{Cos }\theta - 10\text{Sin }\theta\text{ j} + 10$$
$$+ 30\text{Cos }\theta - 30\text{Sin }\theta\text{ j}|$$
$$= |42 - 24\text{j}| = 48.4A$$

Line2 Current can be calculated as the same way and would be = 48.4A. Since the load is balanced, Neutral current = 0.

Industrial/Commercial power supply system

In industrial or commercial setting, where high power rated equipemnt to be operated, a three-phase system becomes a necessary. A common three phase system for small scale industry and commercial applications are 208Y/120V or 240Δ/120V system. Larger industries are supplied with mostly the 480V system. A schematic diagram of 208Y/120V system and 240Δ/120V system is shown as below.
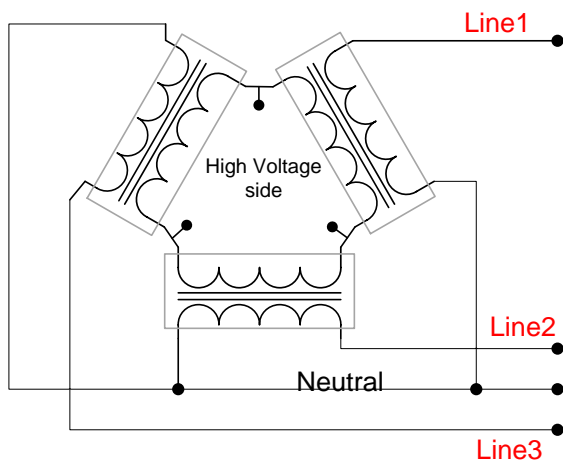
Figure 4.5. Three phase 208Y/120V supply system. Any phase to neutral is 120V, any phase to phase is 208V.
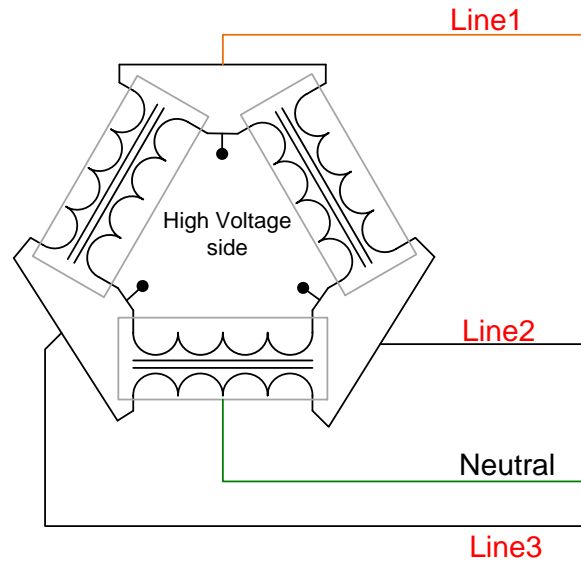
Figure 4.6. Three-phase 240Δ/120V supply system. The voltage between Lines are 240V but only Line2 and Line3 to neutral is 120V. Line 1 is called high leg (or Wild leg) and NEC 110.15 requires identified by orange color insulator.

Example 19

What is the voltage difference between High Leg and neutral for 240Δ/120V supply system?

Solution: The phasor for 240Δ/120V would be as shown below. The length of 'AN' is the voltage between the high leg and the neutral.
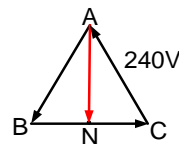
$$AN = 240\,Sin60° \approx 208V$$

National Electric Code (NEC)

NEC handbook is published by National Fire Protection Association (NFPA) as a regionally adoptable standard for the safe installation of electrical wiring and equipment in the United

States. NEC updates its code every three years, and each state requires some version of NEC.

The purpose of the NEC is the practical safeguarding of persons and property from hazards arising from the use of electricity. The Code contains provisions that are considered necessary for safety. Compliance results in an installation that are free from hazard but not necessarily efficient, convenient, or adequate for good service or future expansion of electrical use.

The objective of this notes is to introduce about NEC and briefly discuss the most common used safety rules designing a residential branch circuit and service panel design.

Article 100: Definitions

Ampacity. The maximum Current in amperes that a conductor can carry continuously under the condition of use without exceeding its temperature rating.

Appliance. Utilization equipment, generally other than industrial, that is typically built in standardized sizes or types and is installed or connected as a unit to perform one or more functions such as clothes washing, air conditioning, food mixing, deep frying, and so forth.

Branch Circuit. The circuit conductor between final overcurrent protection device protecting the circuit and the outlets.

Branch Circuit, Appliance. A branch circuit that supplies energy to one or more outlets to which appliances are to be connected, and that has no permanently connected luminaries that are not a part of an appliance.

Branch Circuit, General-Purpose. A branch circuit that supplies two or more receptacles or outlets for lighting and appliances.

Branch Circuit, Individual. A branch circuit that supplies only one utilization equipment.

Branch Circuit, Multiwire. A branch circuit that consists of two or more ungrounded conductors that have a voltage between them, and a grounded conductor that has equal voltage between it and each ungrounded conductor of the circuit and that is connected to the neutral or grounded conductor of the system.

Continuous Load. A load where the maximum current is expected to continue for 3 hours or more.

Ground. The earth.

Grounded (Grounding). Connected (connecting) to ground or to a conductive body that extends the ground connection.

Ground Fault. An unintentional, electrically conductive connection between an ungrounded conductor of an electrical circuit and the normally non-current-carrying conductors, metallic enclosures, metallic raceways, metallic equipment or earth.

Neutral Conductor. The conductor connected to the neutral point of a system that is intended to carry current under normal conditions.

Outlet. A point on the wiring system at which current is taken to supply utilization equipment.

Receptacle. A receptacle is a contact device installed at the outlet for the connection of an attachment plug.

Ungrounded. Not connected to ground or to a conductive body that extends the ground connection. Line1 and Line2 are ungrounded conductors.

Article 110.14 Electrical Connection

Conductors of dissimilar metals shall not be intermixed in a terminal or splicing connector where physical contact occurs between dissimilar conductors (such as copper and aluminum, copper and copper-clad aluminum, or aluminum and copper-clad aluminum) unless the device is identified for the purpose and conditions of use. Must use AL-CU rated wire connectors to splice copper and aluminum wires.
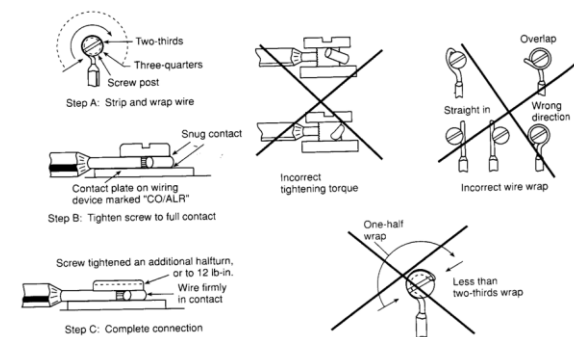
Figure 4.7. The correct and incorrect method of terminating a wire (Adapted from NEC handbook).

Article 200: Wiring Protection

Neutral conductors shall not be used for more than one branch circuit, for more than one multiwire branch circuit, or for more than one set of ungrounded feeder conductors. See NEC 200.4 for more.

An insulated grounded conductor of 6 AWG or smaller is usually identified by a continuous white or gray outer finish. See NEC 200.6 for

more options on identifying a grounded conductor.

A wire can be reidentified to indicate as a neutral/ungrounded conductor by marking tape at its termination and at each location where the conductor is visible and accessible. See NEC 207.C.1 for more details.

Article 210: Branch Circuits

The rating for other than individual branch circuits shall be 15, 20, 30, 40, and 50 amperes. The rating of a branch circuit is determined by the rating of the overcurrent protection device, not by the ampacity of the wire used. See NEC 210.3 for details.

Example:

- A branch circuit using No. 12 copper wire and a 15 A circuit breaker will be classified as a 15 A branch circuit.

- It is against the code to use 20 A circuit breaker in 15 A rated wire (such as 14 AWG)

A multi-wire circuit shall be permitted to be considered as multiple circuits.
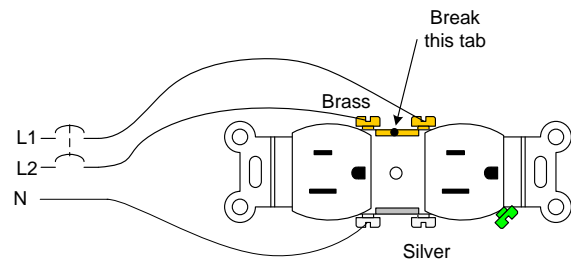
Figure 4.8. A multiwire circuit requires a simultaneous disconnecting of all ungrounded conductors to the multiwire branch circuit.

210.11 Required branch circuits for dwelling units:

1. **Small Appliance Branch Circuits**. Two or more 20-ampere small-appliance branch circuits shall be provided for all receptacle outlets.

2. **Laundry Branch Circuits**: at least one additional 20-ampere branch circuit shall be provided to supply the laundry receptacle outlet(s). This circuit shall have no other outlets.

3. **Bathroom Branch Circuits**. At least one 120-volt, 20-ampere branch circuit shall be provided to supply a bathroom receptacle outlet(s). Such circuits shall have no other outlets.

210.19 Branch circuit ratings

Branch circuit conductors shall have an ampacity not less than the maximum load to be served. Conductors shall be sized to carry not less than the larger of:

(a) Where a branch circuit supplies continuous loads or any combination of continuous and non-continuous loads, the minimum branch-circuit conductor size shall have an allowable ampacity not less than the non-continuous load plus 125 percent of the continuous load.

(b) The minimum branch-circuit conductor size shall have an allowable ampacity, not less than the maximum load to be served after the application of any adjustment or correction factors.

Branch circuits rated at 30, 40, or 50 A cannot be used for lighting or small appliance circuits in residence. Their use is limited to fixed appliances such as electric water heaters, ranges, and fixed space heaters.

210.21(B) Receptacles

1. Single Receptacle on an Individual Branch Circuit. A single receptacle installed on an individual branch circuit shall have an ampere rating not less than that of the branch circuit.

2. Total Cord-and-Plug-Connected Load. Where connected to a branch circuit supplying two or more receptacles or outlets, a receptacle shall not supply a total cordand- plug-connected load in excess of 80% of receptacel rating

3. Receptacle Ratings. Where connected to a branch circuit supplying two or more receptacles or outlets, receptacle ratings shall conform to the values listed below[4]

Circuit Rating (A)	Receptacle Rating (A)
15	Not over 15
20	15 or 20
30	30

If 15 or 20 A branch circuit provides fastened in place (such as grabage disposal) and other receptacles or luminaries, the fastened in place equipment shall not exceed 50% of the branch rating.

Larger appliances such as cooking range, clothes drier, AC units are usually designed to work at 240V.

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[4]NEC table 210.21(B)(3)

210.24 Branch Circuit Requirement Summary

Circuit Rating	15A	20A	30A	40A
Conductor Size	14	12	10	8
OCP[5]	15A	20A	30A	40A
Max Load	15A	20A	30A	40A

Other Branch Circuits Load Calcuations (220.10)

Lighting load: A unit load of not less than 3VA/sq. ft. for residential shall constitute the minimum lighting load. The floor area for each floor shall be calculated from the outside dimensions of the building.

3VA/sq ft includes all receptacle load of 20A or less and bathroom branch circuit load is is included in lighting load[6].

If the number of outlets to be served is known, a better recommendation is to allow no more than 10 outlets for each 15 A circuit and no more than 13 for each 20 A circuit. This is based on an estimated load of 1.5 A per receptacle (single or duplex).

10 outlets on a 15-A receptacle circuit

13 outlets on a 20-A receptacle circuit
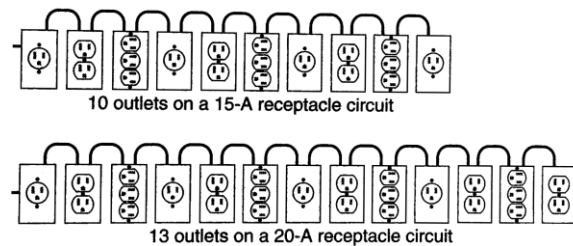
Figure 4.9. Maximum permissible outlets for 15 and 20-amp circuits (Adapted from NEC handbook).

Example 20

A residence need 100 lighting and convenience outlets, not including those to be supplied by the small appliance, laundry, and bath circuits. What ampacity and how many branch circuits are needed to serve this load?

Solution: Since general purpose branch circuit can be supplied either by 15 or 20 circuits, there are several possible combinations you can choose from. A 15-A circuit could have 10 outlets per branch and 20 A circuit can have up to 13 outlets. So, some possible solutions are 10 branches of 15A; 8 branches of 20A; or 4 branches of 20A and 5 branches of 15A circuits.

Electric Cloth Drier (220.54): 5 kW or Nameplate rating whichever is larger.

Electrical Cooking Appliances (220.55):

Because it is unlikely that all portions of the appliances will be in use at full capacity at the same time, sizing for the maximum load would likely produce a larger branch circuit than is needed. The NEC has developed a demand system for one range as follows:

- Nameplate rating up to 8.75 kW: Demand Factor = 80%
- Nameplate rating >8.75 kW to 12 kW: Demand Load = 8 kW
- Nameplate 12kW to 27 kW: Demand Load = 8 kW plus 0.4 kW for each kW over 12 kW

Example 21

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[5] Circuit breaker is usually used as Over Current Protection (OCP) device

[6] For details see NEC 220.14 (J)1.

Find the minimum size of branch circuits rating to supply 11 kW counter-mounted cooking unit and 10 kW wall-mounted oven.

Solution: Maximum load = 11 +10 = 21 KW

Branch circuit size = 8 kW + 0.4*(21-12) = 11.6 kW

At 240V, current = 11.6*1000/240 = 48.3A, Use the next higher rating branch circuit of 50A.

Motors (430.24): Conductors supplying several motors, or a motor(s) and other load(s), shall have an ampacity not less than the sum of each of the following:
(1) 125 percent of the full-load current rating of the highest rated motor, as determined by following table (Table 430.248)

HP	115V	230V
¼	5.8	2.9
½	9.8	4.9
¾	13.8	6.9
1	16	8
1½	20	10
2	24	12

(2) Sum of the full-load current ratings of all the other motors in the group, as determined by table as shown above
(3) 100 percent of the non-continuous non-motor load
(4) 125 percent of the continuous non-motor load.

Example 22

Design a 120 V branch circuit that supplies power to a garbage disposal of ¾ hp.

Solution: Table 430.248 specifies the branch circuit current for ¾ hp motor at 115V to be 13.8A. Branch Circuit Rating = 13.8*1.25 = 17.25A. A 20A branch circuit will be needed.

Service load calculation (Article 220.40)

The calculated load of a feeder or service shall not be less than the sum of the loads on the branch circuits supplied, after any demand factor permitted. Add items (1)-(5) to determine the minimum size of branch circuit.

(1) 220.42 General lighting: 3 VA/ft^2 for general lighting and general-use receptacles.

(2) 1500 volt-amperes for each 2-wire, 20-ampere small appliance branch circuit and each laundry branch circuit.

Add small appliance, and laundry load to calculate general lighting load. For dwelling units, following demand factor for general lighting is permitted:

First 3000 VA	100%
From 3001 to 12,000	35%
Remainder over 12,000	25%

(3) The nameplate rating of the following:

a) All appliances that are fastened in place, permanently connected or located to be on a specific circuit.
b) Ranges, wall-mounted ovens, counter-mounted cooking units
c) Clothes dryers that are not connected to the laundry branch circuit specified in item (2)

d) Water heaters

(4) The nameplate ampere or kVA rating of all permanently connected motors not included in item (3).

(5) Larger of the air-conditioning load or the fixed electric heating load to above calculated load to size the service panel.

A minimum allowed service panel is 100A. Common service panels are rated for 100, 150 or 200A of total current.

Example 23[7]

The dwelling has a floor area or 1500 ft^2, exclusive of an unfinished cellar not adaptable for future use, unfinished attic, and open porches. Appliances are a 12-kW range and a 5.5-kW, 240-V dryer. Assume range and dryer kW ratings equivalent to kVA ratings. Calculate the number of branch circuit required and minimum feeder (Service panel) size.

Solution: General Lighting load: 1500 ft^2@3VA/ft^2 = 4500VA

Minimum numbers of branch circuits required

General lighting: 4500VA/120 = 38A. This requires three 15A or two 20A branch circuit.

Small Appliance: Two 20A circuit.

Laundry: One 20A circuit

Bathrooms: One 20 A branch circuit (no additional load calculation is required for this circuit.)

Range: 12kW range is 8kW design load. With 240V supply current = 33A. A 40A branch circuit is needed for the range.

Dryer: largest of nameplate or 5kW, so 5.5kW. With 240V supply current = 23A. A 30A branch circuit is needed for the dryer.

Minimum size feeder required

General Lighting		4500 VA
Small Appliance		3000 VA
Laundry		1500 VA
	Total	9000 VA
3000VA @ 100%		3000 VA
(9000–3000) VA = 6000VA@35%		2100 VA
	Net Load	5100 VA
Range		8000 VA
Dryer		5500 VA
	Net Calculated load	18,600 VA

Net calculated load for 120/240V, 3-wire, single phase service is 18,600 VA/240V = 78A

Minimum allowable service panel size is 100A. Use 100 A panel size.

Exercise

1. List the specifically required branch circuits in a residential electrical wiring.
2. Show with a diagram, how we obtain 120/240-volt systems and 120/208 volt systems from a 3-phase power line.
3. Two balanced three phase loads are on a 277/480-volt line. One is a Y-connected motor load with Z = 6 + j3 on each phase. The other is a Δ-connected fluorescent light

---

[7] Adapted from NEC Appendix D1

load with Z = 12 – j4 on each phase. Ignore any losses in the transmission lines.

    a.   Sketch the circuit.

    b.   What is the current in one of the lines feeding the motor?

    c.   What is the current in one of the lines feeding the lights?

    d.   What is the real power, reactive power and VA for the total load?

    e.   What is the combined power factor?

4.   Given the 120/240–volt service configuration shown, determine what the current load would be on each of the three lines given the switch positions. The first case has been done for you.
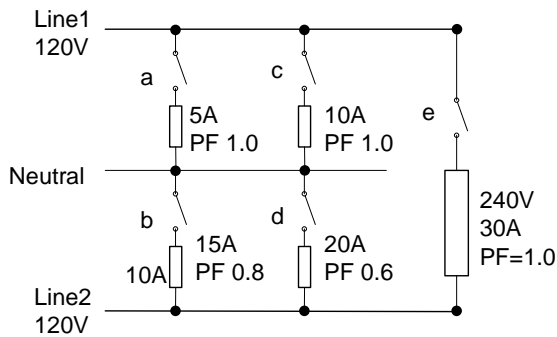
Case	Closed Switches	Load on each line (amp)		
		Hot 1	Neutral	Hot 2
1	a	5	5	0
2	a & b			
3	a & c			
4	a & d			
5	a & e			

5.   A branch circuit needs to supply power one ¾ hp motor fastened in place. Specify the minimum amperage rating for the branch circuit if the motor was a) 115V rated and b) 230V rated.

6.   Specify an amperage rating of a branch circuit that needs to supply power for 230V, two hp waste water pump and one hp feed water lamp used in a bioprocess plant intermittently.

7.   Determine the number of branch circuits along with their required ampacity and minimum required panel size for a farm residence. The dwelling has a floor area of 1500 square foot, exclusive of an unfinished basement not adaptable for future use and open porches. Appliances are

Range, 240 V	12 kW
Clothes dryer 240 V	5.5 kW

8.   For a dwelling that has a floor area of 1500 ft^2 exclusive of the unoccupied cellar, unfinished attic, and open porches, calculate the service equipment rating needed. It has a 12–kW range, a 2.5 kVA water heater, a 1.2 kVA dishwasher, a 9–kVA electric space heater, a five kVA clothes dryer, and a 6 A, 240 V room air conditioner unit for appliances.

9.   For the following 1300 ft^2 residential home as shown below.
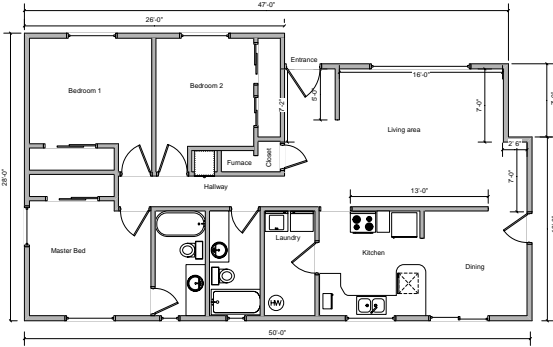
Generate a list of minimum branch circuits required with their ampacity. Also, calculate the service equipment rating needed.

The house has the following electrical equipment:

Electric Range:	10kW@240V
Garbage disposal	1/2 hp@120V
Dishwasher:	1000 VA@120V
Electric clothes dryer	5 kW@240V
Refrigerator	950 VA

True or false

1. Standard household electricity in the United States is rated 110V

2. What is the frequency of the standard household electricity in the United States is 60 Hz.

3. Residential power supply in US has 120V difference between line to line

4. A dwelling unit in US will have both 120V and 240V power supply.

5. As a rule, when wiring in conduit or cable the white wire is always fused or switched.

6. Number 14 AWG wire is used with 20-amp circuits in ordinary wiring.

7. Small appliances in kitchen area needs minimum of 2 branch circuits.

8. The smallest service entrance panel the code allows for a dwelling is 100 amps.

9. Kitchen counter outlets should have a GFCI protection.

10. Number 12 is the smallest wire size permitted in ordinary residential wiring.

11. The demand load for an 11-kW electric range is 11kW.

12. Number 14 AWG wire is used with 20-amp circuits in ordinary wiring.

13. Since the washing machine does not require much power it can be put on the same circuit as the clothes dryer in the laundry area.

14. In a branch circuit, up to 50% of rated load can be a fixed appliance.

15. The kitchen counter must be served by at least two 20-amp general appliance branch circuits.

16. If there are outlets that are below grade level, then NEC code requires these outlets should have a GFCI protection.

17. A branch circuit using No. 12 copper wire (20 A rated) and a 15 A circuit breaker will be classified as a 20 A branch circuit.

18. NEC code guarantees that compliance ensures the most efficient electrical system design.

Chapter 5 Fundamental of electric motors

Charged particles such as electrons experience a force when they move in a magnetic field. Per Lorentz equation for magnetic force:

$$\vec{F} = q\vec{v} \times \vec{B} \qquad (1)$$
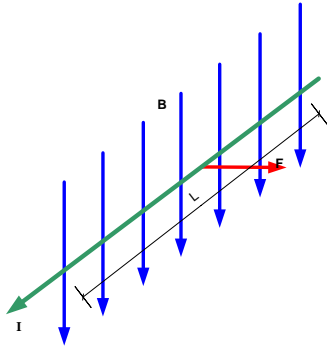
Where, q is the amount of charge in Coulombs, $\vec{v}$ is the velocity of the moving charge in m/s and $\vec{B}$ is the magnetic flux density in Tesla. The cross product provides the direction of Force with respect to the velocity of electric charge and the direction of magnetic field.

If the electrons are moving along a wire of length L at the speed of v m/s, it will take $t = L/v$ seconds for an electron to move from one end to another. For a constant current i flowing through the conductor, $q = it = \frac{iL}{v}$. If we substitute q in equation 1 by iL/v we get.

$$\vec{F} = L\vec{i} \times \vec{B} \qquad (2)$$

Note that the vector $\vec{i}$ points in the direction of electron flow which is reverse of the conventional current flow.

Calculate force on a 1 ft. of conductor coil carrying 1A current perpendicular to the magnetic field of 1T.

Solution: $\vec{F} = L\vec{i} \times \vec{B}$. Since $\vec{i}$ and $\vec{B}$ are perpendicular, $\left|\vec{i} \times \vec{B}\right| = i.B = 1\,A.T$

$$F = 1\,ft \times \frac{1m}{3.28ft} \times 1A.T. = 0.3N$$

Motional EMF

Equation 2 for force also applies to mechanically moved conductors in a magnetic field. If a current carrying conductor in figure 1 is free to move, it will go in the direction of F. The electrons in a moving conductor will experience yet another force in the direction opposite of the electron flow as shown in Fig. 2.
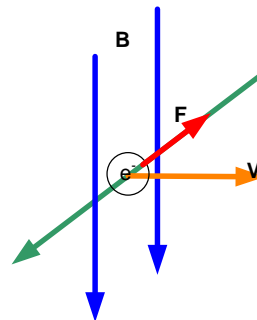
The opposing force on an electron due the physical motion is called *Motional emf (Memf)*. The magnitude of the emf from the force on

electrons can be derived from the definition of voltage.

$$emf = \frac{Force * Distance}{charge} = \frac{q\vec{v} \times \vec{B} * L}{q} \qquad (3)$$

If $\vec{v}$ and $\vec{B}$ are perpendicular to each other, $|\vec{v} \times \vec{B}| = vB$. Means that the motional emf can be written as:

$$Memf = BLv \qquad (4)$$

Since $Power = V.i$

$$Power = Memf \times i = BLvi \qquad (5)$$

Equation 5 provides the fundamental relation between electrical and mechanical power in electric motors. According to this equation, the ways to increase the power output from a motor is by:

i.   Increasing B
ii.   Increasing conductor length (L)
iii.   Increasing speed (v)
iv.   Increasing current (i)

There is a practical limitation on maximum magnetic field strength B that can be achieved for an electromagnet (less than 2T).

The total length L is the product of each conductor lengths and number of conductors. The number of conductors that can be bit in a motor is proportional to the cross sectional area of the motor. Therefore, the conductor L is proportional to the volume of the motor. This can be generalized to a fact that power output from a motor is proportional to the motor volume.
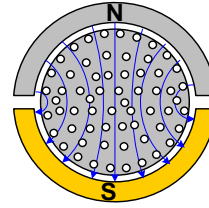
Figure 5.3. Number of conductors that can fit in an electric motor is proportional to motor's cross sectional area.

The speed is proportional to the motor rpm. More power is delivered by a motor running at a higher speed than at lower speed.

The fourth is increasing the current in the coil. There is also a limitation on rated current for a conductor in use. The Higher current generates more heat and may burn the motor.

Example 25

An application uses 8" diameter by 8" long, 1 hp motor to pump from an algae pond. It was determined the motor power was inadequate and needed to be replaced by 3 hp motor to handle the pumping effectively. The space limitation allows increasing the motor diameter only up to 12" diameter motor. If 12" diameter motor was used, what would be the approximate length of this motor?

Solution: We know that for similar operating conditions, the motor power output is proportional to the motor volume. The original motor volume is

$$V_1 = \frac{\pi d^2}{4} * L = 402 \; in^3$$

Approximate volume of the new motor $= 3 * 402 = 1206 \; in^3$

New motor length $= \frac{1206 * 4}{\pi 12^2} = 10.7 \; in.$

Permanent magnet DC motor

As the name implies, a permanent magnet DC motor (or PMDC motor) uses a permanent magnet for the magnetic field. A DC power connects to armature winding. The armature is a part of a motor winding where the motional emf is generated and is connected to the power supply.
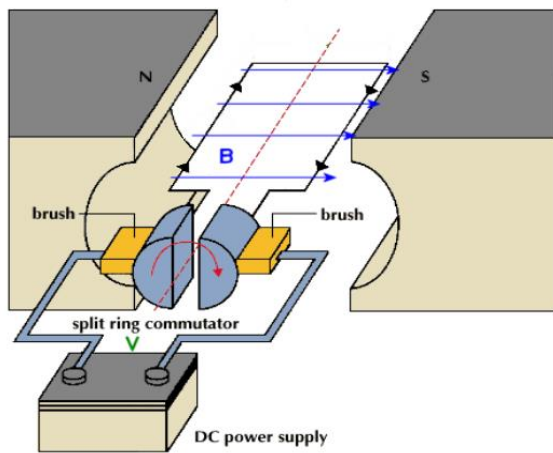
Figure 5.4. Schematics of a permanent magnet DC motor.

The effective conductor length experiencing force depends on the number of turns in the coil and is a constant for a given motor. For a magnetic field B, the relative speed of the conductor (v) is proportional to the armature angular speed(ω). Therefore, the motional emf developed in armature is given by,

$$Memf = kB\omega \tag{6}$$

where, k is a constant depending on armature winding and geometry. If the armature coil has resistance R, the equivalent circuit of a permanent magnet DC motor can be expressed as

$$E = IR + kB\omega \tag{7}$$

where E is the applied voltage across a DC motor. The current in the circuit for an applied external voltage E is given by

$$I = \frac{E - kB\omega}{R} \tag{8}$$
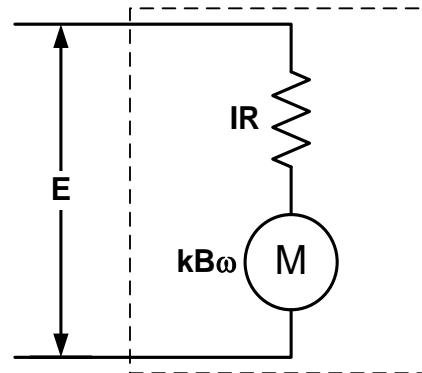
Figure 5.5. Steady-state equivalent circuit of a permanent magnet DC motor

When voltage E is applied when the motor starts, there is no motional emf as ω = 0. According to equation 8, and the initial current is E/R. As the motor spins, motional emf develops opposing the current flow. In theory, at no load, motional emf will eventually grow up to equal to E. In that case, no current will flow through the motor, and no force will develop as per equation 2. For a real motor, however, there is always some friction to overcome even when the motor is running idle. So $kB\omega$ <E and the current I is always greater than 0.

DC Motor Torque

If a motor is driving a load with torque T at an angular speed of ω, then the mechanical power delivered by the motor is $T\omega$. Also, the electrical power that is being converted to mechanical

power = $Memf * I = kB\omega I$. Equating these two,

$$Power = T\omega = kB\omega I \qquad (9)$$

$$T = kBI = kB\frac{E - kB\omega}{R} = \frac{kB}{R}E - \frac{(kB)^2}{R}\omega \qquad (10)$$

Since the value of k, B and R are constant for a given permanent magnet motor; Equation 10 represents a straight line with slope = $-\frac{(kB)^2}{R}$.

This tells us that for a permanent magnet DC motor with supply voltage E, the **'torque linearly decreases as speed increases.'**
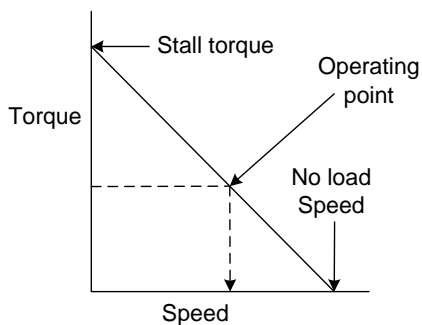
Figure 5.6. Torque-speed curve of a typical permanent magnet DC motor

The torque when the motor is stalled (ω =0) is the maximum torque offered by a DC motor and is equal to $\frac{kB}{R}E$. Therefore, we can also make a statement that for a permanent magnet DC motor **'starting torque is proportional to the applied voltage'**.

Equation 9 can we solved for ω as,

$$\omega = \frac{E}{kB} - \frac{RT}{(kB)^2} \qquad (11)$$

Equation 11 can be used to study effect of armature coil resistance. The higher the value of R, the slower the motor will run at given supply voltage and torque. This is intuitive as higher the resistance, higher the energy loss in the coil. For this reason, **larger motor tends to be more efficient than smaller motor** as larger motor uses thicker wire with relatively low resistance.

One phenomenon not obvious in equation 11 is the effect of magnetic field strength B on the motor speed.

First, let's consider the motor speed at no load condition. At no load, neglecting any resistance, torque produced tends to zero and the second term in right side of equation 11 can be neglected.

The first term tells us that lower the B higher the speed. This means **at no load; the weaker magnet produces higher speed than a stronger magnet**. This phenomenon may be counter-intuitive but if you think considering equation 4, lower the value of B, faster the motor has to spin to produce the same motional emf to counter supply voltage.

Decreasing B, to increase speed is not a good strategy. As we can see from equation 10, at a lower value of B, the stall torque is also lower. Additionally, the torque-speed curve has a lower negative slope, which means the speed reduces greatly with motor loading. As we will see later, reducing B also reduces the motor efficiency.
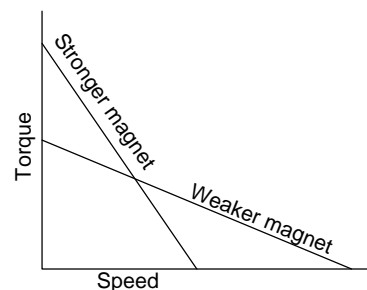
Figure 5.7. The torque–speed curve for strong vs. weak magnet motor.

Example 26

An 80 V DC motor with an armature resistance of 1 ohm is running unloaded at 1800 revolutions per minute (rpm) and drawing 2A current. Estimate the frictional torque.

Solution: Torque produced at no load is the frictional torque.

$$T = kBI = \frac{EI - I^2R}{\omega} = \frac{80*2 - 2^2*1}{1800*6.28/60}$$
$$= 0.83 \ Nm$$

Example 27

A full load current of the motor in above exercise is 30 A. Estimate the full load torque and the motor speed.

Solution:

$T = kBI$

From exercise 1 solution, kB = T/I = 0.83/2 =0.415

$$T_2 = kBI = 0.415*30 = 12.45 \ Nm$$
$$\omega = \frac{E}{kB} - \frac{RT}{(kB)^2} = \frac{80}{0.415} - \frac{1*11.62}{0.415^2}$$
$$= 125.3 \ rad/s$$
$$rpm = \omega * \frac{60}{2\pi} = 125.3 * \frac{60}{6.28} = 1197$$

Example 28

Calculate the torque and current when the motor is stalled.

Solution: When the motor is stalled, motional emf = 0

E = IR, or I = 80/1 = 80 amp

$$T = kBI = 0.415*80 = 33.2 \ Nm$$

Motor efficiency

Multiplying both sides of the equation 7 by I, we get

Power delivered to the motor=

$$EI = I^2R + kB\omega I \tag{12}$$

The first term of the right side is the copper loss which causes the heating of the motor and the second term is the mechanical power output from the motor.

$$motor \ efficiency \ (\eta) = \frac{kB\omega I}{EI} = \frac{kB\omega}{E} \tag{13}$$

This equation shows that **the motor efficiency is higher when operating at higher speed**. However, the motor speed should not be increased by reducing the value of B. The efficiency is proportional to $B\omega$ not just ω. To see the impact of reducing B on efficiency, we can multiply both side of equation 11 with B to get

$$B\omega = \frac{E}{k} - \frac{RT}{k^2B} \tag{14}$$

Equation 14 shows that lower the value of B, the higher is the negative term and hence smaller the value of $B\omega$.

Therefore, we can make a general statement that **higher the value of the magnetic field strength, the higher is the motor efficiency**.

It appears from equation 13 that motor efficiency increases with decreasing operating voltage E, for a given motor speed. However, remember that decreasing voltage also decreases the speed (see equation 11). Therefore, this equation itself does not provide the insight into the effect of E on efficiency.

We need to look at a proportional change in speed to voltage or $d\omega/dE$. If the $d\omega/dE > 1$ then increasing voltage increase efficiency. If $d\omega/dE = 1$ then supply voltage has no effect on efficiency and if $d\omega/dE < 1$ then increaseing supply voltage decreases the efficiency.

For a constant torque application, differentiating equation 11 gives us,

$$\frac{d\omega}{dE} = \frac{1}{kB} \tag{15}$$

Usually $kB < 1$ for a DC motor. Therefore, **the efficiency of a permanent magnet motor increases with increased operating voltage**.

Example 29

Draw speed versus useful torque, current, useful power output and efficiency of the motor in Example 26.

```
% Matlab Code for plotting
E = 80;
R = 1;
N = 0:1800;
w = N/60*6.28;
kB = 0.415;
I = E-kB*w;
T = kB*I;
P = (T-0.83).*w;
E = P./(E*I);
figure(1)
subplot(2,2,1)
plot(N,T)
axis ([0 1800 0 40])
xlabel('Speed (rpm)')
ylabel('Torque (Nm)')
title('Torque')
subplot(2,2,2)
plot(N,I)
axis ([0 1800 0 80])
xlabel('Speed (rpm)')
ylabel('Current (A)')
title('Current')
subplot(2,2,3)
plot(N,P)
axis ([0 1800 0 2000])
xlabel('Speed (rpm)')
ylabel('Power (W)')
title('Power')
subplot(2,2,4)
plot(N,E*100)
axis ([0 1800 0 100])
xlabel('Speed (rpm)')
ylabel('Efficiency (%)')
title('Efficiency')
Will produce the plot:
```
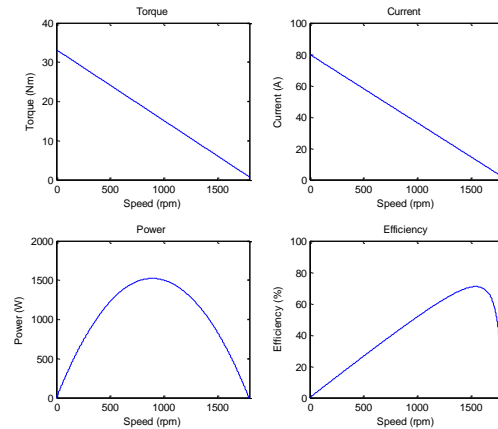

Use of the permanent magnet DC motor

One of the advantages of the permanent magnet DC motor is a linear torque–Voltage relationship for a given motor speed or linear voltage–speed relation for a given torque. A linear relation makes control of a permanent magnet DC motor easier.

Another advantage of a permanent magnet DC motor is the very high speed of rotation. DC motors can run at much higher speed compared to an AC induction motor. Since power is the product of torque and speed,

relatively high power can be generated from a small motor. Therefore, DC motors are used in high speed, high power to weight ratio applications such as unmanned aerial vehicles, prosthetics, and other battery-operated systems.

However, the permanent magnets with genral magnets have relatively low B and can't generate high torque. A neodymium magnets DC motor can produce higher torque, high rated speed and are more efficient. A gear box is usually used to boost the torque in high-torque low-speed applications such as for autonomous vehicles.

Even after using neodymium magnets, the magnetic field strength of a permanent magnet motor is not as high as electrically induced magnets. Therefore, the efficiency of a permanent magnet motor is relatively low. Therefore, the use of permanent magnet motor is limited to small low-cost applications.

Another weak point of DC motor is the mechanical contact between slip ring and brush. A carbon brush usually wears out over time. Additionally, there is usually a spark generated when the contact is disconnected because of the collapsing magnetic field when a coil is disconnected. The sparking makes, brushed DC motor inappropriate for conditions were flammable vapor may be around.

DC Motor Control

One of the advantages of a DC motor is that the torque or speed can easily be controlled by varying the supply voltage. For a constant speed application, equaiton *10* tells us that the output torque is linearly proportional to supply

voltage. For a constant torque application, Equation *11*, tells us that speed is linearly proportional to the applied voltage. The supply voltage for a DC motor can be achieved using an analog or a digital control. Analog voltage control uses an operational amplifier (commonly known as Op-amp) with a variable gain to produce variable DC voltage output. Although this method produces a smooth DC voltage, the circuit is harder to control using modern digital microcontrollers. Therefore, a method called pulse width modulation or PWM is used to generate an equivalent analog voltage using a digital controller.

PWM Control of a DC motor

The use of pulse width modulation allows varying the equivalent DC voltage (analog voltage) using only "ON" or "OFF".

The voltage is varied by controlling the ratio of on-time to the off-time pulse width called *duty cycle*. Figure 8 shows the most basic PWM control of DC voltage. For the first PWM signal, the duty cycle is 0.25 or 25%, for the second PWM signal duty cycle is 50%.
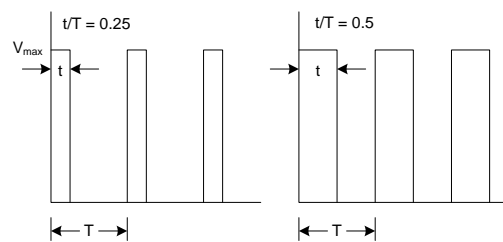
Figure 5.8. PWM signals with a duty cycle of 25% and 50%.

The equivalent voltage of the PWM wave is given by the RMS voltage, not the average voltage.

The RMS voltage or equivalent DC voltage delivering the same power, for a PWM signal is given by:

$$V_{rms} = \sqrt{\frac{1}{T}\int_T V(t)^2 dt}$$

Since $V(t)$ is eitht 0 or Vmax, it integration leads to: $V_{rms} = V_{max}\sqrt{\frac{t}{T}}$

Example 30

If a DC motor is running at full speed. It was desired to reduce the speed by a half for a constant torque application. Determine percentage full speed pulse width for the reduced speed. Assume that the winding resistance is negligible.

Solution: From equation *11*, we have,

$\omega = \frac{E}{kB} - \frac{RT}{(kB)^2}$, Since resistance is negligible, the equation simplifies to $\omega = \frac{E}{kB}$. Since, kB is constant for a motor, $\frac{\omega_1}{\omega_2} = \frac{E_1}{E_2}$

ω is proportional to motor rpm and hence, $\frac{\omega_1}{\omega_2} = 2$. E is rms voltage, which is proportional to $\sqrt{t/T}$. Therefore, $\frac{E_1}{E_2} = \sqrt{\frac{t_1}{t_2}}$. Equating

$2 = \sqrt{\frac{t_1}{t_2}}$ or $t_2 = \frac{t_1}{4}$

Therefore, the to reduce the speed by a half, the pulse width need to be reduced by ¼.

Generating PWM with a microcontroller

PWM signal can be generated using most of the microcontrollers available in the market. A microcontroller is a programmable device that can read inputs from external devices, analyze the input, and then output control signals to external devices. A microcontroller essentially contains three four components; microprocessor, clock, memory, and I/O pins.

A microprocessor is a single chip that executes commands stored in a non-volatile memory on each clock cycle and communicates with I/O pins. The clock produces a sequence of pulses uses to synchronize the information flow. All smart devices use a microcontroller.
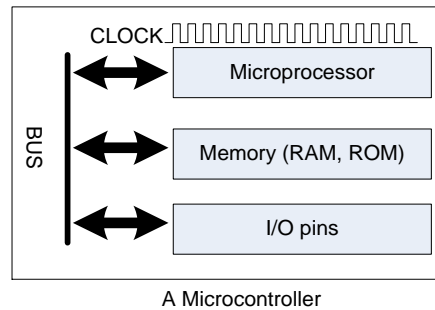
A Microcontroller

Figure 5.9. Major components of a microcontroller

A full description of a microcontroller is out of scope for this handout. We will use Arduino microcontroller to generate a PWM and to control. Arduinos are becoming increasingly popular in control applications because of its low cost and open architecture design. To learn more about Arduino microcontrollers please visit their website https://www.arduino.cc/. A good resource book for beginner is "Make: Getting Started with Arduino 3rd Edition".
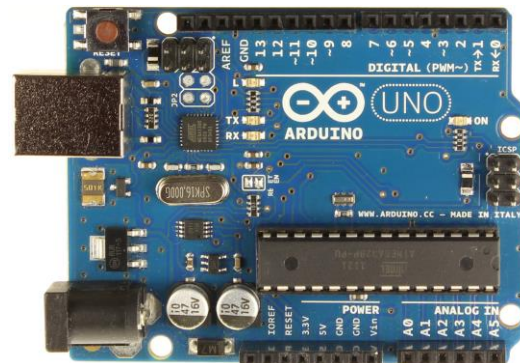
Figure 5.10. An Arduino Uno board, showing PWM capable I/O pins 3, 5, 6, 9, 10, and 11 with ~ symbol.

Arduino uses `analogWrite`(pin,value) function to generate PWM signal. The `pin` is the one of the PWM pins and value is the duty cycle between 0 (always off) and 255(always on).

After a call to `analogWrite()`, the pin will generate a steady square wave of the specified duty cycle until the next call to `analogWrite()`, (or a call to `digitalRead()` or `digitalWrite()` on the same pin). The frequency of the PWM signal on most pins is approximately 490 Hz. On the Uno and similar boards, pins 5 and 6 have a frequency of approximately 980 Hz.

Control Vs. Power signal

Although a microcontroller can generate PWM pulse, it should not be used to power the motor directly. First, most of DC does needs higher voltage than Arduino's maximum of 5V. Second, even if a DC motor is rated at 5V, the power that a microcontroller can provide is inadequate to run a motor at full load. A maximum output current for an Arduino is 40mA. This current is adequate to run not more than a 0.2W motor with 5V supply. Third, when you're powering a DC motor and then cut that power, the motor creates a voltage spike in the opposite direction because of collapsing magnetic field in the foil. This power burst can go back into your Arduino and cause damage. For these reasons, an Arduino should never be used to power the motor directly.

The trick is to use a transistor or a relay to power the motor. A transistor or a relay acts as a power switch that can be controlled by low power control side. A transistor is usually used for lower voltage (usually up to 60V) applications, and a relay is used for high voltage application (60V or higher and for AC). The main difference between a transistor and a relay is that relay's control side is electrically isolated from the power side. A relay will have a separate terminal for controls input and power switch connections, whereas a transistor will have a shared ground terminal.

A transistor would cost much less than a relay. There are two types of transistor that can be used as a switch, TTL, and MOSFET. MOSFET transistor uses low power (about 10 nW) than TTL transistor (about 10 mW) but has a slightly slower response.

Figure 5.11. A transistor or a relay is needed to control the DC motor using a microcontroller. Transistor has a common ground whereas a relay input and output are electrically isolated.

Some common TTL transistors used for switching are TIP 31 and TIP 120 series with compatible pin output as shown in figure 11. TIP 120 is a Darlington Transistor means two transistor stacked together. This allows to

control current load for the same amount of control current. TTL transistors use Base (B), Collector (C) and Emitter (E) pins and connected. MOSFET transistors use Ga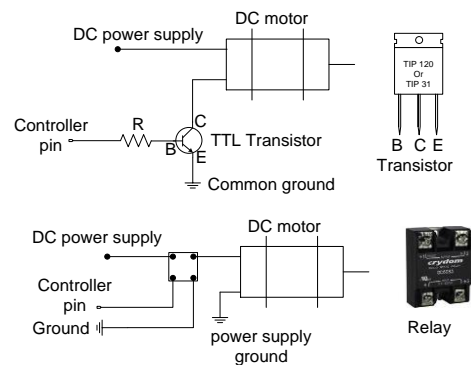te, Drain, and Source for equivalent Base, Collector, and Emitter pins. Some common MOSFET transistors are STP55 and IRF3205.

Dynamic Braking

A PMDC motor can easily be turned into a DC generator. When the armature of a PMDC is mechanically rotated, the motional emf is generated which can be measured across two motor terminals.
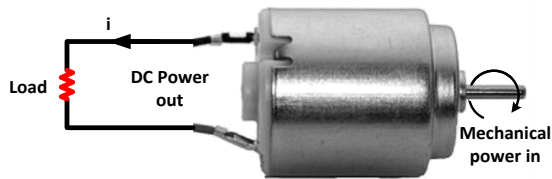
Figure 5.12. A PM DC acts as a generator when the shaft is turned mechanically. This phenomenon can be use used in DC motor braking.

Following Faraday's law, the induced current will be in such a direction to oppose the mechanical rotation. At no load, there is no current flow, and there is no opposing force. When a load, such as a resistor, is added to the output, the current flow makes armature harder to turn. Maximum resistance to mechanical turning is experienced with the output of a DC generator is short-circuited.

Shorting terminals to stop rotation can be used to brake a DC motor. After a DC motor is turned off, the inertial energy keeps the motor running and turns a PMDC motor into a generator. The electrical terminals are then shorted to help the motor stop faster by turning the inertial energy into heat in armature winding. This technique is called the **Dynamic Braking**. Braking is useful in applications when a more rapid stopping of the motor is desirable.

Direction Control of a DC Motor

The direction of a PMDC motor can be reversed by reversing the power supply polarity. However, the motor is being controlled by a microcontroller, an H-Bridge integrated circuit (IC) is used to control the speed. An H-bridge allows voltage to be applied to a circuit in either direction.
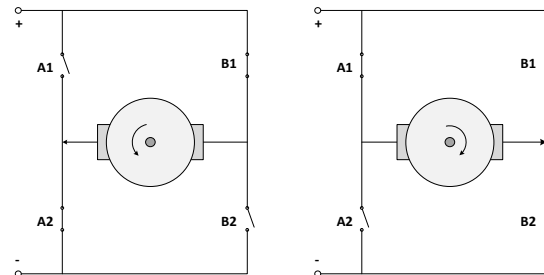
Figure 5.13. Schematics of H-Bridge controlling the direction of PMDC motor spin direction.

During normal operation, if A1 and B2 switches are open while B1 and A2 are closed, the motor will run in one direction. If A1 and B2 are closed while B1 and A2 are open the motor will reverse.   Simultaneous closing of A1 and A2 will cause a circuit. Same goes true for B1 and B2.

An H-bridge such as L293D contains two pairs of H-Bridges as shown below and can be used to control two separate motors of up to 36V. Each motor uses one side of H-bridge.

The pin1 is enable pin. This pin must be connected to 5V to make the left side of H–Bridge responsive. Similarly, Pin 9 is the enable pin for the right side of the bridge. Pin 3 can act as source (+ve voltage) or sink (as ground) depending on whether pin 2 is high (5V) or low (0V). This is equivalent to closing A1 and opening A2 when pin2 is high and opening A1 and closing A2 when pin2 is low. This configuration avoids closing A1 and A2 simultaneously.
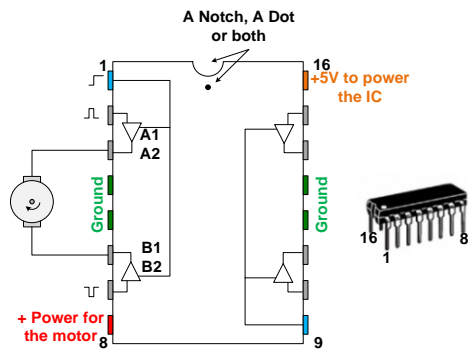
Figure 5.14. The H–Bridge IC L293D schematics.

The same logic applied for pin6 and Pin7. The state of Pin6 is controlled by Pin7. Therefore, when Pin2 is HIGH, and Pin7 is LOW, the motor will run in one direction. If the state of Pin2 and Pin7 is reversed, the motor will move in opposite direction. Disabling Pin1 will stop the motor.

Making both pin2 and pin7 at the same state will also halt the motor faster than disabling pins by braking action. For instance, pulling pin2 and pin7 to LOW will close A2 and B2 switch in Figure 5.13, short–circuiting the output.

Summary about permanent magnet DC motors

From the above discussion we can summarize the following general statements about permanent magnet motor. These statements are generally true for other motors as well, at least in the normal operating range. Therefore, it worth reviewing some of the statements what would help you control the motor effectively.

1. Torque linearly decreases as speed increases.
2. Torque increases with supply voltage.
3. Larger motor tend to be more efficient than smaller motor.
4. At no load, the weaker magnet produce higher speed than stronger magnet, however, the speed reduces at much faster rate as motor load increases.
5. The motor efficiency is higher when operating at higher speed.
6. The higher magnetic field strength produces higher motor efficiency.
7. The motor efficiency increases with increase in the operating voltage.
8. The motor speed is usually controlled by varying the DC voltage using PWM techniques.
9. Motor is dynamically braked when inputs are short circuited.
10. Motor directions can be reversed by reversing the input polarity.

Electromagnetic DC motors

Instead of using the relatively weak permanent magnetic field, a coil called field winding is used to generate the stronger electromagnet. It has been shown that the more powerful magnet translates into higher torque and higher motor efficiency.

There are two types of electromagnetic DC motors, a series motor and a parallel or shunt motor.

Series DC motor

Series DC motor (aka series wound DC motor) uses the electromagnet connection in series with the armature winding as shown in Figure 5.15.
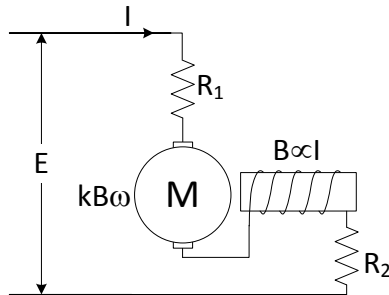
Figure 5.15. Series DC motor. $T = kBI = KI^2$. Since, I is highest when $\omega = 0$. Series DC motor has very high starting torque.

For an electromagnet, the magnetic flux density (B) and the magnetic field intensity (H). The magnetic field intensity H is the driving force for B and given by $H = \frac{ni}{l}$ where, n is the number of turns, I is current and, l is the magnetic path length. H has the unit of ampere–turn/m or Weber. The strength of an electromagnet on the other hand is proportional to the magnetic flux density B.
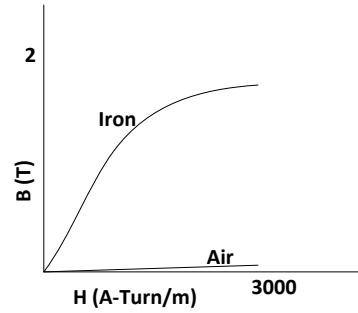
Figure 5.16. B–H curve for Iron Vs. air. Iron core produces about 5000 times' stronger magnet that air core.

For up to about 2000 A-turn/m, the B–H curve can be considered liner. Since H is proportional to the current, B is also proportional to the armature current. The equivalent circuit of the power delivered to the motor can be derived from modifying the Equation from PMDC motor as:

Power provided to the motor

$$=EI = I^2 R + KI^2 \omega \tag{16}$$

Where K is a constant and R is the sum of armature and field coil resistances. Keep in mind that this constant has a different value than used in the equation for PMDC motor.

From equation 1, $I = \frac{E}{R+K\omega}$ $\qquad$ (17)

The torque delivered by a series DC motor can be obtained as:

$$T = KI^2 = \frac{KE^2}{(R+K\omega)^2} \tag{18}$$

Equation 3 tells us that, for a constant motor speed the torque increases in proportion to the square of the applied voltage. For this reason, the starting torque of a series Dc motor is much higher than a PM DC motor. As motor starts to run, however, the torque reduces in proportion to the square of the motor speed making

torque-speed curve non-linear. This non-linear behavior of torque-speed makes speed control difficult especially at higher speed.

The behavior of a series DC motor is demonstrated through following the numerical example.

Example 31

An 80 V series DC motor draws 20 AMPS of current running at 1700 rpm to output 1.5 kW mechanical power.

1. Calculate the torque produced

   Solution: $\omega = rpm * \frac{2\pi}{60} = 177.9$

   $$T = \frac{P}{\omega} = \frac{1500}{177.9} = 8.43 \text{ Nm}$$

2. Calculate the value of constant K.

   Solution: $K = \frac{T}{I^2} = \frac{8.43}{20^2} = 0.0211$

3. Calculate the combined rotor and armature resistance

   $$R = \frac{E}{I} - k\omega = \frac{80}{20} - 0.211 * 177.9$$
   $$= 0.25 \ \Omega$$

4. Calculate the motor efficiency

   $$\eta = \frac{P_{out}}{P_{in}} = \frac{1500}{80 * 20} = 0.94$$

5. Plot Torque, Current, Power, and efficiency vs. speed for operating range of 100 to 1800 rpm.

Solution: Following MATLAB Code will produce the plot:

```
E = 80;
I = 20
N = 1700;
w = N/60*6.28;
T = 1500/w;
K = T/I^2
R = E/I-K*w
e = K*I*w/80
%Plotting
N = 100:1800;
w = N/60*6.28;
T = K*E^2./(R+K*w).^2;
P = T.*w;
I = E./(R+K*w)
e = P./(E*I)
plot(N,T)
figure(1)
subplot(2,2,1)
plot(N,T)
xlabel('Speed (rpm)')
ylabel('Torque (Nm)')
title('Torque')
subplot(2,2,2)
plot(N,I)
xlabel('Speed (rpm)')
ylabel('Current (A)')
title('Current ')
subplot(2,2,3)
plot(N,P)
xlabel('Speed (rpm)')
ylabel('Power (W)')
title('Power')
subplot(2,2,4)
plot(N,e*100)
xlabel('Speed (rpm)')
ylabel('Efficiency (%)')
title('Efficiency')
```
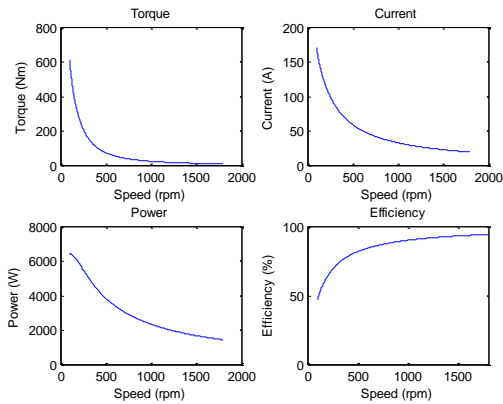
Parallel DC Motor

Parallel DC motor (aka shunt DC motor, or shunt wound DC motor) connects the field coil in parallel with armature coil. To limit the heating of the field coil, the current i2 is limited using a thinner wire with higher resistance. Use of thinner wire makes space for more turns to fit in, thus increasing H for the same i2.
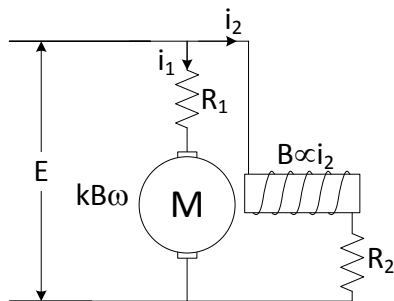
Figure 5.17. A schematic of a shunt motor.

Shunt DC motor can maintain speed for a wide range of torque. As a result, they are an excellent choice of motor when constant speed operation is necessary.

For a PMDC we had the relation, $T = \frac{kB}{R}E - \frac{(kB)^2}{R}\omega$ and we saw that Torque and speed has linear relation for a given voltage E with slope $= -\frac{(kB)^2}{R}$. Shunt motor performance is similar to this PMDC except that B depends on motor speed.

As speed increases, the motional emf increases and the voltage across armature coil also increases. The higher the armature voltage, higher is the current i_2 and higher is the B. However, even when $\omega = 0$ there will be some current in field coil because of R_1.

A general equation $B = k_0 + k_1\omega$ will approximately model the relation between B and ω . Substituting B in equation above, the torque–speed curve for a shunt DC motor approximately follows,

$$T = k_0 E + k_1 E\omega - k_2\omega^3 \tag{19}$$

Example 32

For a 24V parallel DC motor, the following load torque data was obtained. Find the approximate torque–speed curve for this motor.

Speed (rpm)	ω	Torque(lb–ft)
0	0	6
1400	146.6	16.3
1200	125.7	29.1

Solution:

Plugging the values for 0 speed in equation 4, we get,

$k_0 E = 6$ or $k_0 = 0.25$

Now from next two data points we get equations,

$10.3 = k_1 E * 146.6 - k_2 * 146.6^3$ and

$23.1 = k_1 E * 125.7 - k_2 * 125.7^3$

Solving for $k_1 E$ and k_2, we get

$k_1 E = 0.5$ and $k_2 = 2 \times 10^{-5}$

Therefore, the Torque-Speed relation is:

$$T = 6 + 0.5\omega - 2 \times 10^{-5}\omega^3$$

Example 33

Plot the Speed-Torque curve for the motor in above example for speed 0 to 1500 rpm.

Solution:

The following Matlab code will produce the plot

```
rpm = 0:1500;
Omega = rpm/60*2*pi;
T = 6+0.5*Omega-2e-5*Omega.^3;
xlabel('Speed (rpm)')
ylabel('Torque (Nm)')
title('Torque')
```
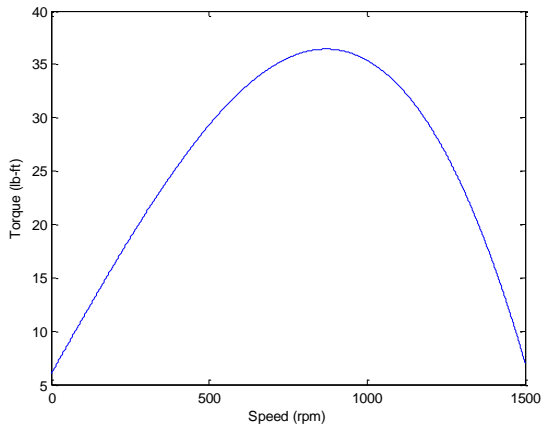

Selecting a DC motor

Usually, the selection of a motor deepens on its Torque-Speed characteristics. A general torque speed characteristics of three type of DC motors are shown below:
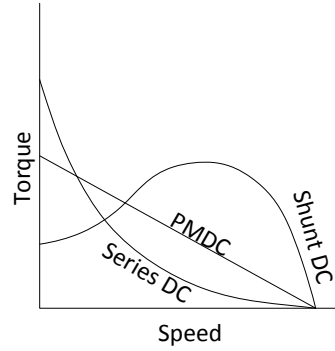
For a high-torque low-speed application, a series DC motor is a good choice. An example of such use is a screwdriver, lathe machine, rock crusher, paper shredder, and prosthetics. Series DC motor speed varies quite a bit for a small change in torque at high speed.

PMDC are easy to control because of its linear speed-torque characteristics over the entire range. They are comparatively cheaper. However, because of the low value of B, PMDC motors are not as efficient and produces relatively small torque.

Shunt DC motors are an excellent choice for fairly constant speed application at high speed. Relatively small speed change occurs over a large range of torque. However, shunt DC motor has a low starting torque. The motor is an excellent choice for applications such as conveyor belt, escalators, pumps, fans, etc.

There is also a hybrid motor with both series and shunt field called compound wound motor that has a superior starting torque yet maintains relatively constant speed for a large range of torque. These motors are useful for applications such as air compressor, scooter, wheelchair, etc.

Controls of Electromagnetic DC motor

Electromagnetic motor speed can be controlled with PWM signal as in the case of the PMDC motor.

One notable difference between a PMDC motor and Serial DC motor is that the rotational direction does not reverse when the power supply polarity is reversed. As shown in the figure below, when the power supply polarity is reversed, the direction of both current (i) and Magnetic field (B) changes. Since the Force is the cross product of B and i, the direction of the force will keep pointing in the same direction.
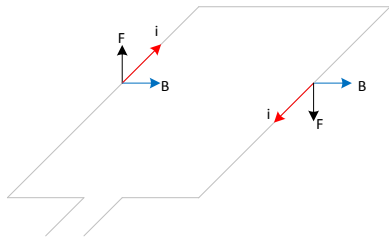
For that reason to reverse the motor direction, only one of either field winding or armature winding is switched.

Series DC motor is similar in construction to the Universal Motor, which can run on either AC or DC power supply.

Exercise

1. Calculate force on a 1000 ft. of conductor coil carrying 1A current perpendicular to the magnetic field of 2T.

2. Calculate the force and, torque developed, and current in the following coil in a permanent magnet DC motor.
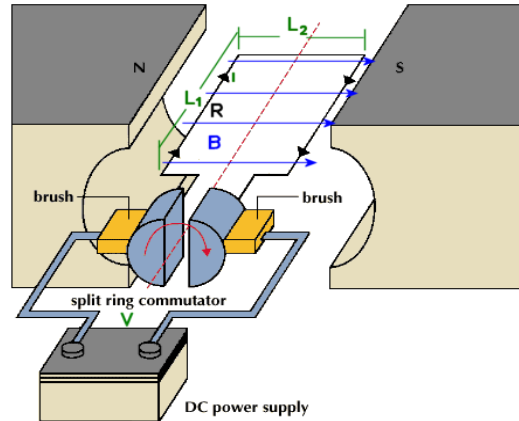
Assume that the coil is not rotating. The specifications are

L_1 = 4 in (0.1m), L_2 = 4 in (0.1 m), B = 1.5 T., Loop resistance (R) = 1 Ω, V = 5V, Number of turns (n) = 1

3. If the coil is allowed to rotate in above exercise, and rotating at 1000 rpm, calculate the current in the coil, force on L_1 and torque developed.

4. An application uses 10" diameter by 12" long, 3 hp motor to pump from a bioreactor. It was determined that the motor power was too high and needed to be replaced by a 1 hp motor to handle the pumping effectively. What would be the approximate volume of the new motor?

5. An 80 V DC motor with an armature resistance of 1 ohm is running unloaded at 40,000 revolutions per minute (rpm) and drawing 0.5A current.

   a. Estimate the frictional torque.

   b. A full load current of the motor in above exercise is 30 A. Estimate the full load torque and the motor speed.

c. Calculate the starting torque and current for this motor.

d. Draw speed versus useful torque, current, useful power output and efficiency of this motor.

e. If the motor is operated at 60 V instead of 80V, at full load, what would be the new motor efficiency?

6. What is the difference between average and RMS voltage?

7. Give some reasons why PWM controller output should not be used to power a DC motor?

8. What are the ways you can improve a DC motor efficiency?

9. Methanol, a flammable chemical is used in biodiesel production. What kind of hazard may you face using a DC motor in biodiesel production?

10. What are the advantages of using stronger neodymium magnet in a permanent magnet DC motor?

11. Motional emf is the product of ___, ____ and ____.

12. Explain how DC motors stop faster when terminals are shorted after power off?

13. What are the four ways of getting more power out of a DC motor?

14. What is an H-bridge? Can you use H-bridge just to control a DC motor speed? How many motors will you be able to control only speed (not direction) with L293D H-bridge?

15. A 50 V series DC motor draws 8 AMPS of current running at 1200 rpm to output 0.5 hp mechanical power.

    a. Calculate the torque produced

b. Calculate the combined rotor and armature resistance

c. Plot Torque, Current, Power, and efficiency vs. speed for operating range of 0 to 1500 rpm.

16. For a 24V parallel DC motor, the following load torque data was obtained. Find the approximate torque-speed equation for this motor and plot the torque-speed relation for this motor from 0 to 1500 rpm.

Speed (rpm)	ω	Torque(lb-ft)
0	0	15
1400	146.6	10.6
1200	125.7	25.6

17. A hydraulic gate on a dam needs to open slowly but against a very high initial torque. What kind of DC motor, A PMDC, series wound, shunt wound, or compound wound would better fit this application?

18. In a lumber mill, a machine needs to carry a log back and forth to make a multiple cuts. The motor must produce high torque to stop the motion in one direction and accelerate fast in reverse direction. Also, the motor needs to maintain the almost constant speed when the log is being fed through the saw at relatively high speed. What kind of DC motor, A PMDC, series wound, shunt wound, or compound wound would better fit this application?

19. A conveyor belt in a food processing plant carries food waste up to load a dump truck. The waste is deposited on the conveyor randomly, but the belt needs to maintain almost constant speed at all loads. The belt

does not need much torque to start. What kind of DC motor, A PMDC, series wound, shunt wound, or compound wound would better fit this application?

20. A toy car wheels do not need much torque, but it should be able to control the rotational speed easily and should be as low cost as possible. What kind of DC motor, A PMDC, series wound, shunt wound, or compound wound would better fit this application?

21. For a permanent magnet DC motor with following nameplate[8] data. Express torque, current, and efficiency in terms of rotational speed (ω) and supply voltage (V). Use the conversion factor, 1Nm = 8.85 lb-in and 1 rad/s = 9.55 RPM. What is the efficiency of the motor at full load?

HP	¼ at 1725 rpm	RPM	0 to 1725
Volts	80	Field	PM
Amps	3	SF	1
Torque	9.0 lb.in at 1725	Duty	Cont

22. An application in biomechanics requires the constant speed of 955 RPM at load varying from 1 lb-in to 15 lb-in. The speed is controlled to 955 RPM by varying the voltage. Express the voltage, current, and efficiency in terms of torque for this application. Check to see if the maximum voltage, current, and power output does not exceed the motor specifications. Plot using

technology the motor efficiency (y-axis) vs. power output (x-axis).

23. In above biomechanics application, it was determined that speed should be reduced by half (477.5 RPM) while maintaining the power output to a constant by doubling the torque output (from 2 lb-in to 30 lb-in). How does the motor efficiency is impacted by this decision? Check to see if the maximum voltage, current or power output does not exceed the motor specifications. Superimpose the plots using technology the motor efficiency vs. power output to the previous case.

24. Above motor is used in a robotic arm application. The arm is holding a 5 lb force with an arm length of 2" at the position shown below. What supply voltage do you need so that it will be just enough to produce the needed torque? What is the efficiency of the motor for this application and How much power is lost as heat?
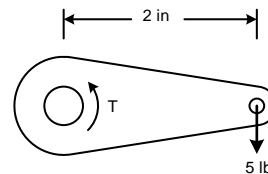
25. In the above application, a gearbox with speed reduction ratio of 10:1 is used (10 rpm of the motor to 1 RPM of gearbox output). Assuming no power is lost when gears are not moving, calculate the power

---

[8] HP = Horse Power; ARM = Armature; RPM = Revolutions per minute; PM=Permanent magnet; SF = Service factor; CONT. = Continuous

loss in the motor to maintain the same force after the gearbox use.

True or False?

26. Permeability of a given magnetic material is the same across all magnetic flux density.

27. Motional emf always opposes the flow of electron.

28. Permanent magnet DC motor will also run in AC because of the commutator.

29. Permanent magnet motor has a permanent magnet in its rotor.

30. For a DC series motor, starting torque produced is proportional to the applied voltage

31. For a PMDC, torque linearly decreases as speed increases.

32. In a permanent magnet DC motor, weaker magnet produces higher speed at no load.

33. DC motor produces higher torque at higher speed.

34. For a constant supply voltage, A PMDC motor provides higher torque at lower speed.

35. For a DC series motor, starting torque is proportional to the cube of the applied voltage.

36. DC motor efficiency is higher when operating at higher speed.

37. The stronger magnet produces higher motor efficiency.

38. For the same power output, motor efficiency increases with increase in the operating voltage.

Chapter 6 Induction motor

Induction motors are the industry workhorse. Almost all electric motors used in industry are induction motors because of its relatively low cost, robustness and the fact that the motor runs on AC power, avoiding the need to convert AC to DC.

All induction motors run on AC. Unlike DC motor, there is no reason to provide power to armature winding, meaning no slip ring and brush are required. This eliminates the spark and one of the main reason induction motors last longer. The power to the armature is delivered through induction, as in the case of a transformer, hence the name induction motor.

Principle of operation

At the core of every induction motor lies a squirrel cage like a rotor. To understand the workings of an induction motor, let's consider a mechanism with two magnetic poles and a squirrel cage rotor as shown in Figure 5.12.
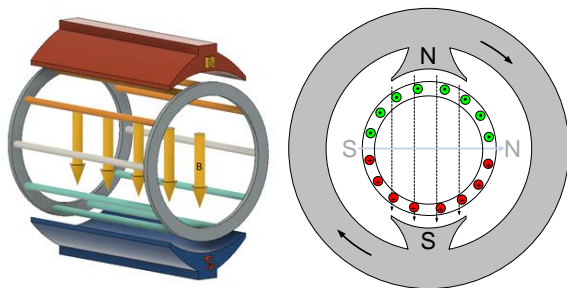
Figure 6.1. Relative motion of the outer magnet makes magnetic lines cut the squirrel cage bars. The direction of cut is opposite for the bars under N-pole and S-pole. The cage rotates along with rotating magnetic field.

Imagine that the outer magnet is being rotated mechanically. The relative speed of outer

magnet makes the magnetic N–S lines cut the cage bars creating a time varying magnetic field on the bars. According to Faraday's law, the varying magnetic field creates an electric current in the bars. The bars under the North Pole will have a current flow in the opposite direction than the bars under the South Pole because magnetic field cuts the bars in the opposite directions. End rings connect the cage bars. Therefore, a current loop is formed in the squirrel cage, and this induced current creates its own magnetic field perpendicular to the direction of the outer magnetic field. According to Lenz's law, the induced magnetic field will be in the direction to oppose the rotation of the outer magnet. The direction of the induced North pole is shown in Figure 5.12. The repulsion between like poles and attraction between opposite poles make the squirrel cage rotate in the same direction as the rotating magnetic field.

The speed of the outer magnetic field rotation is called 'Synchronous Speed.' In all induction motors, the outer magnetic field is produced by an electromagnet, and the field is electrically rotated. The synchronous speed of rotation depends on the supply frequency and the number of poles in the motor. The synchronous speed of rpm for an induction motor is given by

$$Synchronous\ speed = \frac{120f}{\#poles}\ \text{(rpm)} \qquad (20)$$

Example 34

Find the synchronous speed of a four pole induction motor connected to 60Hz power supply.

Solution:

Synchronous speed = 120*60/4=1800 rpm

Motor slip

The current developed in squirrel cage bars is because of the relative speed between outer magnetic field and the rotor speed. If both the cage bar and outer magnet is rotating at the same speed there would be no induced current, and hence no rotating force will be generated, and rotor will slow down. In practice, all induction motor rotor rotates at speed less than synchronous speed. The percent difference between synchronous speed and rotor speed is called slip and is given by the equation:

$$slip\ (s) = \frac{Synchronous\ speed - rotor\ speed}{Synchronous\ speed} \quad (21)$$

In terms of angular speed if the synchronous speed is ω_n and rotor speed is ω, the slip can also be expressed as:

$$s = \frac{\omega_n - \omega}{\omega_n} \quad (22)$$

Example 35

A 4-pole motor sold in the US indicates that full load speed is 1725 rpm. What is the motor slip at full load?

Solution:

Since the US has 60 Hz power supply, the synchronous speed of the motor is 1800 rpm.

Slip = (1800−1725)/1800 = 0.04 or 4%

Maximum efficiency = 1−.04 = 0.96 or 96%.

Modeling an Induction motor
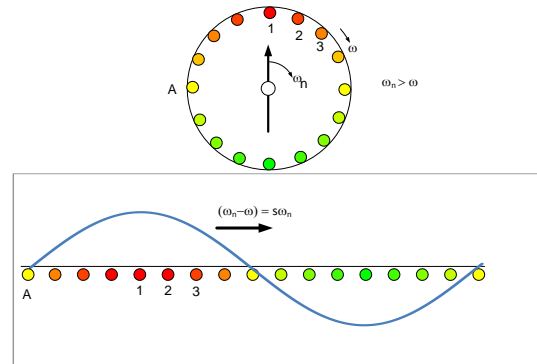
Figure 6.2. Slip cause a given cage bar to be under N-pole and S-pole alternately. This makes induced current in cage bar to alternate at frequency $s\omega_n$.

The induced voltage in the squirrel cage bar is proportional to the slip. No voltage is generated when slip is 0 (rotor speed = synchronous speed), and the maximum voltage is generated when s= 1 (rotor is stationary). If V_2 is the induced voltage when the rotor is stationary, then the voltage generated at a slip s would be sV_2. Also, the generated emf will have the angular velocity of $s\omega_n$.

To understand electrical to mechanical energy conversion, we can make an analogous comparison to a transformer. For an ideal transformer, the following relation between primary and secondary winding holds:

$$\frac{V_1}{V_2} = \frac{N_1}{N_2} = \frac{I_2}{I_1} \quad (23)$$

If an electrical load $Z_2 = R_2 + jX_2$ is connected to the secondary, the secondary current $I_2 = \frac{V_2}{R_2 + jX_2}$. The equivalent primary current $I_1 = \frac{N_2}{N_1} I_2 = \frac{N_2}{N_1} \frac{V_2}{R_2 + jX_2} = \frac{V_1}{(R_2 + jX_2)\left(\frac{N_1}{N_2}\right)^2}$. This is equivalent to having an impedance $Z'_2 =$

$(R_2 + jX_2)\left(\frac{N_1}{N_2}\right)^2$ in the primary side as shown in the figure above.

The induction motor is similar to a transformer except that the magnitude and frequency of induced voltage V_2 also depend on the slip. When the rotor is locked, the motor is no different than a transformer. A transformer can be thought as an electric motor with slip = 1.

The reactance of the squirrel cage depends on the frequency of the induced voltage. Since the reactance of an inductor $X_L = j\omega L$, the reaactance is proportional to the frequency which in turn is proportinal to slip.

If the squirrel cage rotor has impedance $Z_2 (= R_2 + jX_2)$ at slip = 1, the effective reactance of the rotor bar at slip s $is\ R_2 + jsX_2$.

Reduced induced voltage and lower reactance are the main differences between a transformer and an induction motor.

The current on the secondary side is $I_2 = \frac{sV_2}{R_2+jsX_2} = \frac{V_2}{\frac{R_2}{s}+jX_2}$. This secondary current produces the primary current $I_1 = \frac{N_2}{N_1} I_2 = \frac{N_2}{N_1}\frac{V_2}{\frac{R_2}{s}+jX_2} = \frac{V_1}{\left(\frac{R_2}{s}+jX_2\right)\left(\frac{N_1}{N_2}\right)^2}$
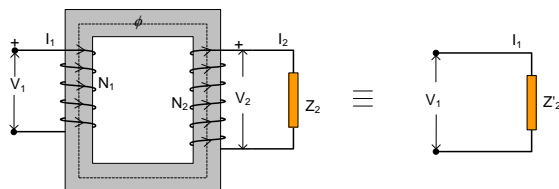
Figure 6.3. A load of Z₂ in secondary side is equivalent to connecting a load of Z'₂ directly to the primary.

To the primary, this is equivalent to having an impedance $Z'_2 = \left(\frac{R_2}{s} + jX_2\right)\left(\frac{N_1}{N_2}\right)^2$ in the

primary side as shown in the figure above. Note that only difference in the transformer equivalent circuit and the induction motor equivalent circuit is that the primary sees the resistance of the secondary increased by $\frac{R_2}{s}$ in case of motor.

Induction motor efficiency

The power delivered to the the rotor (secondary side) can be calculated from the equivalent circuit. The total real power delivered to the secondary is given by:

$$P = |I_2|^2 \frac{R_2}{s} \tag{24}$$

Of this power, the $|I_2|^2 R_2$ is lost as heat. The remaining power $|I_2|^2 R_2 \frac{(1-s)}{s}$ is converted to the mechanical power. Hence the motor efficiency is

$$\eta = \frac{|I_2|^2\frac{R_2(1-s)}{s}}{|I_2|^2\frac{R_2}{s}} = 1 - s \tag{25}$$

Therefore, **the theoretical efficiency of an induction motor is 1-s**. This is a surprisingly simple equation that tells us that the efficiency of an induction motor only depends on the motor slip. Theoretical efficiency is the maximum efficiency that a motor will ever have with no other losses, for all practical motor the actual efficiency is less than theoretical efficiency.

Example 36

A 4 pole motor sold in the US indicates that full load speed is 1710 rpm. What is maximum motor efficiency at full load?

Solution:

Slip = (1800–1710)/1800 = 0.05

Maximum efficiency = 1–.05 = 0.95 or 95%.

Torque-speed curve

The torque produced in the rotor can be calculated as:

$$T\omega = |I_2|^2 \frac{R_2(1-s)}{s} \tag{26}$$

Since the rotor angular velocity $\omega = (1-s)\omega_n$,

$$T = |I_2|^2 \frac{R_2}{s\omega_n} \tag{27}$$

$$T = \left|\frac{sV_2}{R_2+jsX_2}\right|^2 \frac{R_2}{s\omega_n} = \frac{sV_2^2}{R_2^2+s^2X_2^2}\frac{R_2}{\omega_n} \tag{28}$$

The induced voltage $V_2 = \frac{N_2}{N_1}V_1$, even though the effective turn ratio $\frac{N_2}{N_1}$ is not as clear to see in an induction motor as in a transformer, it is a constant (k) for a given motor geometry. Therefore,

$$T = \frac{ksV_1^2}{R_2^2+s^2X_2^2}\frac{R_2}{\omega_n} \tag{29}$$

The equation tells us that for a **motor operating at a constant slip, the produced torque is proportional to the square of the supply voltage.** This behavior is similar to a series DC motor.

The effect of slip on torque is not clear in equation 10 as the term 's' appears both in the numerator and the denominator. However, the maximum torque is produced when $\frac{s}{R_2^2+s^2X_2^2}$ is maximum. Differentiating this term with respect to 's' and equating it to zero, we get, $s = \pm\frac{R_2}{X_2}$. Eliminating negative value of s, *the torque is maximum when* $s = \frac{R_2}{X_2}$. The maximum torque produced in an induciton motor is given by:

$$T_{max} = \frac{kV_1^2}{2X_2\omega_n} \tag{30}$$

Since the slip where peak torque occurs depends on the cage bar resistance to reactance ratio, induction motors are designed to have a peak torque at various slip. There are four broad cage bars design classes from design 'A' through design 'D' with distinct torque speed characteristics. The induction motors are standardized for their performance and size by National Electrical Manufacturers' Association (NEMA). NEMA motor guide MG–1 (section II, Part 12) specified that Design D has the lowest starting torque of 275% of the full load torque. A typical torque–speed curve for design Ba nd D motor is shown below.
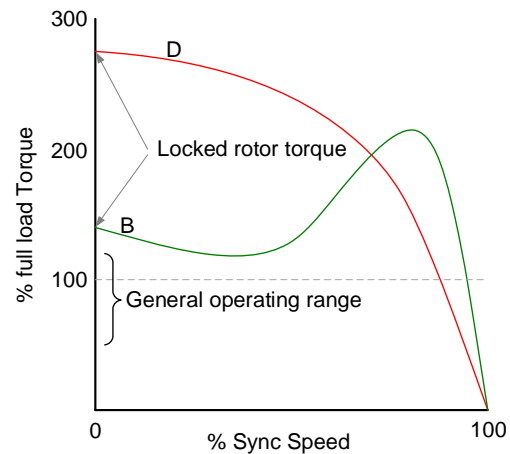

Figure 6.4. The torque–speed curve for design B and D motors.

There is a tradeoff between starting torque and motor efficiency. At a 100% load, design D motor has much lower % sync speed (higher slip) than design B motor. Since motor efficiency is equal to 1–s, design D motor will run less efficiently at full load than design B motors.

For that reason, design D motor selection should be limited to applications where a high initial torque is needed and for con continuous use, such as a garage door operation. Design D motor should be avoided for continuous use motors as they run inefficiently and runs hotter than comparably rated design B motor. Design B motor is the most commonly used motor because of its relatively high starting torque (aka locked rotor torque) and small slip at full load.

Torque-Speed in the operating range

Induction Motor when running normally, has an approximate linear torque–speed relation. This can be seen from Figure 6.4 that for both design B and D motors, the motor speed increases linearly with decreasing torque in the operating range up to 0 load. Since motor slip increases from right to left in that figure, the slip and torque has a linear relation as well.

For that reason, the how much the motor is loaded, and how consequently the power draw can be estimated simply measuring the motor speed. This is a quick way to determine the motor loading without measuring the power with a motor.
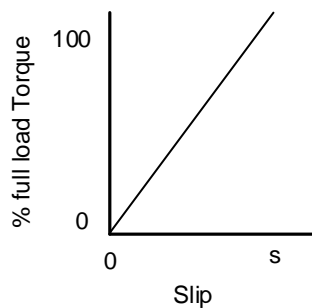
Example 37

A 1 hp 1710 rpm at full load induction motor is running at 1760 rom. What is the approximate power output from the motor at this speed?

Motor startup current

At startup, the motor slip is 100%. The effective rotor impedance $\frac{R_2}{s} + jX_2$, subsequently reduces to its lowst value while the induced voltage in rotor sV_2 goes to its highest value. This situation is equivalent to having a short circuit in a transformer secondary coil. This causes a high startup current commonly known as inrush current in primary side and, almost all of that power goes into heating the motor. Therefore, a stalled induction motor causes a fire hazard.

The ratio of startup current to full load current is usually printed in the motor nameplate at motor code letter. Smaller motors and design C or D motors require higher starting to full load current ratio compared to standard motors (design A or B motors). Standard motors with less than one hp have code letter L or not reported. Motors larger than 15 hp usually have code letter G. The KVA per hp for common code letters are shown in the table below (NEC Table 430.7(B).

Motor Code	kVA/hp
L	9–9.99
K	8–8.99
J	7.1–7.99
H	6.3 – 7.09
G	5.6 – 6.29

NEC 430.6 specifies for general motor (usually design A or B motor) applications, NEC Table 430.248, Table 430.249, and Table 430.250 shall be used to determine the ampacity of conductors or ampere ratings of switches, branch-circuit short-circuit and ground-fault protection, instead of the actual current rating marked on the motor nameplate. This is similar to the branch circuit for residential wiring discussed earlier. A separate motor overload protection shall be based on the motor nameplate current rating.

For high torque or low-speed motors (Design C or D motors), the rated current shall be locked-rotor current, and this nameplate current shall be used to determine the ampacity of the branch circuit conductors.

Example 38

Calculate the approximate locked-rotor current for a 1/2 hp 240 V motor with a motor code L.

From Table above, an L motor code implies 9 to 10 kVA/hp.

Solution:

Lowest startup current =

$$9000\frac{VA}{hp} \times \frac{1}{2}hp \times \frac{1}{240V} = 18.75\ A$$

Highest startup current =

$$10000\frac{VA}{hp} \times \frac{1}{2}hp \times \frac{1}{240V} = 20.83\ A$$

The starting current will be between 18.75 and 20.83A.

Example 39

A 30 hp 480V, 3-phase motor has code letter G. Determine the maximum locked rotor current for this motor.

Solution:

Max. locked rotor kVA $= 30 \times 6.29 = 188.7$

Max Locked rotor current =

$$\frac{Max.\ locked\ rotor\ kVA}{\sqrt{3}\ kV} = \frac{188.7}{\sqrt{3}\ \times 0.48} = 227A$$

Motor speed controls

Unlike DC motor an induction motor speed should not be controlled varying only the voltage. This can be explained by equation 10. The equation tells us that for a given slip, the torque is proportional to the square of the supply voltage. The torque speed curve for reduced voltage is shown in Figure 6.5.
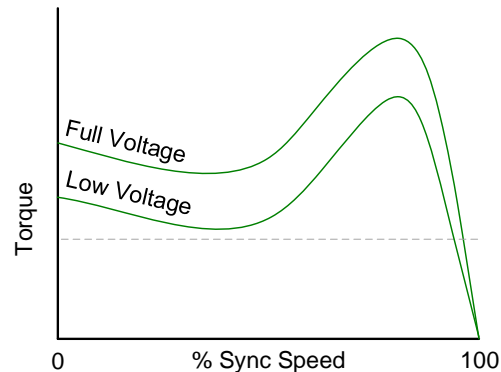

Figure 6.5. Running an induction motor at reduced voltage increases slip and reduces efficiency.

For a given load, the reducing voltage will reduce the speed, but reduced speed means increases slip and reduced efficiency. Reduced efficiency means more of the electrical energy converts to heat causing the motor to overheat. Normally a motor should not be allowed to run

at less than 90% of the rated voltage or else the motor may overheat.

Motors nameplate usually has insulation class information (A =105°C, B=130°C, F=155°C or H=180°C). The class gives the maximum temperature of the winding that the insulation is rated for. If the temperature exceeds this rating, the motor life would be reduced. As a rule of thumb for every 10°C increase in winding temperature above the rated value, the motor life is reduced by a half. Motor surface temperature does not reflect the winding temperature, and care must be taken. **Normally if the ambient temperature is 40°C or below and the motor is not overloaded and running at full voltage, the motor should not overheat.**

Additionally, even if the motor is not overheated, from Figure 6.5, the speed can not be reduced beyond the slip where the peak torque occurs. The peak torque occurs at 20% slip for design B motors. As soon as the applied torque is above this value, the motor will promptly stall.

Therefore, for an induction motor to control speed, synchronous speed needs to be controlled to keep the motor efficiency unchanged. Since $N = N_s(1 - s)$, the speed can be reduced by reducing the synhronos speed without affecting the motor efficiency.

However, changing the synchronous speed, both ω_n the reactance X_2 in equation 10 will change proportionately and as a result the output torque will also change. If we would like to keep the same torque speed curve, the supply volage must also be adjusted.

Looking at equation 11 for maximum torque, to keep the T_{max} constant, the voltage has to be changed at the same fraction of the frequency.

$$T_{max} = \frac{k(f.V_1)^2}{2(f.X_2)(f.\omega_n)} \tag{31}$$

When supply frequency and voltage are changed in proportion, the torque–speed curve will shift hence reducing the speed for a given torque without changing the slip.

A variable frequency drive (VFD) or also known as Variable Speed Drive (VSD) is a device used to control the motor speed by varying the supply frequency and the voltage in proportion. When supply frequency and voltage are changed in proportion, the torque–speed curve will shift hence reducing the speed for a given torque without changing the slip.
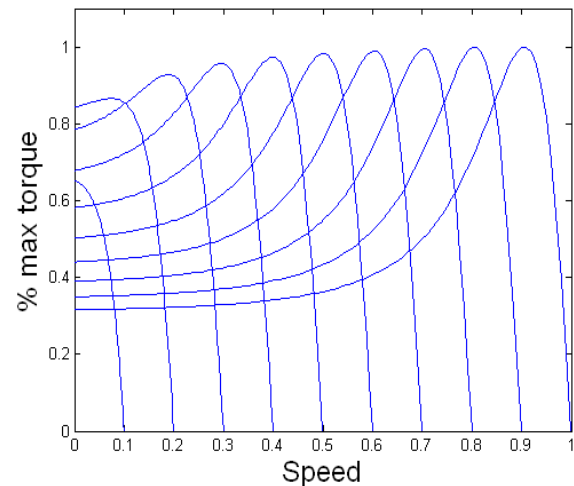

Figure 6.6. A typical speed-torque relation of VFD-driven motor at a various fraction of design synchronous speed. When supply frequency and voltage are changed in proportion, the torque-speed curve will shift.

In Figure 6.6, Note that when the speed is reduced less than about 15% of the design synchronous speed, the motor can no longer deliver the rated maximum torque. This may cause the motor to stall at very low speed. So the VFD speed control should not be used to run the motor at very low speed (as a rule of thumb not less than 20% of design speed). A combination of gearbox and VFD may provide a better speed control in those situations.

Example 40

For a three phase motor with nameplate as shown below

HP	½	Hz	60	PF	72.5
Volts	208	RPM	1725	Code	K
Amps	1.9	PH	3	Design	B

a) Calculate the slip at full load of the motor.

Solution:

Synchronous speed is greater than RPM. For a 60 Hz motor, only two possible synchronous speed that is greater than RPM is 3600, and 1800 RPM. Synchronous speed is the next higher speed than given RPM, in this case, 1800 RPM.

$$Slip = \frac{1800 - 1725}{1800} = 0.043$$

b) Calculate the motor efficiency at full load

Solution:

$P_{in} = \sqrt{3}\,V.I.PF = \sqrt{3} * 208 * 1.9 * 0.725\ = 496.3W$

$P_{out} = 0.5\ hp = 0.5 * 746\ W = 373W$

$Efficiency\ = \frac{373}{496} * 100\% = 75.2\%$

c) Calculate the torque developed at full load.

Solution:

$$373 = T * \frac{1725}{9.55}$$

T = 2.06 Nm = 8.85*2.06 Lb–in = 18.3 lb.in = 18.3*16/12 Oz–ft = 24.4 Oz–ft

d) Calculate the starting current for this motor.

Solution:

For motor code K, the locked rotor current is 8–8.99 KVA/HP. We can take the mid-range value to estimate the average inrush current as

kVA = 8.5/2 = 4.25 kVA. Since the motor is running at 208V

Inrush current = 425000/208 = 20.4 A.

Digital control of induction motor

VFD are digital device and produces only ON–OFF signal. The PWM technique can be used with time varying pulse width to generate an equivalent sine wave of various frequencies.

Pulse width modulated sine wave is generated by turning on and off the DC voltage to have the same RMS voltage as a sine wave.
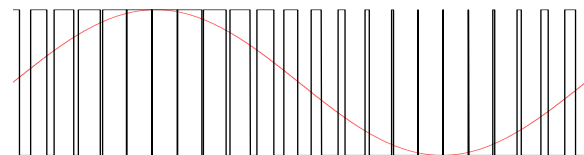

Figure 6.7. PWM simulated sine wave. The pulse width if proportional to the magnitude of the sine wave.

VFD controller converts the incoming AC voltage to a DC voltage first and then produces the AC of varying voltage through PWM

modulation. Therefore, a VFD is also called an inverter drive for this reason.

Induction motor types

Three phase induction motor

A three phase induction motor has an inherent rotating magnetic field. So when a three phase motor is connected to a power supply, the motor would automatically start in one direction.

When connecting a three-phase motor for the first time care should be taken on the direction of motor rotation as it is not possible to predict the rotation direction. If the motor turns in the opposite of the desired direction swapping any two phase will reverse the spin.
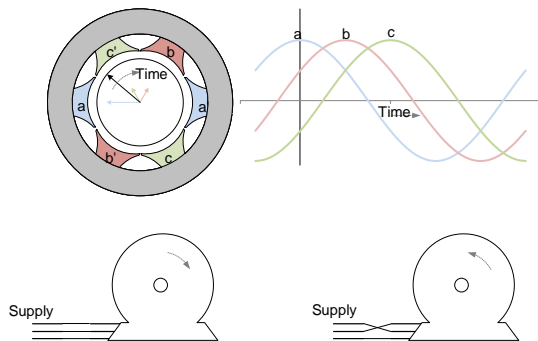

Figure 6.8. A three phase motor's resultant magnetic field rotates with time. Switching any two phases will reverse the rotation direction.

Single phase induction motor

Contrary to a three phase motor, single phase motors do not have a rotating magnetic field. Instead, it has a pulsating magnetic field.

If an AC power is applied when the motor at stationary, the varying current does induce emf in the rotor, but the current in all cage bars flows in the same direction. This would

generate equal force to all cage bars in the same directions. As a result, no net moment is generated to start the motor. The motor will just sit and hum with large inrush current.
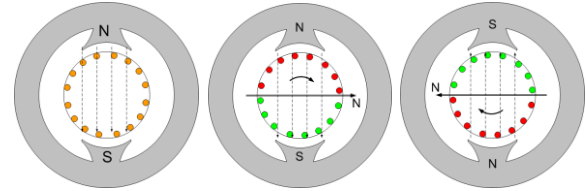
Figure 6.9. A single phase induction motor has no net starting torque. However, if rotor spins in a direction, the motor will keep spinning in that direction due to the induced loop current.

If the rotor is rotated by some means, the relative motion generates current in the opposite direction under the North Pole and the South Pole; This will cause the induced magnetic pole. As discussed earlier, the induced magnetic field will cause the rotor to keep spinning in that direction. The rotational direction is not affected by a pulsating magnetic field, as induced magnetic field reverse when outer magnetic poles are reversed as shown in Figure 6.9.

Common ways to start a motor is to create a time lagged secondary coil that acts as a second phase to create a rotating magnetic field. Single phase motors are usually categorized based on its starting mechanism. There are mainly three types of starting mechanism,

1. Capacitor start
2. Split phase
3. Shaded pole

Capacitor start

Capacitor start motor has a secondary coil in series with the secondary coil. The secondary

coil is turned off once motor picks up the speed using a centrifugal switch

Centrifugal switch
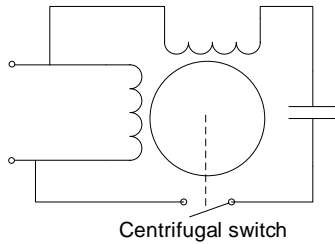
Figure 6.10. Schematics of a capacitor start the motor. Centrifugal switch is opened when the motor reaches certain threshold speed. The switch is closed back when the motor stops.

The capacitor creates a phase lag. Two pulsating winding with a phase lag creates a rotating magnetic field.

The motor rotation can be reversed by switching the polarity of primary or the secondary coil.

Example 41

For the following capacitor start the motor, the the resistance, reactance, and capacitance of a coil and capacitor are as shown below.
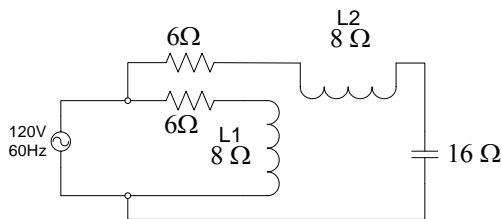
a) Calculate the magnitude and phase of the current in L1 and L2.
b) Plot the current wave for L1 and L2 for a complete cycle.
c) If the magnetic flux density is =0.1*coil current, assuming that coil L1 and L2

are perpendicularly placed. Plot the resultant magnetic field in MATLAB for a complete cycle

Solution:

a)

Impedance in branch L1= 6+8j,

Current in branch L1 = 120/(6+8j)

Magnitude of current in L1 = 12A

Phase of current in L1 = –53.1°

Similarly,

Impedance in branch L2 = 6–8j.

Magnitude of current in L2 = 12A

Phase of current in L2 = 53.1°

b)

The following MATLAB code will plot a complete cycle of current in branch L1 and L2

```
t = linspace(0,1/60,100);
I1 = 12;
I2 = 12;
Ph1 = -53.1*pi/180;
Ph2 = 53.1*pi/180;
Omega = 2*pi*60;
IL1 = 12*sin(Omega*t+Ph1);
IL2 = 12*sin(Omega*t+Ph2);
plot(t,IL1,t,IL2);
xlabel('Time (s)')
legend({'I in L1','I in L2'});
```

will produce the plot

c)
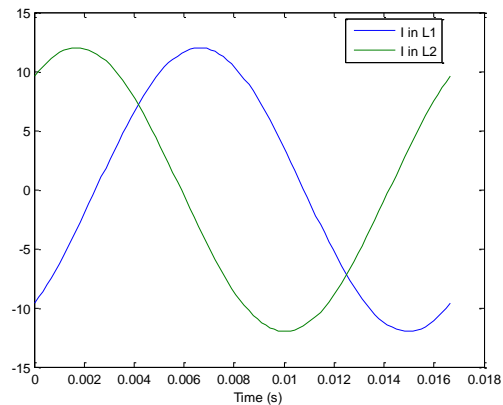
B in L1 = 1.2 with phase angle –53.1°

B in L2 = 1.2 with phase angle 53.1°

```
t = linspace(0,1/60,100);
I1 = 12;
I2 = 12;
Ph1 = -53.1*pi/180;
Ph2 = 53.1*pi/180;
Omega = 2*pi*60;
IL1 = 12*sin(Omega*t+Ph1);
IL2 = 12*sin(Omega*t+Ph2);
B1 = 0.1*IL1;
B2 = 0.1*IL2;
x = [0 B1(1) 0 0 0 B1(1)];
y = [0 0 0 B2(1) 0 B2(1)];
ht = plot(x,y,'linewidth',2);
axis(1.5*[-1 1 -1 1]);
xlabel('B1')
ylabel('B2')
for j = 1:length(t)
    x(2) = B1(j);
    x(6) = B1(j);
    y(4) = B2(j);
    y(6) = B2(j);
    set(ht,'XData',x)
    set(ht,'YData',y)
    pause(.1)
    drawnow
    F(j)=getframe;
end
movie2avi(F,'Animation.avi','compre
ssion','None','fps',10)
```
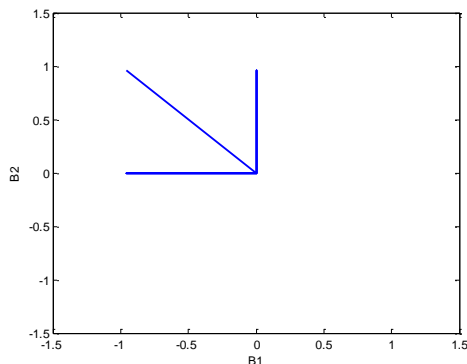Will plot B1, B2 and the resultant B vector. Noteice that even if B1 and B2 are pulsating vectors the resultant vector is a rotatinng vector.

Split Phase motor

Split phase motor has a similar principle as capacitor start motor. Instead of using a capacitor, secondary coil of higher resistance is used to create a phase angle. The secondary coil as a result has a lower current. The resultant rotating magnetic field is much weaker in split phase motor than a capacitor start motor.

Like in a capacitor start motor, the rotation can be reversed by switching the polarity of primary or the secondary coil.

Shaded pole motor

Shaded pole motor has a small copper loop shading the magnetic flux in that portion. A current is generated in the copper loop because of the time-varying magnetic field. The induced current opposes the change of magnetic flux. As a result, the magnetic flux under shaded pole is delayed compared to non-shaded portion. This causes a small phase lag in the magnetic field between non-shaded and shaded portion. Which is equivalent to rotating magnetic field although this is the weakest of all that we have discussed so far.
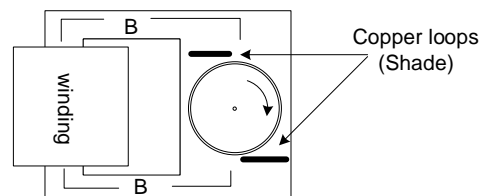

Figure 6.11. Schematics of a shaded pole motor. The shaded loop current is proportional to the rate of change of magnetic flux.

A shaded pole motor is the least expensive one and can be used where low starting torque is acceptable such as a small fan or pump motor.
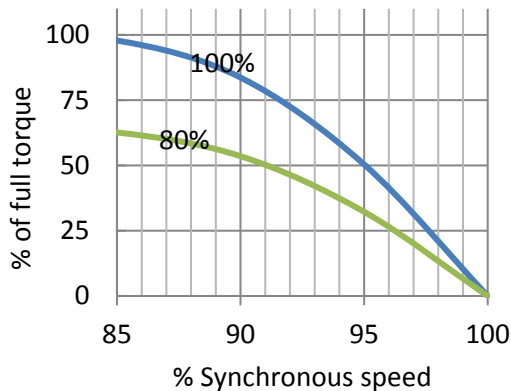
One disadvantage of the shaded pole motor is that the motor direction cannot be easily reversed. The motor will always rotate in the direction of shaded to unshaded part of the motor.

Exercise

1. The most common number of poles in an induction motor are 2, 4 and 6. What are the synchronous speeds for each of these poles?

2. If a motor rated speed is 1725, what do you concluded about the number of poles for this motor and its synchronous speed?

3. Explain why reducing only voltage or frequency to control the motor speed would not work?

4. What is the purpose of a capacitor in three phase motor?

5. What is the purpose of a capacitor in a single-phase motor auxiliary winding?

6. If a European motor designed to run at 240 V,50 Hz is brought to the US and powered with 240V,60 Hz power supply. What kind of performance difference do you expect in the US compared to one in Europe? Base your answer looking at equation 10 and 11.

7. How would you reverse a three-phase motor rotation direction?

8. Explain in your own words why a single-phase motor cannot start on its own as a three-phase motor without any additional windings?

9. Which of the single-phase motors discussed in this handout cannot be reversed for rotational direction?

10. How does a comparable sized capacitor start induction motor have a higher starting torque than a split phase induction motor?

11. For a three-phase motor with nameplate as shown below

HP	3/4	Hz	60	PF	0.8
Volts	208	RPM	1710	Code	K
Amps	3	PH	3	Design	B

a) Calculate the slip at full load of the motor.

b) Calculate the motor efficiency at full load

c) Calculate the torque developed at full load. Given that it is a design B motor, what is the approximate starting torque? [Hint: NEMA B design has starting torque approximately 150% of the running torque]

d) Calculate the starting current for this motor.

e) If the motor is running at 1750 rpm, approximately how much power the motor is producing?

f) The torque speed curve of an induction motor is shown in the figure below. At normal condition, the motor is loaded at 50% of the full torque. Estimate the reduction in theoretical efficiency of the motor when the motor is operated at 80% of the rated voltage.

12. Calculate the locked rotor torque to full load torque ratio for the following motor with locked rotor torque data given below. Use the conversion factor, 1Nm = 8.85 lb-in and 1 rad/s = 9.55 RPM.

Name Plate Info

Motor Type: AC Capacitor start
Rated hp : 1/4
Rated Volt: 115
RPM: 1725
Rated Amps: 3.7

Locked rotor test measured values:

Arm length: 9.5 in.

Locked rotor force: 3.5 lb.

13. A HVAC cooling tower fan runs at full speed when temperature of the condenser reaches 100°F and turns off when the temperature drops below 90°F. For a particular month, it was found that the fan comes on at full speed every 15 minute, runs for 5 minute, then shuts off, and then comes on again in 15 minutes. It was determined that if the fan were to run at 50% of the measured speed, the temperature of the condenser would remain between 90 and 100°F and there would be no need for the motor to turn off. If a VFD (97% efficient) was recommend regulating the fan speed to 50% of the current measured speed. Estimate the annual electricity cost saving and demand charge saving. Electricity costs $0.07 per kWh and demand charge is $5.00 /kW. Assume that the motor power factor does not change significantly from full load in either case. The HVAC system runs 16 hours a day, 6 days a week. (Hint: Use affinity law to estimate the power requirement with VFD]

Nameplate information:

HP=75

V = 460   I = 82.5

EFFICIENCY = 94.1 RPM=3555

Measured data:
V = 480V   I=76.1A   PF =
0.87   RPM = 3560

14. A motor was driving a heavy metallic return-line conveyor belt for hog fuel (mostly low grade tree bark). The return-line carried the hog excess to the boiler. The motor was running at full load driving the

conveyor belt at 1.5 linear ft/s. However, it was observed that the conveyor belt was carrying only little bit of hog fuel (less that 10% of the conveyor's capacity).

It was determined that it would be enough to run the belt at 1/10 th of the speed and still be adequate to carry the return fuel. So the industry was suggested to reduce the speed of conveyor belt to 1/10th of the current speed. Speed reduction could be achieve using a VFD or adding a gearbox. Since this was not a variable speed application, the team suggested using an additional 10:1 gearbox (98% efficient) to reduce the speed. Assume that the belt would require a constant torque at both speeds. Also assume a linear torque slip relation, and that the motor efficiency stays the same. Calculate the annual energy cost saving (@$0.07 per kWh). Also Calculate the new belt speed.

Nameplate information:

HP = 50   VOLTS = 460

AMPS = 57   RPM=1775

15. A motor (50 kW power input at full load) runs at full load to fill a tank with water when level drops to 10% mark and stops at 90% mark. It was determined that it takes 5 minutes to fill the tank from 10% to 90%

mark and it takes 15 minutes for tank to deplete from 90% to 10% mark before pumps starts again. It was recommended that the pump should run at 25% speed all time using a variable frequency drive (97% efficient) to meet the flow demand which would keep the tank level constant at 50% level. Estimate the average power saving in kW by taking this measure. Assume that motor efficiency does not change at reduced speed. Also assume that affinity laws hold true in this case that is power required by pump is proportional to the cube of the speed.

16. Which has greater starting torque, a capacitor start motor or a shaded pole motor?

17. What is the purpose of a motor centrifugal switch? Show by a sketch how it serves a useful role in the motor circuit.

True or False?

18. All induction motor must have some slip for it to produce torque.

19. One of the drawbacks of three phase motors is that it is inherently noisier than a single-phase motor delivering the same torque.

20. As a general rule, larger motors are more efficient in energy conversion than smaller motors.

21. Power factor of an induction motor is lowest when it is running at no load and highest when the motor is stalled.

22. Rotor resistance of design 'D' induction motor will be higher than that of design 'A' of same rated power.

23. Out of four different ways of controlling induction motor speed, changing the voltage is the best way to control the speed.

24. If you are using dual element time delay fuse to protect a single-phase induction motor, its rating can be up to 1.25 times the rated current for the motor.

25. Capacitor start induction run motors will be noisier in operation than capacitor start capacitor run motors.

26. Efficiency of a motor running at full load is 100%.

27. A motor running at 5% slip have efficiency greater than 95%.

28. Induction motor slip must increase with loading within normal operating range.

29. Running an induction motor at lower than rated voltage may overheat the motor.

30. Variable frequency drive changes both supply Voltage and frequency in proportion for a constant torque application.

31. A motor with nameplate RPM 3550 is most likely a 2-pole motor.

32. You cannot electrically reverse the rotational direction of a shaded pole motor.

Chapter 7 Motor Controls and PLC Programming

Motor Branch Circuit

The requirement for a motor branch circuit specification is covered in NEC Article 430. This note serves as introduction to the code using commonly encountered residential and industrial situation.

NEC 430.6(A)(1): Conductors that supply a single motor used in a continuous duty application shall have an ampacity not less than 125% of the motor full-load current. Table 430.247 to 430.250 shall be used to determine the ampacity of conductors or ampere ratings of switches, branch circuit short-circuit and ground-fault protection, instead of the actual current rating marked on the motor nameplate.

For Full load current shall be determined A branch rating for a motor typically is much larger than the full load current rating of the motor. For instance, the NEC code Table 430.248 specifies that for a branch circuit with a single motor running at 115V, 125 percent of the full-load current, as determined by National Electrical Manufacturers Association (NEMA), the current rating is 16A. In contrary, the full load current, for a one horsepower motor is about 6.5A (746W/115V).

NEC, required branch circuit rating less than the locked rotor current because of the way thermal circuit breakers work. Figure 7.1 shows a simplified current to trip time for an inverse time circuit breaker. Meaning, less time to trip for higher current. These are the kind commonly used in the residential electric panel.

Notice that a circuit breaker can handle the much higher current for a short time without tripping.
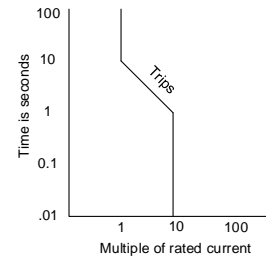
Figure 7.1. A simplified circuit current to trip time for a thermal inverse-time circuit breaker.

Be aware that the higher than full load rated branch circuit means the circuit may not trip when the motor is overloaded for a long time. An overload condition may arise from a condition such as failure to start from a locked rotor, a single-phase condition on a three-phase motor, or mechanical overload. Overloaded motor runs hotter and may cause fire hazard or ruin the motor. If one of the phases in a three-phase motor is disconnected, the motor will still run and is called single-phase condition. The current drawn from each line will almost double, and the motor will begin to overheat.

In general, there are four components of any motor control circuit that needs to be addressed to address adequate safety. They are:

Overload protection (NEC 430-III)
NEC 430.32 (A): Each continuous duty motor greater than 1 hp shall be protected against overload. 430.32 (B): One hp or less motor that is permanently installed or that could start

automatically or not within sight of the operator shall be provided against overload.

430.32(A)(1): An overload device shall trip at 115% of full load current or at125% of the rated current if SF is 1.15 or higher. Motor overload device that can restart automatically after tripping shall not be installed.

On power failure, a motor may shut down. After an unintentional shut down, it is extremely important that the motor does not suddenly start up again. Such an event may lead to a serious accident.

Branch Circuit Short-Circuit and ground fault protection (NEC 430-IV)

The motor branch circuit short-circuits and ground fault protective device for a single-phase motor shall not exceed 250% of the inverse time breaker (Circuit breakers are typically inverse time breaker). Ampacity rating for branch circuit in NEMA follows this requirement. However, the code allows using the higher rated protection to carry startup current without tripping the protection device.

430.53: More than one motors and other loads are permitted on 120V, 20A or less or 1000V or less, 15A or less branch circuit with the following restrictions:

i) The full load rating of each motor does not exceed 6 A.
ii) The rating of the branch circuit protective device is not exceeded.
iii) Each motor has its own overload protection.

Motor Controllers (NEC 430 VII)

The motor controller is the collection of components to control the motor operation such as starting, stopping, speed control and reversing. Motor controller unit may have integrated overload protection. A Variable Frequency Drive (VFD) is an example of a motor controller, but it could also mean a simple switch.

NEC 430.81 specifies that for a motor 1/8 hp or less this is normally left running and is constructed such that it cannot be damaged by overload or failure to start, the branch circuit disconnecting means is allowed as a motor controller.

430.81 (C) For a stationary motor, less than 2 hp and running below 300V, (1) a general switch having an ampere rating not less than twice the full load current of the motor or (2) a general snap-switch rated for AC motor where full load current is no more than 80% of the ampere rating of the switch.

Figure 7.2. A manual motor starter without or with thermal protection or a VFD are examples of motor controllers. VFD usually has built-in overload protection.

Disconnecting means (NEC 430 IX)

NEC 430.103: A disconnecting means shall open all ungrounded supply conductors (This includes neutral). Disconnecting means shall not close automatically.

An individual disconnecting means within the sight from controller location shall be provided for each controller. A disconnecting means for the motor shall be located in sight from the motor location and the driven machinery.

The controller disconnecting means is permitted to serve as the disconnecting means for the motor if it is in sight from the motor location.
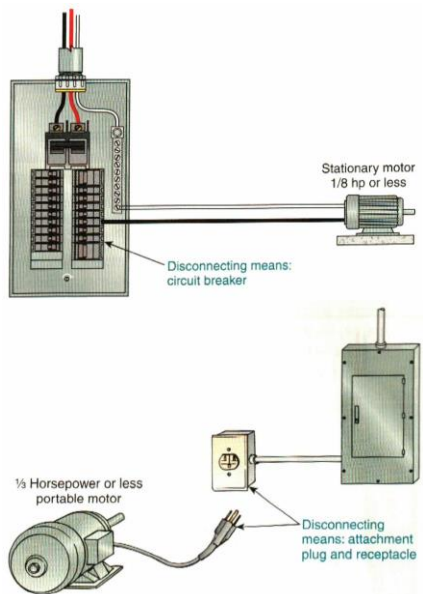

Figure 7.3. A branch circuit overcurrent device serves as a disconnecting means for a stationary motor less than 1/8 hp. Plug and receptacle serve as disconnecting means for 1/3 hp or less. (Source: 2014 NEC handbook).

NEC 430.109 specifies the type of allowable disconnecting means. A commonly used disconnecting means are 1) Motor circuit switch 2) Manual motor controller additionally marked "Suitable as Motor Disconnect" 3) Molded case circuit breaker, typically

combines, a thermal overload, short circuit and ground fault in addition to a manual switch.

For a stationary motor 1/8 hp or less, the branch circuit overcurrent protection device is permitted to serve as the disconnecting means.

For a stationary motor, less than 2 hp and running below 300V, (1) a general switch having an ampere rating not less than twice the full load current of the motor or (2) a general snap-switch rated for AC motor where full load current is no more than 80% of the ampere rating of the switch 3) A listed manual motor controller having a hp rating not less than the rating of the motor and marked "Suitable as Motor Disconnect"

Motor Controls Circuit

Designing a motor control circuit includes developing the circuit diagram for motor controls. NEMA motor controls usually have all the protection devices to meet the NEC code. Circuit diagram of a single phase NEMA motor controller looks like this:
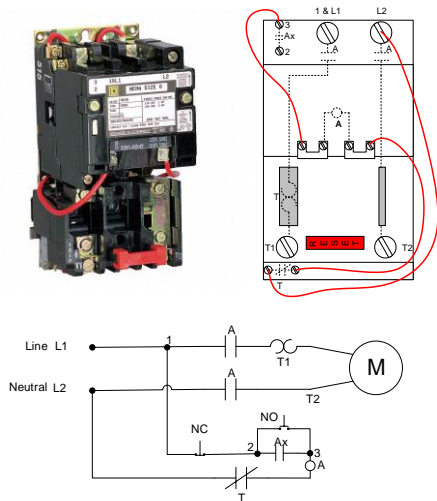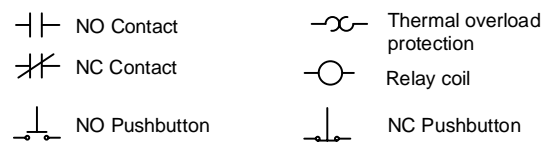

Figure 7.4. Internal circuitry and connection diagram of NEMA single phase motor starter. Dotted lines indicate internal connections. Solid wires are external connections.

The circuit above uses a standard ANSI electrical symbols. NO stands for Normally Open and NC stands for Normally Closed.

Relay switch or Contact is a type of switch controlled by a relay coil. When a relay coil is energized, all the relay switches associated with that coil is activated. Activating a NO relay switch will close it and activating a NC relay switch will open it. A relay could be a

mechanical device that physically connects the relay switches, or it could be an electronic relay. Electronic relays are becoming more common as it makes no noise, more efficient and can switch extremely fast.
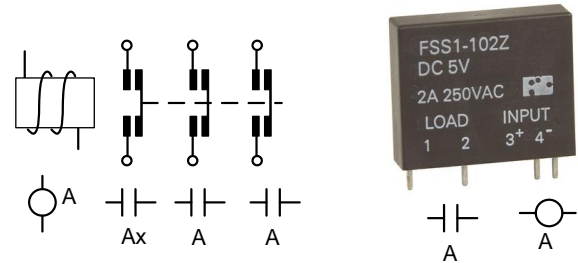
Figure 7.5. Mechanical relay (left) and electronic relay (right). When coil A is energized, all relay switches (Contacts) are activated.

Thermal overload protection for a motor disconnects a circuit when it senses a long period of overcurrent. Thermal overload is a mechanical device that disconnects a circuit on high current flow. A good video explaining a working principle of thermal overload protection is found at www.youtube.com/watch?v=1xbsxEWxo3A

In the above motor starter circuit diagram, when a START switch (a NO switch) is pressed momentarily, the coil 'A' is energized. This will close three relay switches that are named 'A' or 'Ax'. Since both relay switches, 'A' are closed, the motor is powered. Activated 'Ax' relay switch will keep the circuit closed even when the NO switch is released. When STOP switch (A NC switch) is pressed, the power to relay 'A' is disconnected and it de-activated all relay switched disconnecting the power to the motor.

During a power down, the relay 'A' is deactivated. When power restored, the relay is not activated until start switch is pressed hence preventing auto staring of the motor.

Staggered Starting of Motors

As we have seen, electrical motors draw several folds more current at startup compared to the normal operating condition. If several motors were needed in a process, the combined current could either trip the circuit breaker or may cause unnecessary electrical peak demand. Usually, industries pay a separate demand charge in addition to electrical usage, and monthly demand charge is based on the highest demand for that month. Demand charge per kW is typically 100 times higher than cost per kWh of electrical usage.

To avoid this peak demand, a stagger-start or time delay start with the motors should be implemented when possible. The time delay between motors can be achieved by use of a delay timer.
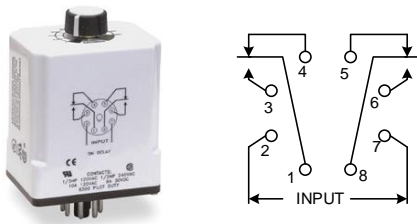

Figure 7.6. On time delay switch.Terminal 1&3 are NO. When terminal 2 and 8 (input) is energized, and 1&3 are closed after a certain time delay.

A schematic of an On-delay timer is shown in figure 6. This timer has two separate switches. When the input is not energized, the terminal pairs 1&4 are closed, and terminal 1&3 are

open. Once the input is energized (enabled), the timer starts. When the timer is done, terminal 1&4 opens and 1&3 closes. The same action happens between terminals 8, 5 and 6.

A physical connection of staggered started two motors is shown in Figure 7.7.
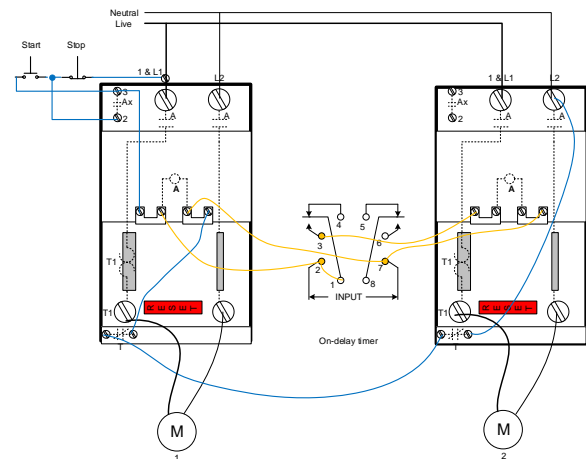
Figure 7.7. The physical connection of two motors with an on-delay timer. The second motor runs only after a certain time delay. Notice that auxiliary coil in the 2nd motor controller is not used, and thermal protection is connected in series.

When the start switch is pressed, the coil in motor 1 (left) is activated immediately. The blue lines make the closed circuit. That also starts the timer and after a certain time delay, the coil A of the second motor is activated.

Notice that coil A of both motors is connected in parallel. Therefore, the auxiliary coil in motor 2 controller is not needed. When stop switch is pressed, the circuit in left coil A is broken. That will timer input to go lo and both motors are turned off immediately. Also, the thermal relays are connected in series, so that tripping of any

thermal relay will break the left coil A and both motors are turned off immediately.

Programmable Logic Controllers

A PLC (Programmable Logic Controller) is defined as miniature industrial computers that contain hardware and software used to perform control functions. PLC is widely used today in industrial automation and controls. If there is industry present, the chances are, there is a plc present.

A controller consists of two basic sections: the central processing unit (CPU) and the input/output (I/O) interface system. The CPU, which controls all system activity, can further be broken down into the processor and memory system. The input/output system is physically connected to field devices (e.g., switches, sensors, etc.) and provides the interface between the CPU and the information providers (inputs) and controllable devices (outputs).

Programs are typically written using ladder logic, a language that closely resembles a relay-based wiring schematic. Usually the program "reads" input data from connected field devices using its input interfaces, and then "executes" or performs the control program.

Initially, devices that controlled the discrete attributes ON-OFF of a machine were known as Programmable Logic Controllers (PLCs). This tended to emphasize that the main functionality of these systems was LOGIC operations. As technology has advanced, so have the PLC capabilities. Today's PLC can do not only digital

input-output but also analog input-output and communications such as emails and texts to the operator for information or about an error.

PLC is almost identical in purpose to a popular microcontroller such as Arduino or STAMP; the only difference is that the PLC is industrial grade and uses ladder login as a programming language.

PLC Components

PLC has input and output (usually wire terminals in a module) to communicate with outside world. The PLC itself consists of CPU, Memory and Communication lines. A power supply is also an integral part of a PLC. The PLC is mounted on metallic rails and usually resides inside a metallic cabinet.
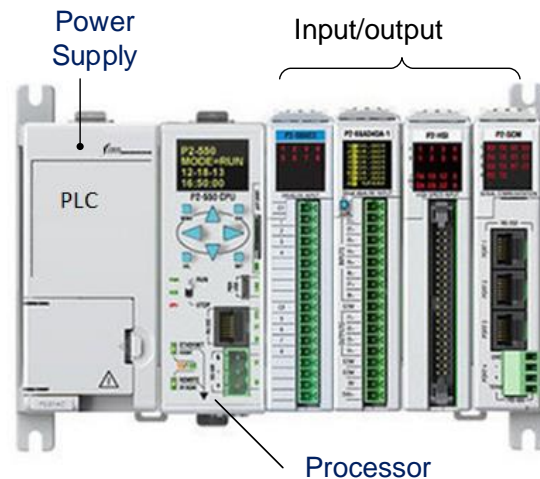

Figure 7.8. Components of a PLC

The reading of inputs and controlling is done through logic programming, stored in memory inside of the processor (Figure 7.8). Almost all the industrial control programs are done in a program called 'ladder logic.'

PLC Programming

Each input and output terminal has a unique address in memory. Depending on PLC brand, the naming convention of input and output may be little different. However, any modern-day PLCs allow assigning a convenient name instead of default name to any input-output memory addresses for easy recognition. Most of the PLC inputs are switches or contacts, and most of the PLC output is relay **coil** or load. The relay coil is the switch that PLC controls to control high power equipment such as a motor.

Ladder logic program

Relay-based control systems need to be considered in two separate segments: the control circuits and the power (or load) circuits.
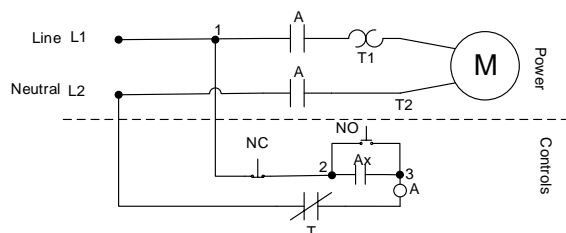
Figure 7.9. Motor start circuit can be divided into power and controls circuit. The PLC is simulating the control portion of the circuit.

If we look at the motor starter circuit, we can divide this circuit into power and controls circuit (Figure 7.9). Ladder logic program simulates the control portion of the circuit very closely (Figure 7.10), and as a result, it is a very popular form of language in industrial controls. Industrial automation professions view ladder logic as more robust, and visual than text-based language.
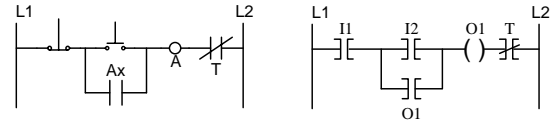
Figure 7.10. Control circuit on the left is same as above. The ladder logic program or right resembles the actual control circuit. The switches are represented by contacts.

In the ladder logic diagram, the state of an input terminal is copied in memory and can be addressed using logic instructions. For reading binary inputs such as push button usually two types of instructions are used:

XIC (Examine if Closed): This instruction is TRUE if the input is High. It is represented as][in ladder logic diagram (Figure 7.11). This is the most common instruction. When this instruction is connected to another logic bit, this instruction is the same as a connected bit.

XIO (Examine if Opened): This instruction is TRUE if the input is Low. It is represented as]/[in ladder logic diagram. When this instruction is connected to another logic bit, this instruction is the opposite of the connected bit.

The ladder logic diagram in figure 10 uses XIC instructions I1, and I2, representing input switches connected to terminals 1 and 2. For motor control circuit, they are connected to STOP and START switches respectively. O1 represents the output terminal 1 that is connected to relay coil A. The physical auxiliary relay switch is no longer needed and replaced by programmatical contact switches controlled by O1. The actual connection of circuit would look like as shown below.
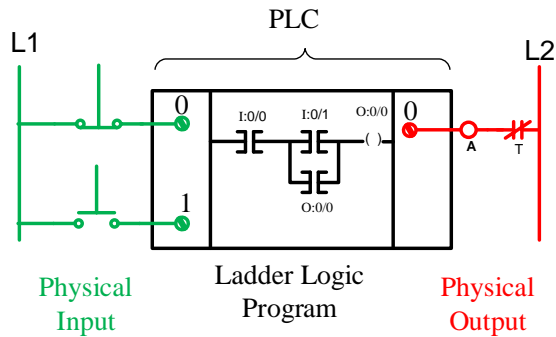
Figure 7.11. Physical connection and ladder logic for a motor control circuit. XIC instruction is true when the input is high.

Some PLC vendors (Such as Allen-Bradley) has a specific way of addressing I/O ports. The input port is indicated as I:x/y where I is for input, x is port number and y are terminal numbers in that port. The counting starts from 0. So, I:0/0 is the 1st input terminal in the first port. I:0/1 is the second terminal in the first port. They scan the status of the physical input in almost real time (generally every few milliseconds), and they update their state in memory.

When the external switches are connected, and the program as in Figure 7.11, is downloaded into PLC, in the PLC senses that input 0 is High and XIC representation I:0/0 becomes True. However, unless START switch is pressed, input 1 is Lo, So XIC I:0/1 is False. Output of True AND False is False, and O:0/0 goes Low. When the START switch is pressed, the XIC I:0/1 goes True. Output of True AND True is True and O:0/0 goes high. That would also make the relay contact O:0/0 True. So even after START switch is released, the logic True AND (False OR True) is True and coil O:0/0 remains High.

Pushing stop switch would make the logic False AND (False OR True). The output is False and output O:0/0 goes Low.

Some rules in developing a ladder logic:

1. Only one load per line or "rung".
2. Each load must be connected to the L2 line (right side of the ladder) either directly or through overload contacts.
3. Control devices such as push buttons, switches, or contacts, must be connected between L1 and the load.
4. Coils and their contacts must be labeled with the same letter/numerical identifier.

Example 42

Draw a physical connection to PLC and the ladder logic diagram that will turn motor if either of two NO switches connected to the input I:0/1 and I:0/2 is pressed.

Solution: Since either of two switches will turn the motor on, the logic should be placed in parallel. Either pressing push switch 1 or 2 will turn complete the circuit to energize coil A.
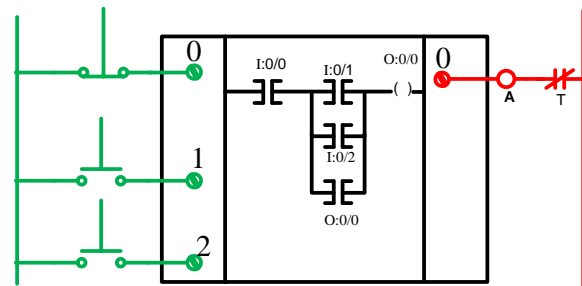
Example 43

Draw a physical connection to PLC and the ladder logic diagram that will turn ON the

motor and a pilot light when NO switches connected to the input I:0/1 is pressed.

Solution: Since two loads cannot be placed on the same rung they must be placed in parallel
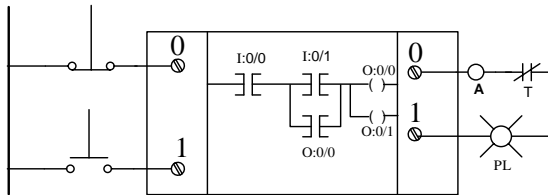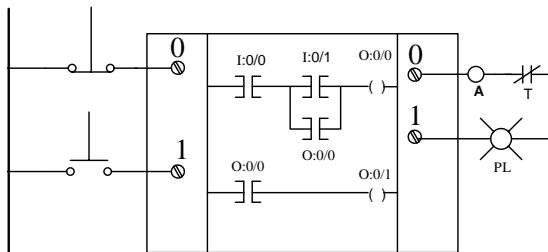

Alternatively, the internal relay O:0/0 can be used to activate the second branch as shown below:

Delay Timers

One frequently used feature is ladder logic is a timer to control the timing of a process sequence. The number of available times depends on the PLC unit. For example, Allen-Bradley MicroLogix 1500 PLC has 256 timers.

Depending on PLC vendor, the memory address of these times varies a little. Allen-Bradley PLC uses memory file# to access different memory locations. All times are in file 4. Hence, timers are addressed as T4:x where x could be a number between 0 to 255. Following the convention, T4:0 would be the first of 256 timers.

There are three variations of delay timers; On-delay timer (TON), Off-delay timers (TOF) and

retentive timer (RTO). TON is the most commonly used one, and its functionality is briefly described below.

Programming a timer:
TON has two variables to set:

Time base: (0.01 to 1s) is the incremental value of time steps in seconds.

Preset (1- 32,767): is the number of time steps

So, if Time base is set to 0.5 and Preset value is set to 10, the timer is set for 0.5*10 = 5 seconds.

Each timer has three bits (represented by relay contacts in ladder logic) reserved in memory to indicates its status. They are referred to as EN, DN, and TT. EN bit is true when the timer is energized. DN bit is true when the time is done. And TT bit is true when the timer is timing, that is between the time the timer is energized until the time is done. The timing diagram of TON action is shown in figure 7
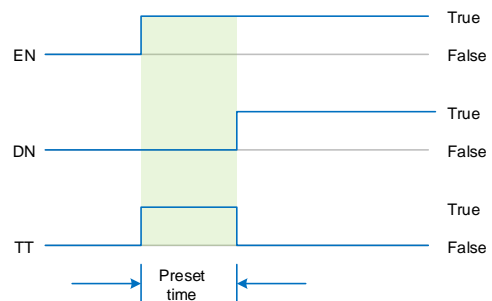
Example 44

Draw a physical connection to PLC and the ladder logic diagram that will turn ON the motor 1 and when NO switches connected to the input I:0/1 is pressed (for the circuit shown in staggered start section). The motor 2 should turn on after a time delay of 5 seconds.

Solution:
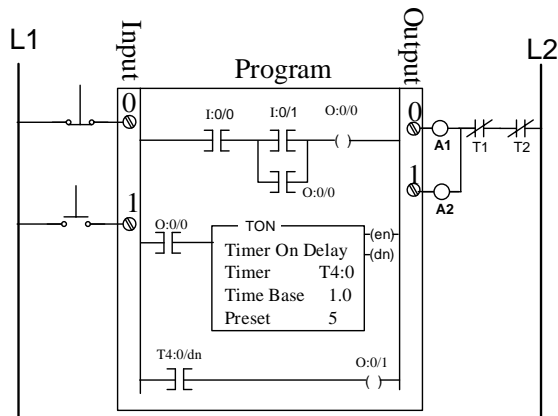
Repeating timed sequence

A cascading timer is the one when one timer starts after another timer is done. For a self-repeating timed sequence of event, the timer loop must be restarted. This is achieved by resetting the first timer when the last timer is done.

Example 45

Program a flashing red light (connected at 0:0/0) that should turn on and off every second.

Solution: One of the possible solutions is to two cascading times.

Initially, Timer2 (T4:2) is off, hence $\overset{T4:2/DN}{\dashv\!/\!\vdash}$ is true as it is an XIO instruction. That makes Timer1 (T4:1) to start. The light is ON when Timer1 is ti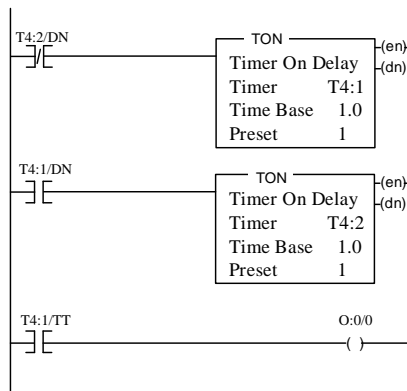ming. When Timer1 is done it is not timing, so the light turns off, and Timer2 starts. When Timer2 is done, it makes $\overset{T4:2/DN}{\dashv\!/\!\vdash}$ false. This turns off timer1 and $\overset{T4:1/DN}{\dashv\!\vdash}$ instruction goes False. This, in turn, makes Timer2 turns off and $\overset{T4:2/DN}{\dashv\!/\!\vdash}$ again goes True, which, resets timer1 and the process repeats.

Exercise

1. Explain why the branch circuit rating for a motor is greater than motor full load current but less than locked rotor current?

2. What should be the minimum branch circuit rating in percentage for a motor with a service factor of 1.15?

3. For a stationary motor ___ hp or less, the branch circuit overcurrent protection device is permitted to serve as the disconnecting means.

4. A regular 15 A rated light switch is used as a disconnecting means for a stationary 1 hp motor at 120V. Does this meet NEC code? Justify your answer.

5. What is a relay and when a relay is used in a motor control circuit?

6. Envision some real or hypothetical operating conditions that may activate the thermal overload protection of a motor.

7. What is the purpose of stager starting electric motors? List two possible consequences of not stagger starting electric motors.

8. What does PLC stand for? What language is normally used to program a PLC?

9. Draw a ladder logic diagram that will turn on the red light connected to O:0/0 to warn operators for 5 seconds before turning on the conveyor belt motor connected to O:0/1. The user pushes start and stop button to control the motor.

10. Imagine a traffic light at an intersection. Program ladder logic to operate the lights as shown below. The sequence should repeat indefinitely.

NS	EW
Red 1 s	
Red 9 s	Green 7 s
	Yellow 2 s
Red 1 s	
Green 7 s	Red 9 s
Yellow 2 s	

NS_R is connected to: O:0/0
NS_G is connected to: O:0/1
NS_Y is connected to: O:0/2

EW_R is connected to: O:0/3
EW_G is connected to: O:0/4
EW_Y is connected to: O:0/5

Chapter 8 Controls system

A **control system** is a device or set of devices to regulate the behavior of other devices or systems. A control system may consist of subsystems and processes assembled for obtaining the desired performance by controlling the input.

Control system is an integral part of our lives, from your home heating system to robotics. The human body itself is one of the best control systems out there, and yet no artificial controls system has been able rival human body controls system. All living organism has controls system. They control in many ways such as growth, metabolic activities, and movements.

In the simplest sense, we can image a control system as one input and one output from a system. The system output is controlled by changing the input to the system. One example of the system could be this classes you are taking, the output could be the grade you get, and input is the time you spend to learn. You can control your grade by changing the time you put in for this course.
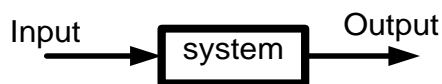

Figure 8.1: We can control the system output (behavior) by changing the system input(s)

Modern applications of controls system

Today, controls systems find widespread application in every aspect of engineering including biological engineering such as

process automation of , variable rate chemical application (also known as variable rate technology) in precision agriculture, medical applications such controlled drug delivery, prosthesis, pollution control, an unmanned aerial vehicle from crop scouting and much more.
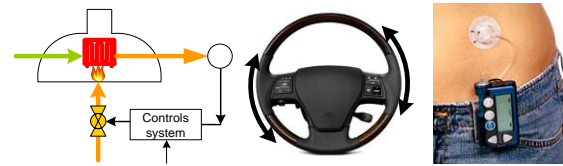
Figure 8.2: Controls examples include Process controls, auto guidance, and controlled drug delivery.

Types of control systems

There are two main classes of control systems, open loop, and closed loop or feedback control system. In open loop control system, the control signal does not depend on the state of the current output. Open loop controls are use when an external disturbance is low, and the sequence of operation are well defined. Traffic light system is a good example of open loop system, the sequence of lighting is predetermined and external disturbances to the system are low. A distinguishing characteristic of open loop system is that it cannot compensate for any unforeseen or random disturbances. The input sequence is preprogrammed with some expected behavior of the system. Some other open loop examples are:

A Robotic task with no sensor. This would work if the task is simple such as going from point A to point B following pre-calculated steps and disturbance is not expected.

Prescribed insulin dose without monitoring the actual glucose level. Mostly this has been in practice and works because the human body can tolerate higher margin of error for these drugs and response to the drug is well understood through previous studies.

The advantage of open loop system is that it is relatively simple and costs less. The disadvantage is its inability to correct for error. A closed loop system, on the other hand, has a feedback mechanism to determine the error for correction. A sensing system and control units are two additional components needed to a closed loop system. A simple feedback system is shown in figure 3.
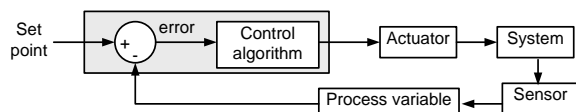

Figure 8.3: A typical close-loop control system.

In figure 3, set point (SP) is the targeted value of a variable interest (such as temperature or position), which could well be a constant such as set temperature in a thermostat; or it may be continuously changing such as in desired hand position in the prosthetic arm. A process variable (PV) is the property of a process that is being controlled such as temperature. The term PV is also used to indicate the Present Value of a Process Variable. The difference SP-PV is the error, and the error is used by the control

algorithm to act on the system through an actuator to drive the error to zero.

Some examples of closed loop systems are:

Robot following a path with feedback: Feedback system such as computer vision (as a sensor) will be required to avoid obstructions.

Medicine dose (such as insulin) based on current glucose level. A biosensor to measure glucose level and an insulin injection system (actuator) will be needed.

System modeling

Since the objective of a control system is to maintain the output variable at desired level by changing the input variable, understanding of the linkage between input and output is needed. Let us look at a simplest possible relation between input and output as y = x that is output from the system is equal to input to the system, and there is no time lag between input and output.
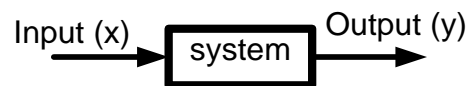

Figure 8.4: One of the simplest systems to control is when Output = Input, and there is no time lag between output and input

This system will be easiest to control. For any desired output (y) we give the input (x) equal to the output. To compensate for any external disturbances, which causes Δy deviation in output, we change input x by Δy to compensate for the disturbance.

Unfortunately, the systems are not as simple as the example above. The real system deviates from above example in the following important manners:

1. There is some time lag between response (change in y) and change in control signal x. This is called **DEAD TIME**.

2.   The output y does not reach the desired value instantaneously because of its own inertia (in a general sense). The time after dead time to reach 63.3% (one time constant) of its final value from step input change is called **PROCESS DELAY** or **LAG TIME**.

Example:

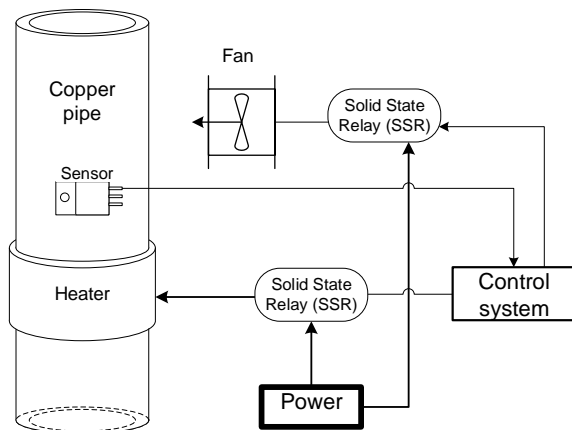Consider a classical temperature control system:

Figure 8.5: A temperature control system

➢ The surface temperature of the pipe must be maintained at certain temperature T. The temperature is the process variable (PV) in this case.

➢ The surface temperature is measured using a temperature sensor.

➢ Control system compared PV with Setpoint (SP) and calculated the difference. The difference is the error (e).

➢ Based on Error value the control either turns on the heater or the fan at a certain speed.

The problem is to control the power to the heater or to the fan so that error is as close to 0 as possible. The better the control system, the faster error will reach 0, without overcorrecting, while keeping the system stable.

The main hurdles in achieving the best control systems are:

1. Unknown input-output relation. That is how much temperature will chance with different level of heater and fan speed. This may be further complicated by the unknown amount of disturbances such as fluctuating flow rate.

2. The dead time between the control signal is sent, and the temperature starts to rise. This includes the time for the heater or fan to come to full speed or the block heater to warm up.

3. The lag time between the start of rising the temperature and reaching the set point.

A mathematical model is a useful tool in system understanding. The best control algorithm can be developed if the exact relation between input and output is known. The first step in controls problem is therefore to understand input-output relation.

System understanding through Modeling:

There are mainly two methods to understand the system and determine the system response

1. Theoretical consideration

2. Experimental method

The first step in understanding the system is through some theoretical consideration of the system. Theoretical analysis reveals the underlying input-output relationship and provides an explanation of system behavior.

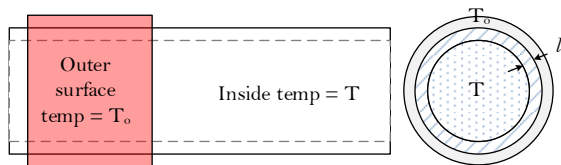In this example, we will make a mathematical model of the above system.

Figure 8.6: A temperate controls system

Let's make some assumptions to simplify the system model.

1. The pipe can be considered thin.

2. The temperature of the heater surface is T_o is constant.

3. The temperature of the fluid inside of the pipe (T) is homogeneous.

4. Heat transfers are mainly through conduction of the copper pipe.

Heat transfer rate $= \frac{kA}{A}(T_o - T)$

Where k is the thermal conductivity of copper, A is the area of the average surface area of contact between the heating unit and inside the fluid. L is the pipe thickness, and k is the thermal conductivity.

This transferred heat increases the inside fluid temperature. The heat absorption of increasing fluid temperature is:

Heat absorption rate $= mC\frac{dT}{dt}$

where 'C' is the specific heat capacity of the fluid. Since all the heat transferred is absorbed by the fluid,

$$mC\frac{dT}{dt} = \frac{kA}{l}(T_o - T) \quad \text{Or}$$
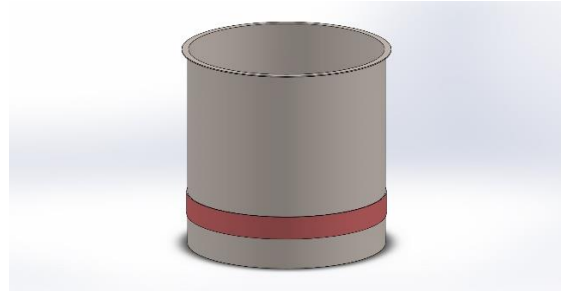
$$\frac{mCl}{kA}\frac{dT}{dt} + T = T_o$$

This is the first order model of the heat transfer. From this equation, we can predict the temperate inside the pipe T as a function of controlled outside temperature.

_____

Example 46

_____

A biodiesel plant needs to heat the oil to 60°C before the reaction. A band heater around the reactor is used to heat the oil.

The reactor is a ¼" thick grade 316 stainless steel (with a thermal conductivity of 15 W/m°K. The oil being heated is vegetable oil with the specific heat capacity of 2 J/g°K. The volume of oil inside the reactor is 100 gallons with a specific gravity of 0.93 g/ml. The reactor has 26" radius, and the band heater is 6" wide that goes all around the reactor. The oil is continuously stirred to maintain the uniform temperature. Assume the tank is insulated and there is no heat loss.

Establish a first order differential equation relating the oil temperature over time.

Solution:

$$\frac{mCl}{kA}\frac{dT}{dt} + T = T_o$$

The right side of the equation ahs unit of temperature, so each term on the left side of this equation also has to have the same unit of temperature. Converging all given parameters to the same units makes it easy to check the compatibility. In this case, we will convert them into SI units.

$$m = 100 \text{ gal} \times \frac{3.78L}{1gal} \times \frac{0.93kg}{L} = 351.5 \text{ kg}$$

$$C = 2\frac{J}{g\,°K} = 2000\frac{J}{kg\,°K}$$

$$l = 0.25 \text{ in } \frac{1m}{39.37in} = 6.35 \times 10^{-3}m$$

$$k = 15\frac{W}{m\,°K}$$

$$A = 2\pi \cdot 26\text{in} \cdot 6\text{in} \cdot \left(\frac{1m}{39.37in}\right)^2 = 0.63m^2$$

$\frac{mCl}{kA}$ has a unit of $\frac{kg \cdot \frac{J}{kg°K} m}{\frac{W}{m°K}m^2}$. Since W = J/s, this gives the term unit of second. The term $\frac{mCl}{kA}\frac{dT}{dt}$ then has the unit of temperature if dt is measured in seconds. Pay attention that the choice of the unit gives the unit for time. For this case t is in seconds. Also, all temperatures must be in the same units. In this case, we will use °C.

Substituting these value, we get,

$$470\frac{dT}{dt} + T = T_o$$

From this equation, we can predict the temperate inside the reactor over time, given the initial condition.

Example 47

Solve the differential equation from the above example as a function of time. The band heater temperature gets to 70°C. Initial temperature of the oil was 20°C.

Solution:

Although we can solve above differential equation several ways, we will use the Laplace transform to solve this equation. The Laplace transform provides an insight into system behavior that no other solution techniques can provide.

The left side parameter m, C, k, A and l describes the system and determines how the system will behave to external forces. In this

case, how the temperate T with changes with time. These are all system specific parameters in question and are independent of external forces.

The right side of the equation T_o is the external force forcing the system to change. If the system is excited from equilibrium and let it settle with no external forces applied, the way system would settle is captured by the equation:

$$\frac{mCl}{kA}\frac{dT}{dt} + T = 0$$

The solution of this part is called the "natural response" (aka **homogeneous solution** in mathematics). When the external force is applied to the system, the system responses to the external force following the equation

$$\frac{mCl}{kA}\frac{dT}{dt} + T = T_o$$

The solution of this equation is known as "forced response" (aka **Particular solution** in mathematics).

Solving model responses with Laplace Transformation

Instead of doing a two-part solution, we can take the Laplace transform of the differential equation which transforms a differential equation into an algebraic equation with a complex variable 's'. The algebraic equation is easy to work with and understood. See appendix for a brief review of the Laplace transform. Refer to Laplace transform table in appendix to aid your understanding of this transformation.

$$\mathcal{L}\left\{470\frac{dT}{dt} + T\right\} = \mathcal{L}(70.u(t))$$

$$470\,(sF(s) - f(0)) + F(s) = \frac{70}{s}$$

$$470\,(sF(s) - 20) + F(s) = \frac{70}{s}$$

$$F(s) = \frac{9400s + 70}{470s^2 + s}$$

This function does not lend itself to apply an inverse Laplace transform from the Appendix directly. Usually, a partial fraction is necessary to turn this equation into one of the standard forms available in inverse Laplace transform table. There are several helpful documents to explain the mechanics of partial fraction such one found at mathcentre.ac.uk. Alternatively, MATLAB has a built-in function to do a partial fraction for you.

```
num = [9400 70];
den = [470 1 0];
[r, p, k] = residue(num,den)
```

Will produce

```
r = [-50    70]
p = [-0.00213    0]
k = []
```

This MATLAB result is interpreted as

$$F(s) = \sum \frac{r_i}{s - p_i} + k \qquad \text{or for this case,}$$

$$F(s) = \frac{-50}{s + 0.00213} + \frac{70}{s}$$

Since, $0.00213 = 1/470$, Taking the inverse Laplace transforms using table in Appendix produces,

$$T = \left(70 - 50e^{-\frac{1}{470}t}\right).u(t) \text{ or}[9]$$

$$T = \left(70 - 50e^{-\frac{1}{470}t}\right) \text{ for } t \geq 0$$

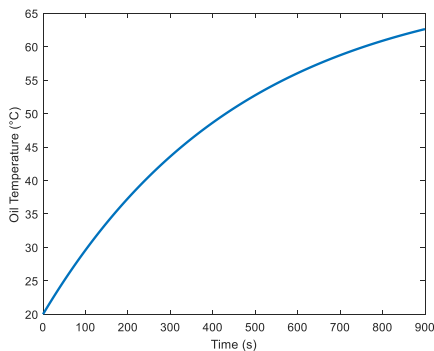*Example 48*

Plot the temperature vs. Time for this system from 0 to 15 minutes

Solution:

$T = \left(70 - 50e^{-\frac{1}{470}t}\right)$. Following MATLAB code will plot the temperature over time.

```
t = 0:15*60;
T = 70-50*exp(-1/470*t);
plot(t,T,'linewidth',2)
xlabel('Time (s)')
ylabel('Oil Temperature (°C)')
```
Will produce the plot:

LTI System

A Linear Time Invariant or commonly known as LTI system is the system that has a linear input and output relation, and the relationship does not change over time. A linear system must satisfy the following criteria.

1. Superposition: Output response of sum of two inputs is the sum of the output of individual inputs.
2. Homogeneity: The system output to system input ratio is constant.

For an LTI system, the output is completely understood if output is understood for a unit input, commonly known as **step input,** or an impulse response.

Most of the controls theory is developed around the LTI system; it is beneficial to convert a non-linear system into the linear system by using some mathematical transformation, that the system control theory could be effectively applied.

Example 49

Is the system y = 3x + 1 is an LTI system?

Solution:

If this is an LTI system, it will satisfy the superposition and homogeneity properties.

Superposition: If input x_1 makes the output y_1, input x_2 makes the output y_2, and input $(x_1 + x_2)$ makes the output y_3, then superpositon criteria is satisfied if $y_3 = y_1 + y_2$. Check;

$$y_1 = 3x_1 + 1$$
$$y_2 = 3x_2 + 1$$
$$y_3 = 3(x_1 + x_2) + 1$$

Or,

$$y_3 = 3x_1 + 3x_2 + 1$$

---
[9] See appendix for definition of u(t)

But,

$$y_1 + y_2 = 3x_1 + 3x_2 + 2$$

$y_3 \neq y_1 + y_2$ and the system is not linear.

Example 50

Is the system $470\frac{dy}{dt} + y = x$ linear?

Solution:

If this is an LTI system, it will satisfy the superposition and homogeneity properties.

Superposition: If input x_1 makes the output y_1, input x_2 makes the output y_2, and input $(x_1 + x_2)$ makes the output y_3, then superpositon criteria is satisfied if $y_3 = y_1 + y_2$. Check;

$$470\frac{dy_1}{dt} + y_1 = x_1$$

$$470\frac{dy_2}{dt} + y_2 = x_2$$

If we add these two equations,

$$470\left(\frac{dy_1}{dt} + \frac{dy_2}{dt}\right) + (y_1 + y_2) = x_1 + x_2$$

Or, $470\left(\frac{d(y_1+y_2)}{dt}\right) + (y_1 + y_2) = x_1 + x_2$

But,

$$470\frac{dy_3}{dt} + y_3 = x_1 + x_2$$

Comparing two equations,

$y_3 = y_1 + y_2$. The property is satistied.

Homogeneity: If input x_1 makes the output y_1, and input x_2 makes the output y_2, where $x_2 = ax_1$. The homogeineity property is satisfied if $y_2 = ay_1$ for any constant value of a.

$$470\frac{dy_2}{dt} + y_2 = x_1$$

But,

$470\frac{dy_1}{dt} + y_1 = x_1$, multiplying both side of this equation by a,

$a\left(470\frac{dy_1}{dt} + y_1\right) = ax_1$, Or,

$470\frac{day_1}{dt} += ax_1$, Comparing this with y_2, we get,

$$y_2 = ax_1$$

The system is an LTI system.

Linearization

As mentioned earlier, it is beneficial to linearize the system to effectively apply well-understood control system theories. One common way to linearize the system with intercept like y = 3x + 1 is to take derivative on both sides to get rid of a constant term. The control theory is applied to new variables Δy, and Δx. Usually, they are expressed in percentage change for a desired operating range.

Example 51

Linearize the system y = 3x + 1 for an operating range of x = 0 to 1.
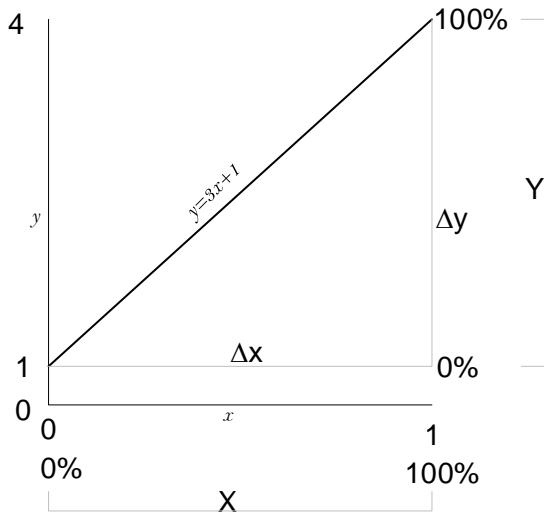
Solution:

$$dy = 3dx \ \ or \, \Delta y = 3\Delta x$$

Figure 8.7: Linearization from y = 3x + 1 to Y = X

A 100% change in Δx (0 to 1) corresponds to 100% change in Δy (0 to 3). If variable X is % chenge is Δx and variable Y is % chenge is Δy then,

Y=X is a new equation, both variable X and Y changes from 0 to 100%.

Example 52

Show that Y = X is an LTI system.

Solution:

If this is an LTI system, it will satisfy the superposition and homogeneity properties.

Superposition: If input x_1 makes the output y_1, input x_2 makes the output y_2, and input $(x_1 + x_2)$ makes the output y_3, then superpositon criteria is satisfied if $y_3 = y_1 + y_2$. Check;

$$y_1 = x_1$$
$$y_2 = x_2$$
$$y_3 = x_1 + x_2$$

It is easy to see that, $y_3 = y_1 + y_2$

Homogeneity: If input x_1 makes the output y_1, and input x_2 makes the output y_2, where $x_2 = ax_1$. The homogeineity property is satisfied if $y_2 = ay_1$ for any constant value of a.

$$y_1 = x_1$$

Multiplying both side of this equation by a,

$$ay_1 = ax_1$$

$470\frac{day_1}{dt} += ax_1$, Comparing this with y_2, we Again, it is easy to see that $y_2 = ax_1$

The system is an LTI system

System Transfer function

For an LTI system, as mentioned earlier, the output of the system is known if the response of any input is known. Let us analyze the first order system that we developed earlier.

$$\frac{dy}{dt} + ay = x$$

With initial value of $y = y_0$,
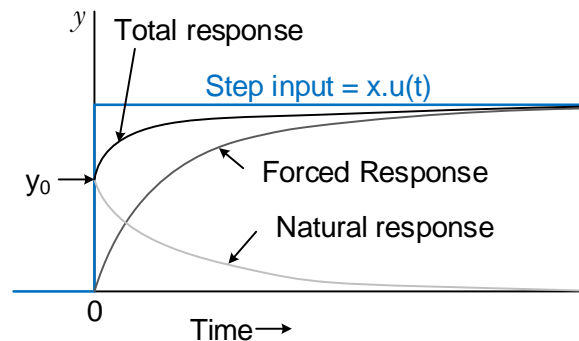

Figure 8.8: System response (aka total response) is the sum of natural response and the forced response.

If the initial value was 0, the Laplace transform of the forced response is given by:

$$Y_1(s) = X(s) \cdot \frac{1}{s + a}$$

If there was no external force, the system output is given by

$$Y_2(s) = \frac{y_0}{s+a}$$

The total response is given by

$$Y(s) = \frac{X(s)}{s+a} + \frac{y_0}{s+a}$$

This equation captures, a component of system output from 1) Initial condition 2) Natural response and 3) forced response. Also, taking Laplace transform converts a differential equation into an algebraic equation that are easy to visualize and solve. The input–output of the system can be expressed as a block diagram:
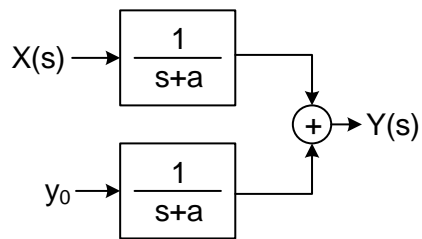

Figure 8.9: Laplace transforms differential equation into algebraic equation (called transfer function). Each term can be multiplied and added to determine the system response.

Example 53

For the differential equation $470\frac{dT}{dt} + T = T_o$, plot the system response, Forced response and Total response. The initial value of T is 20°C and T_0 = 70.u(t). Plot these responses from t = 0 to 15 minutes.

From analogy, a = 1/470 and x = $\frac{70}{470}$u(t).The forced response part is:

$$Y_1(s) = \frac{70/470}{s} \cdot \frac{1}{s+1/470} \ or,$$

$$Y_1(s) = \frac{70}{s} \cdot \frac{1}{470s+1} = \frac{70}{470s^2+s}$$

The system response part is:

$$Y_2(s) = \frac{20}{s+\frac{1}{470}} = \frac{9400}{470s+1}$$

The total response is:

$$Y(s) = Y_1(s) + Y_2(s) = \frac{9400s+70}{470s^2+s}$$

This is the same at the one we derived before. The following MATLAB code with generate the transfer function and plot (using MATLAB Controls Systems toolbox).

```
t = 0:20:900; %Generate time in s
Y1 = tf(70,[470 1 0]);%see tf help
Y2 = tf(9400,[470 1]);
Y = Y1+Y2;
y1 = impulse(Y1,t);%see help
y2 = impulse(Y2,t);% for inpulse
y = impulse(Y,t);
tm = t/60; %Convert to minute
plot(tm,y1,'-.r');
hold, plot(tm,y2,'.b');
plot(tm,y,'k');
xlabel('Time in minute')
ylabel('Temperature °C')
legend('Forced','System','Total')
grid on
```

Controls System
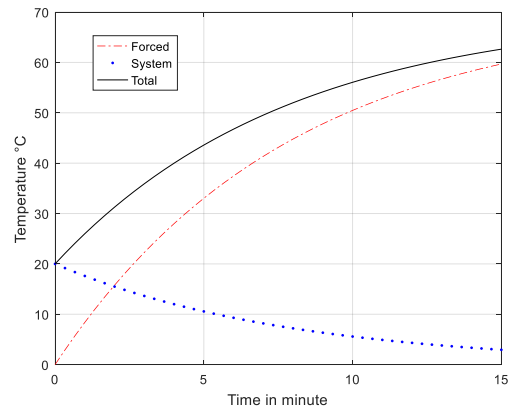
Transfer function

In many engineering problems, the system output from initial condition is not much of an interest, because, the effect of initial condition wears out rather quickly or the system may be at rest at t = 0 i.e $f(0) = 0$.

It is important to notice that rest state may not be absolute 0 state. For, instance, in above example of heating the oil tank, the rest state may be the ambient temperature. If the oil is left alone for a long time, the temperature will in equilibrium with the ambient temperature. The initial temperature of the oil given at 20°C could be the ambient temperature, in that case, there is no system response. If the ambient temperature is considered 0 input, then according to out linearity, the output temperature of the oil must also be 0. When the system is at equilibrium at t = 0, the equation reduces to

$$Y(s) = X(s) \cdot \frac{1}{s+a}$$

$$X(s) \longrightarrow \boxed{\dfrac{1}{s+a}} \longrightarrow Y(s)$$
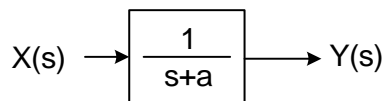
The ration $\frac{Y(s)}{X(s)}$ is called the transfer function and characterizes the system output for an impulse response.

Example 54

In the above oil heating example, if the equilibrium temperature is 20°C, linearize the input–output function in terms of percentage change.

Let the temperature be expressed as

$$T' = \frac{T-20}{70-20} \times 100\%$$

The new equation is:

$$470 \frac{dT'}{dt} + T' = 100. \, u(t)$$

$$470 \, (sF(s) - f(0)) + F(s) = \frac{100}{s}$$

$$470 \, (sF(s) - 0) + F(s) = \frac{100}{s}$$

$$F(s) = \frac{100}{470s^2 + s}$$

The following MATLAB plot generate the plot.

```
t = 0:20:900; %Generate time in s
Y1 = tf(100,[470 1 0]);
y1 = impulse(Y1,t);
y = y1/2+20;
tm = t/60;
yyaxis left
plot(tm,y1);
ylabel('T'' %')
hold
yyaxis right
plot(tm,y,'k');
xlabel('Time in minute')
ylabel('T °C')
grid on
```
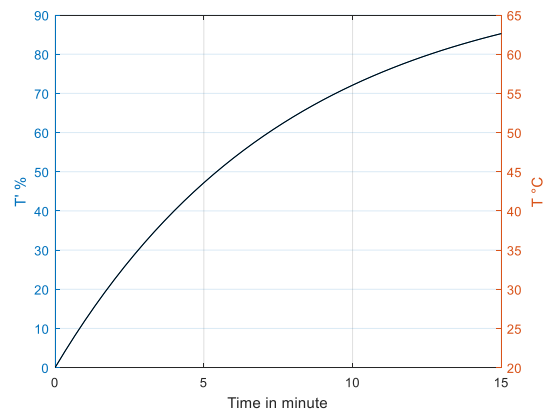

Compare this plot to one from previous exercise. The same total response curve is the same.

tf function builds the transfer function from numerator and denominator coefficients. The **impulse** function calculates the response from an impulse funciton $\delta(t)$ which has Laplace transform value of 1 (See Appendix).

Example 55

Find the transfer function for the system

$$470\frac{dT'}{dt} + T' = 100.\,u(t)$$

$$\frac{Y(s)}{X(s)} = \frac{1}{470s + 1}$$

This function represents the output for an impulse input.

Once a transfer function is found, the reposne of this system to any imput can be easily determined.

To find the step response from a transfer function you can use a MATLAB funcitno **step**.

Example 56

Determine the step response to the above system with step of 100.

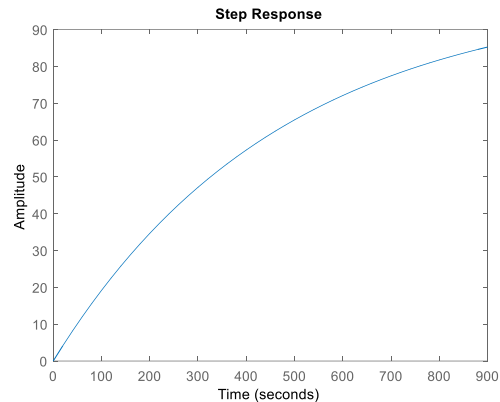For a linear system, step size of 100 produces 100 times more of step size of 1.

So, the transfer funciton is written as

```
F = 100* tf(1,[470 1]);
```
The following code then plots the step response of this transfer function

```
F = 100*tf(1,[470 1]);
t = 0:20:900;
step(t,F)
```

will generate the same plot, comparable to the plot from example 9.

The time constant

The *Time Constant* is one of the main characteristics of the *first order system* which tells us how rapidly the system will respond to applied external force. If a constant force is applied to a first order system, in one time constant the system variable will change about 63% ($= 1 - e^{-1}$) of difference between final and initial values. This can be best described by an analogy of a jumping from. A frog must travel from start to finish. In every jump, the frog travels 63% of the remaining distance. However, it takes one full time constant to make each jump.

Figure 8.10: Time constant of a first order system. In each time constant, the frog is 63% closer.

Taking this analogy to our equation, we can re arrange above equation as $T = 20 + 50\left(1 - e^{-\frac{1}{470}t}\right)$. The start value of T is 20 which can be obtained by plugging in t=0. The final value of T = 70, which can be obtained by plugging in t = ∞. So, after first time constant, the

temperature will reach $20 + 50(1 - e^{-1})$=51.5°C.

Example 57

What is the time constant for the system in above example?

Solution:

It is easy to see that $T = 20 + 50(1 - e^{-1})$ when t = 470s. Therefore, the time constant for this system is 470s.

In general, if a first order system has the form

$$\tau \frac{dy}{dt} + y = f(x)$$

Then the time constant for this system = τ. If the time constant and start and final values are known for a system, the temperature at any time can be written as:

$$T = \text{Initial} + (\text{Final} - \text{Initial})\left(1 - e^{-\frac{1}{\tau}t}\right)$$

1. How long it will take to reach the fluid temperature to 60°C?

   $T = 20 + 50\left(1 - e^{-\frac{1}{470}t}\right)$, plugging in T = 60, we can solve for t as $t = 756.4s$

   The following MATLAB code will solve for the equation numerically,

   ```
   % create an anonymous function

   f=@(t) 40-50*(1-exp(-1/470*t))
   t=fzero(f,100) %100s is a guess
   ```
   will produce

   ```
   t= 756.4s
   ```

Exercise

1. Establish a mathematical model for the following capacitor charging system when SPDT switch is connected to terminal 'a.'
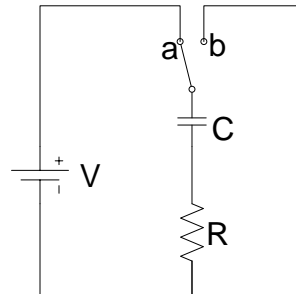
a) If R = 1 MΩ, C = 10 μF, what is the time constant for this system? How long will it take to charge 99% of the full capacity?

b) Create a diagram using technology for the voltage across capacitor and current in the circuit as a function of time for t = 0 to 30 s.

c) After 100 s, the switch is flipped to terminal 'b'. Create a diagram using technology for the voltage across capacitor and current in the circuit as a function of time for t = 30 to 60 s.

2. Establish a relationship between translational velocities of the piston for the following engine system with a constant force F from the ignition. m is mass of the piston and b is the resistance proportional to the piston velocity.
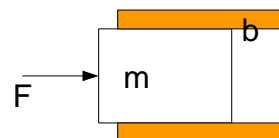
a) What is the time constant for this system if m = 5 lb and b = 10 lb.s/ft

b) Compare your model with the equivalent electrical system from the previous problem, how are they similar, how are they different?

3. In the following waste treatment facility, chemical concentration (mg/gal) within the tank can be considered uniform. The initial chemical concentration inside the tank was 0 mg/gal, the concentration of effluent coming in is 10 mg/gal. The volume of the tank is 10,000 gallon. The fluid coming in rate is equal to fluid going out is equal to 50 gal/min. Establish a dynamic model of how the concentration of chemical inside the tank increases over time. How long will it take to reach the chemical concentration of tank to 5 mg/gal?
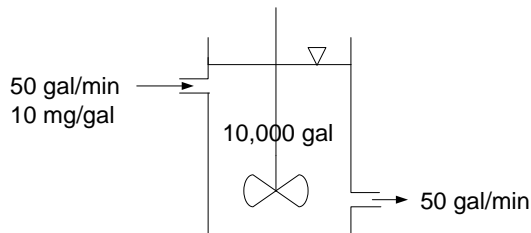
4. Represent the system in terms of Laplace transform variable 's' and write down the transfer function.

5. Plot the chemical concentration over time for this system using MATLAB or any other technology for a day (24 hours)

6. Assume that the inflow was from a biochemical industry. The industry generates 10mg/gal of chemical concentration in their wash water and washing takes 1 hours. The wash water then enters the treatment facility. Plot the chemical concentration of effluent for 24 hours using technology.

7. Is the system $Y = X(1 - e^{5t})$ linear? Where x is input, and y is output. Draw the output for the input function for t = 0 to 10 s. Assume both X and Y = 0 for t<0.

$$x = \begin{cases} 1 \ for \ 0 \leq t < 5 \\ 3 \ for \ 5 \leq t \leq 10 \end{cases}$$

8. Represent the above system using Laplace transform variable 's'.

9. The hormone insulin is responsible for glucose metabolism. Hormones are in general are the control systems of the living organisms. Research has shown the rate of change of glucose level in the blood is proportional to both glucose level itself and the level of insulin. This just means that while insulin is a major factor in reducing blood glucose, there are some other factors in play that also consumes blood sugar whose rate is proportional to glucose level itself. The differential equation for glucose and insulin is given by:

$\frac{dG}{dt} = -0.1G - 0.8H$ the negative sign of A indicates that the glucose level decreases over time in proportion to both glucose and insulin level. Let G = 100% and t = 0. Plot

the glucose level over time until it reaches 20% with and without insulin.

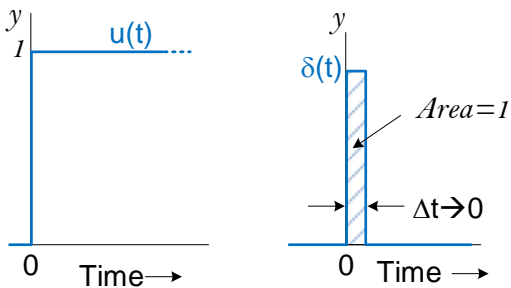1. For a stable system, all value of s (as a Laplace transform variable) has to be less than or equal to zero.

2. The system y = 3x+1, where x is input to the system and y represents system output is not a linear system.

3. Linear system must follow the property of superimposition.

Appendix

Laplace transform

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt$$

Laplace transform table

Step function u(t) and Dirac Delta function $\delta(t)$

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\dfrac{1}{s}$
$t^n.u(t)$	$\dfrac{n!}{s^{n+1}}$
$e^{-at}.u(t)$	$\dfrac{1}{s+a}$

$\sin(\omega t).u(t)$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos(\omega t).u(t)$	$\dfrac{s}{s^2+\omega^2}$

Laplace transform properties

$\mathcal{L}\{f(t)\}$	$F(s)$
$\mathcal{L}\{f'(t)\}$	$sF(s) - f(0)$
$\mathcal{L}\{f''(t)\}$	$s^2F(s) - sf(0) - f'(0)$
$\mathcal{L}\left\{\int_0^t f(t)\,dt\right\}$	$\dfrac{F(s)}{s}$
$\mathcal{L}\{kf(t)\}$	$kF(s)$
$\mathcal{L}\{k_1f_1(t) + k_2f_2(t)\}$	$k_1F_1(s) + k_2F_2(s)$
$\mathcal{L}\{e^{-at}f(t)\}$	$F(s+a)$F1
$\mathcal{L}\{f(t-a)\}$	$e^{-as}F(s)$
$\mathcal{L}\{f(t) * g(t)\}$	$F(s) \cdot G(s)$

$*$ Convolution operator

If $f(0) = 0$, meaning if the system has 0 output at t = 0, then differentiating a function is equivalent to multiplying by s to the Laplace transformation.

First order differential equation

$$\tau \frac{dy}{dt} + y = f(t)$$

Time constant = τ

Chapter 9 Feedback controls

Unlike open loop controls, the feedback or closed loop controls continually monitors current process variable value (y) and compares it with set point (x) to calculate the error (e). The error is used by the controller to set the level of the control signal to drive error towards zero.
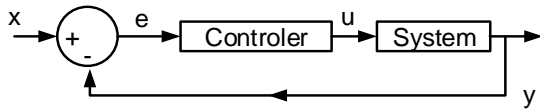
Figure 9.1: Generalized feedback control loop.

In general, u and y are in different units. For instance, the control signal u may vary from 4mA to 20mA current signal, whereas y may be in °F that varies from 30°F for u=4mA to 90 °F for u = 20 mA. For a generalization of the process controls calculations, it is customary to express both u signal and PV in percentage and they are linearly mapped so that u% = PV% as shown in figure 2.
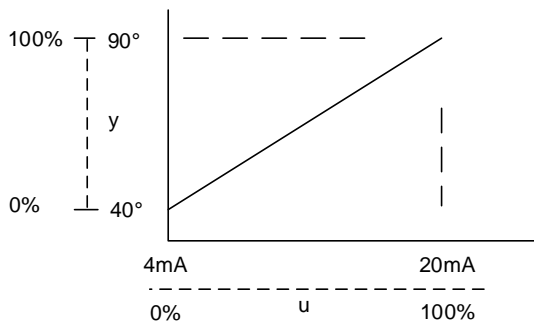

Figure 9.2: Usually both u and y are converted to % so that u% = y%

The controller is the brain of a feedback control system. The controller takes in two values; the SP and the current value of process variable P. Based on these two pieces of information, processor adjust the system through an actuator (in the previous example, heater ON or OFF signal) so that process variable value moves towards set point.

The primary objective of a control system is to get to the set point as soon as possible with the minimum amount of overshoot and without rendering the system unstable. The controller puts out u, so that the output y goes to desired value. We will start the discussion with the most common PID Controller.

PID Controller

Most controllers are PID type. PID is an acronym for **Proportional Integral and Derivative**. We will examine the characteristics of the each of the proportional (P), the integral (I), and the derivative (D) controls, and how to tune them to obtain the desired system response. We will start our discussion with a proportional controller.

The P-controller

The control algorithm is $u = K_p e$, where K_p is, a constant known as proportional gain. This is a simple algorithm where the strength of correcting signal is proportional to the error.

Although theoretically, the control signal can be any value depending on the error, practically they are limited. For instance, you can turn on the heater to 100% of its capacity

and no more. No error value cannot make the heater to turn beyond its full capacity. The range where the control signal is proportional to an error called the **proportional band**.
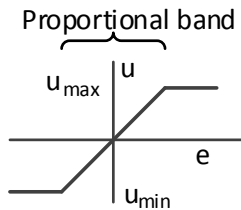

Proportional band

Figure 9.3: The controller output 'u' is proportional to error 'e' within the proportional band and is saturated (maxed out) outside of it.

The **Proportional gain** is the percent change in u per unit change in error within the proportional band. This is equal to the slope of the proportional process curve and = u/e. Note that proportional band is independent of the set point.

Process Block Diagram

As discussed earlier, the use of Laplace transform allows converting a differential equation into an algebraic operation. If the lower-case letters x, e, u, and y represent the time domain signal and X, E, U and Y represents corresponding Laplace transforms, the equivalent block diagram of the proportional controller can be represented as shown in the figure below. F is the LTI system transfer function.
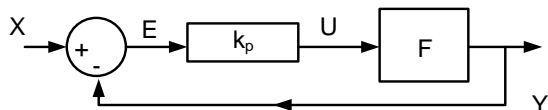

Figure 9.4: A proportional control algorithm. For this system, Process gain (G) = k_p

$U = (X - Y) \cdot k_p$ and $Y = U \cdot F$, Or,

$$Y = X \frac{K_p F}{1 + K_p F} \qquad (32)$$

And

$$U = X \frac{k_p}{1 + k_p F} \qquad (33)$$

Example 58

For a system represented by the transfer function, $F = \frac{1}{470s+1}$, it was desired to maintain the output (x) at 50% (=45°C). A proportional gain is used to implement this algorithm.

Plot the set point x, oil temperature (y) and controller output (u), over time for 15 minutes with a proportional gain (k_p)value of 2.

Solution:

The set point is a constant value of 50%. The set point can be plotted using the following MATLAB code:

```
t = 0:900; %Generate time in s
x = 50;
```

X(s) in equation 1 is the Laplace transform of the SP. Since the SP is s step function of value 50, $X(s) = \frac{50}{s}$. The step function in MATLAB can be generated using the function 'step'. The Matlab code to generate Y_{PV} is:

```
kp = 2;
F = tf(1,[470 1]);
Y = kp*F/(1+kp*F);
y = x*step(Y,t);
```

Similarly, for the controller output u,

```
U = kp/(1+kp*F);
u = x*step(U,t);
```

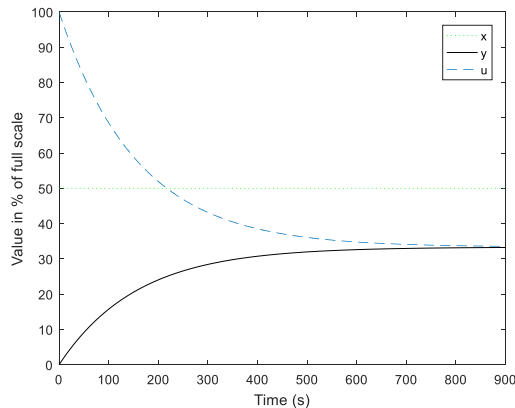Finally, the following code will plot the values,

```
plot(t,t*0+x,'g:');%Green dot line
hold on
```

```
plot(t,y, 'k');% Black line
plot(t,u,'--')% dash line
legend('x','y','u')
xlabel('Time (s)')
ylabel('Value in % of full scale')
```

will produce the following plot

Steady State Error

One characteristic of a proportional control is that steady state error is not zero for SP other than 0%.

Expressing both u and PV in percentage, for a steady state condition, u = PV. Then,

$$y = k_p * e = k_p * (x - y) = k_p * x - k_p * y$$

Or, $y = \dfrac{k_p}{1+k_p} * x$

Since, $k_p < k_p + 1$, for a proportional control, at steady state PV is always less than SP[10]. The difference is called the **Steady State Error**.

Steady state error $= x - y = \dfrac{1}{1+k_p} * x$

Example 59

Calculate the PV(y) and the steady state error when k_p = 2 and SP(x) is set to 50%

---

[10] PV < SP does not hold true for transient response because of dead time and process lag.

$$y = \frac{k_p}{1+k_p} * x = \frac{2}{3}50 = 33.4\%$$

Steady state error = 1/3*50% = 16.7%

You can verify the steady state error from the plot in the above example

Dead time

In the above example, the output of the system changed as soon as the input was applied. However, in a real system that may not be the case. The output starts to change only after a certain time after the input is applied causing the 'input delay' or there may be a delay in feedback system causing the 'output delay.' The delays are commonly known as **dead time**. An input delay may be from:

a. Delay between controller signal and the start of actuator: This phenomenon is common when multiple components in an actuator need to be checked for safety before turning on the actuator. A good example is a household gas furnace. After the control signal tells the furnace to turn on, several components need to be checked such as checking the pilot lamp is on before opening the gas valve. Once gas starts to flow, the temperate of the combustion chamber may be still too low for the blower motor to turn on, and there is a travel time for hot air from the furnace to the room vent before it starts to warm up the room.

b. Sensor is placed far away from the actuator. In the above example of making biofuel,

imagine that the sensor was placed away from the band heater and the fact that in practice a complete uniform temperate inside the reactgor is unlikely, there is going to be dead time. That is the time between the heater on to the temperature rise.

Common causes of output delay are:

c. Sensor is not sensitive enough to pick up the slight changes: Sensors usually have some tolerance, dead band and other factors that prevent the sensor from real time feedback. In out example, if the thermometer used in the process has a resolution of 1°C, the temperature rising from 19.5°C to 20.49°C recorded as 20°C. The feedback system would not know the temperate is rising until it reaches a certain threshold.

d. Inadequate sampling rate: is a frequent cause of dead time. In the above biofuel example, it may be deemed that sampling every 5 minutes is adequate. This introduces up to 5-minute lag in sensor reading to actuator output.

Modeling with a Dead time

Whatever may be the cause of the dead time the time between the control signal and the start of the system response may be lumped together as a single dead time D.

$$U = X \frac{k_p}{1 + k_p F}$$

With delay, u lags by some amount of time τ. From the Laplace transform properties, if

$\mathcal{L}\{f(t)\} \rightarrow F(s)$, then $\mathcal{L}\{f(t - \tau)\} \rightarrow e^{-\tau s} F(s)$,

The time delay can be modelled as:

$$Y = U'.F = e^{-\tau s} U \cdot F = e^{-\tau s} X \frac{k_p F}{1 + k_p F} \qquad (34)$$

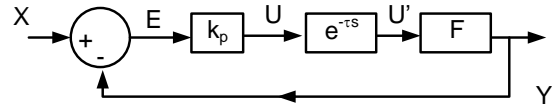
Figure 9.5: Proportional control with dead time τ.

For a single input–single output (SISO) system, like the one shown in figure 5, the effect of input delay is the same as the effect of the output delay.

Example 60

For a system represented by the transfer function, $\frac{1}{470s+1}$, it was desired to maintain the output (x) at 50%. A proportional gain is used to implement this algorithm. A time delay of 5 minutes was observed between the step input and the start of the increase of the temperature.

Plot the x, oil temperature (y) and u, over time for 30 minutes with a proportional gain (k_p) value of 2.

Solution:

$$F = \frac{1}{470s+1}, \ \tau = 300s$$

$$Y = e^{-300s} X \frac{2F}{1+2F} \qquad (35)$$

The following MATLAB code will generate the model and generate the graph

```
t = 0:30*60; %Generate time in s
x = 50;
kp = 2;

F = tf(1,[470 1],'Inputdelay',300);
Y = kp*F/(1+kp*F);
y = x*step(Y,t);

U = kp/(1+kp*F);
```
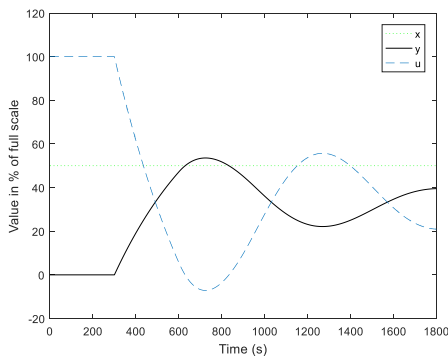
```
u = x*step(U,t);

plot(t,t*0+x,'g:');%Green dot line
hold on
plot(t,y, 'k');% Black solid line
plot(t,u,'--')% dash line
legend('x','y','u')
xlabel('Time (s)')
ylabel('Value in % of full scale')
```

Will produce the following plot

Characteristics of a control system

✓ **Rise time (T_r)**: The time for PV to go from 0.1 to 0.9 of the SP.

✓ **Peak Time(T_p)**: The time to reach the first peak.

✓ **Percent overshoot**:

$$\frac{Peak\ value - SP}{SP} \times 100\%$$

✓ **Settling time (Ts)**: The rime to reach within a tolerance range of SP(usually 2%).

✓ **Steady state error (e_s)**: The difference between SP and the final value of PV

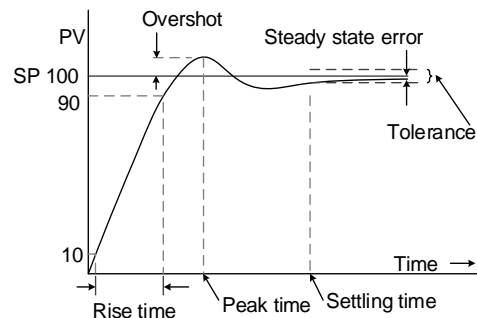A typical response of a controller is shown in the following figure:

Figure 9.6: A typical response of a proportional controller

Tuning a P-Controller

Setting the value of the proportional gain is called controller tuning. There is always a tradeoff in the desirable characteristics of the controller response. For instance, decreasing the rise time usually increase the percentage overshoot. The Higher gain will reduce the steady state error but higher the gain, higher will be the oscillation. In general, higher gain means shorter rise time and peak time, longer settling time and smaller steady state error. Too high of a gain in a system may cause system instability.

MATLAB provides a PID tuning app to provide the visual feedback of the system for a given system at various gain values. Type `pidtool` in MATLAB to use the tool. Alternatively, you can plot the system response at various k_p values and pick the one that satisfies the system performance criteria.

Effect of dead time

Longer the dead time, harder it is to control the system. Longer dead time results in bigger oscillations. The amount of overshoot increases with gain. Since PV lags in time, in our biofuel example, by the time the oil reaches SP to make

error value zero, the heater temperate is already too high. Even if the heater was turned off at that point, the higher temperate might drive the system temperature higher than SP.

Effect of process lag or time constant

A system with shorter process lag (or time constant) will rise or decay at much faster rate. Faster response means the system is more sensitive to input. A massive system with high inertia has a longer time constant. The home heating system is a good example. The amount of head absorbed as specific heat capacity is large compared and as a result, once desired temperature is reached it may take a long time to fall the temperature below an acceptable level. The system with a larger time constant is easier to control. An ON–OFF control may be satisfactory for this kind of system. Contrary to this example, controlling a system with a shorter time constant will be much harder to control and ON–OFF control may cause too much fluctuation.

The step response method

It is not always possible to make a mathematical model of the system or know the parameter values in advance. In addition, the assumptions made in deriving the mathematical model is not always accurate. Even in the simple example we have made several assumptions such as temperature within the tank is uniform, heat is only transferred through conduction and that thermal conductivity is constant. In reality, these may be somewhat different than ideal assumptions. **In most practical application, system response to input needs to be determined experimentally.**

One most common experimental method is measuring the system output for a step input. This approach is known as 'step response method'.

In step response method, the controller is switched to manual mode. The control variable (SP) is set to zero, and the system is allowed to come to equilibrium. Once the system is in equilibrium, u is set to a certain constant value and output is measured. As an example, figure 7 shows the measured biofuel temperature after the heater was turned fully ON. The step response test is conducted in open loop mode so that u is kept at a constant value. The curve shown in figure 7 is a common type of response you will encounter in process controls.
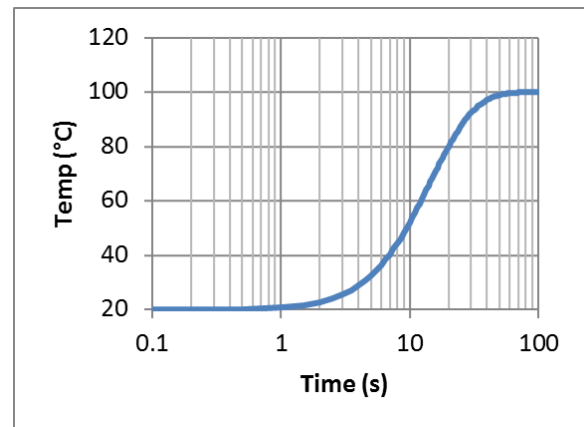

Figure 9.7: Response of cascaded 1st order systems

We can calculate the approximate dead time and time delay of the system by using a graphical method:

1. Draw a tangent line at the steepest point of the response curve, the point where it intersects the time axis is the dead time.

2. Mark a point which is 63.3% of the difference between final and initial value away from initial value.

Tuning a P controller using Ziegler–Nichols method

The Ziegler–Nichols method is a commonly used empirical method of tuning a PID controller. The method provides a good starting value of k_p for tuning. The controller may be fine-tuned from that point using trial and error method.

The method had a major influence on the practice of PID control for over half a century. The methods are based on the characterization of process dynamics and simple equations.

In this method; feedback–loop of the controller is turned off. First the system is brought to rest. A unit step input is applied and the PV is measured over time. The response is them plotted and two dynamic parameters are derived as shown in figure 8.
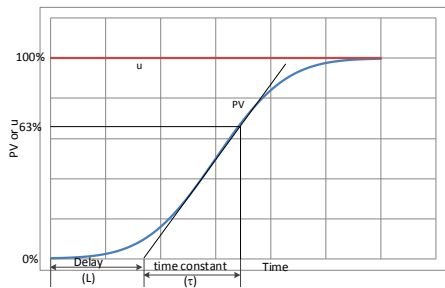

Figure 9.8: PID tuning using Ziegler-Nichols method

For a proportional controller, the method suggests the value of $k_p = \frac{\tau}{L}$.

One drawback of Ziegler–Nichols method is that the method does not work for a system with no dead time (L=0). A second Ziegler–Nichols method can be used in those cases.

The P gain is increased until it reaches the critical gain, Kc, at which the output of the loop loop starts to oscillate. Kc and the oscillation period Pc are used to set the gains:

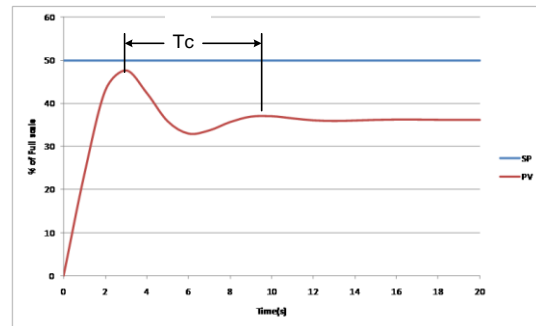Figure 9.9: Parameters for tuning a PID controller using self-oscillation method. Slowly increase k_p until response starts to oscillate. Kc is the critical gain (=k_p when oscillation begins).

$$kp = 0.5Kc$$

Example 61

For the step response in figure 7, find

a. Dead time
b. Process delay (or time constant)
c. The value of k_p for a proportional controller using Ziegler–Nichols method

Controller

A typical response of PI controller is shown below:

Figure 9.10: Typical response of a PI controller
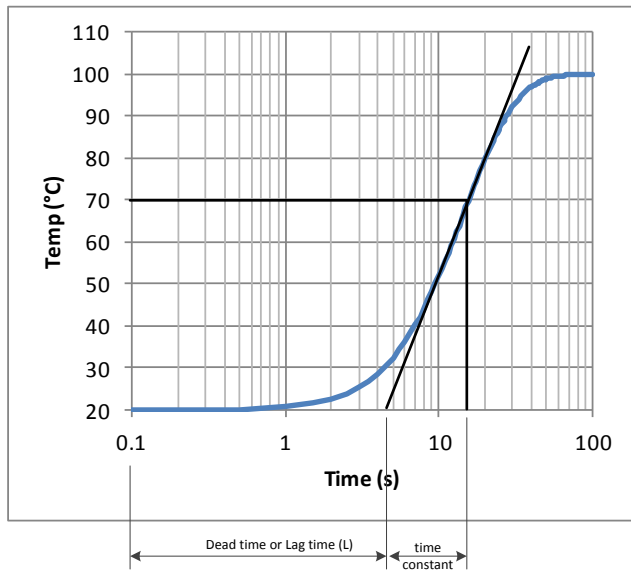
Solution:

Dead time (L)~ 4.5 s

Mark the point at $(100-20)*63.3\%+20 = 70.6°C$ and find the corresponding time. This time – Dead time is the time constant or the process lag (τ)= 17–4.5 = 12.5 s
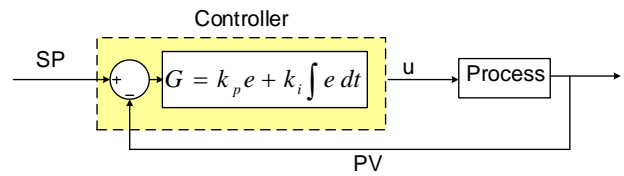
$$k_p = \frac{\tau}{L} = \frac{12.5}{4.5} = 2.8$$

PI-controller

A major drawback of proportional control, as we have seen, is the steady state error, i.e. PV never reaches SP. In order to overcome this drawback, the PI controller integrates the error over time and some fraction of the accumulated error is added to the proportional controller. For a PI controller,

$$u = k_p e + k_i \int e \, dt$$

This accumulated error drives the PV towards SP until the error is zero.

Effect of Integral gain

The integral of error term gives the net area between SP curve and PV curve. When an error is large, the error value accumulates more rapidly and hence the PV value moves more rapidly to SP. For the same value of k_p, addition of $k_i \int e \, dt$ makes the response much faster. However, since the integral term accumulates errors from the past, it may cause PV to overshoot SP. Therefore, the value of k_p is kept lower for PI controller than for P controller to reduce the overshoot.

Higher value of k_i, makes the faster response but tends to increase the oscillation. Smaller value of k_i reduces the overshoot (or oscillation), at an expense of slower rise time.

In summary, a PI controller eliminates the steady-state error, reduces the rise time, but it may increase the overshoot and may make the transient response more oscillatory.

Tuning a PI controller

If the model is known, the PI controller can be tuned for

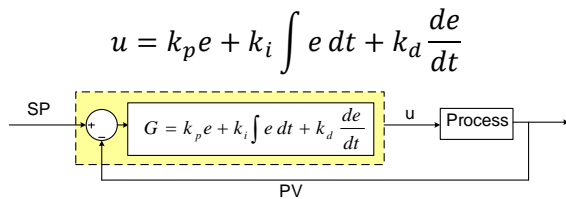## Tuning a PI controller using Ziegler–Nichols method

Step response method:

$$k_p = 0.9 * \frac{\tau}{L} \quad k_i = 0.27 * \frac{\tau}{L^2}$$

Self Oscillation method:

$$k_p = 0.4 \, K_c \quad k_i = 0.8 T_c$$

PID controller

Derivative controllers tend to improve the transient response of PI controller by adding the derivative term. The rate of change of the process error is calculated by determining the slope of the error over time (i.e., its first derivative with respect to time) and multiplying this rate of change by the derivative gain k_d.

$$u = k_p e + k_i \int e \, dt + k_d \frac{de}{dt}$$

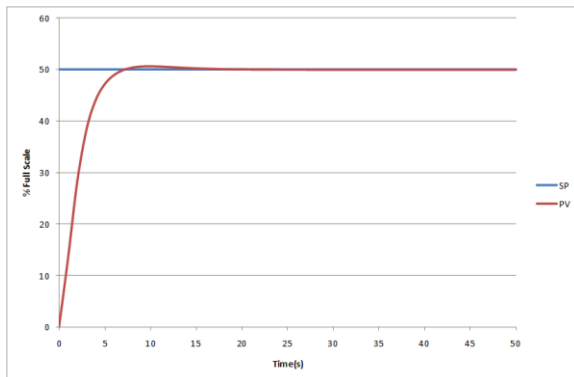A typical response of PID controller is shown below:

Figure 9.11: Typical response of a PID controller

Tuning a PID controller using Ziegler–Nichols method

Step response method:

$$k_p = 1.2 * \frac{\tau}{L} \quad\quad k_i = 0.6 * \frac{\tau}{L^2} \quad\quad k_d = 0.6 * \tau$$

Self Oscillation method:

$$k_p = 0.6 \, k_c \quad\quad k_i = 0.5 T_c \quad\quad k_d = 0.125 \, T_c$$

Effect of Differential gain

The differential term takes into account of the rate of change of error. Since error = SP-PV, and for a constant value of SP, $\frac{de}{dt} = -\frac{dPV}{dt}$. The slope of error curve is negative of the slope of the PV curve. Therefore, the differential term de/dt, tend to slow down the rate of change of the controller output. For an oscillating PV over SP, the PV curve slope is maximum when PV intersects SP curve (This is analogous to sine curve, the sine curve has maximum slop when it intersects the x-axis). Therefore, the negative effect is greatest when SP is near PV, which in turn helps reduce the magnitude of oscillation.

It should be noticed that if the coefficient of proportional and derivative gains is unchanged, derivative term slows the response of the system. Therefore, for a faster response, usually the proportional gain and integral gain needs to be increased.

Unlike integral term, the differential term will not have any effect on steady state value. At steady state, PV does not change with time and hence $\frac{de}{dt} = 0$. Therefore, while PI controller works fine without a differential term, PD controller will always have the same steady state error as P controller.

It is very important to note that the differential term only depends on previous two error values and usually *dt* is very small. Any small noise in

measurement is amplified by the factor 1/dt. Out of three terms in PID controller, the differential term is highly susceptible to noise. Therefore, in a noisy environment, it is prudent not to use the differential term or use it with a very small value of k_d, with caution. Differential gain can cause a process to become unstable if the noise and the derivative gain are sufficiently large.

Example 62

Calculate the differential term for past two error e1 = 2.0 and e2 = 1.99 measured 0.001 second after e1. Assume k_d = 1:

Differential term = $k_d * \frac{de}{dt} \cong k_d * \frac{e2-e1}{\Delta t} = \frac{-0.01}{0.001} = -10$

If because of noise, error e1 was recorded as 1.99 and e2 = 2.0 calculate the differential term

Differential term = $k_d * \frac{de}{dt} \cong k_d * \frac{e2-e1}{\Delta t} = \frac{0.01}{0.001} = 10$

As you can see the error of 0.01 in measurement caused the differential term to change from –10 to +10. This may cause the system to go unstable.

Summary of the PID coefficient response

	T_r	O_s	T_s	e_s
k_p	↓	↑	↑	↓
k_i	↓	↑	↑	0
k_d	↑	↓	↓	No Change

↓ Decrease, ↑ Increase, T_r: Rise time, O_s: Overshoot, T_s: Settling time, e_s: Steady state error.

Note that these correlations may not be exactly accurate, because k_p, k_i, and k_d are dependent on each other. In fact, changing one of these variables can change the effect of the other two. For this reason, the table should only be used as a reference when you are determining the values for k_p, k_i, and k_d.

Higher order models

Often, a system consists of several first order model coupled together to form a higher order model. In above biofuel example, we assumed that heater temperature reaches 20 °C as soon as the heater is turned off. However, practially we know that the heater temperature does not return to room temperature as soon as you turn it off. The heater itself can be represented as a first order model cascaded with the biofuel system to make a higher order model.

Example 63

The heater surface temperature can be represented as a differential equation:

$$15\frac{dT_h}{dt} + T_h = p$$

T_h is the heater temperature. P is proportional to the power input to the heater. The model of the biofuel reactor is:

$$470\frac{dT}{dt} + T = T_h$$

Derive the combined model when two systems are cascaded.

Solution:

When two systems are cascaded, the combined model is:

$$15\frac{d}{dt}\left(470\frac{dT}{dt}+T\right)+470\frac{dT}{dt}+T=p$$

$$7050\frac{d^2T}{dt}+485\frac{dT}{dt}+T=p$$

Example 64

For a second order system

$$50\frac{d^2T}{dt^2}+15\frac{dT}{dt}+T=100$$

Plot the oil temperature as a function of time. The initial temperature was at 20% and temperature increase rate was 0 at t=0.

Solution:

Taking the Laplace transform,

$$50\left(s^2F(s)-sf(0)-f'(0)\right)$$
$$+15\left(sF(s)-f(0)\right)+F(s)$$
$$=\frac{100}{s}$$
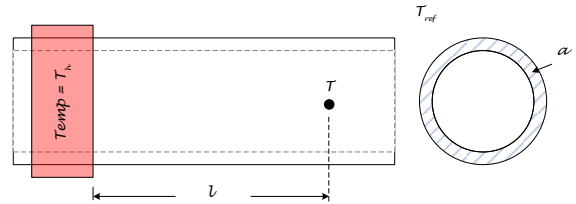$$50s^2F(s)-1000s+15sF(s)-300+F(s)$$
$$=\frac{100}{s}$$
$$50s^2F(s)+15sF(s)+F(s)$$
$$=\frac{100+300s+1000s^2}{s}$$
$$F(s)=\frac{100+300s+1000s^2}{s(50s^2+15s+1)}$$
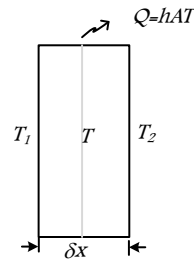

Example 65

A pipe heating system is shown below. The heater band temperature can be controlled by a PWM controller.

Establish a model how the temperature (T) at a distance l from heater increases with time for step input for heater temperature.

Solution

Let us consider a small section of the pipe.

Assume that $T_1 > T_2$ and the heat is being transferred from left to right. The temperature at the middle is T. The net inflow of heat into the system is $\dot{Q}_{in}=\frac{ka}{\delta x}(T_1-T_2)$, where k is the thermal conductivity, a is the cross-sectional areas of the pipe. The convective heat loss rate from the pipe is proportional to the surface temperature and surface area (A). The $\dot{Q}_{in}$ raises the pipe temperature and from the heat balance, $\dot{Q}_{in}=mc\frac{dT}{dt}+hAT$, where m is the mass of the pipe section, c is the specific heat capacity, h is the convective heat transfer coefficient. Combining these two equation, we get,

$$mc\frac{dT}{dt}+hAT=\frac{ka}{\delta x}(T_1-T_2)$$

For a very small section of pipe, we can assume that T is the average of T_1 and T_2.

$$mc \frac{d\frac{(T_1 + T_2)}{2}}{dt} + hA \frac{(T_1 + T_2)}{2} = \frac{ka}{\delta x}(T_1 - T_2)$$

For a step input of T_1, $\frac{dT_1}{dt} = 0$. Therefore, the above equation simplifies to

$$mc \frac{dT_2}{dt} + \left(\frac{2ka}{\delta x} + hA\right) T_2 = \left(\frac{2ka}{\delta x} - hA\right) T_1$$

This is the time domain model of the system. We can take the transfer function of this model and cascade it to find the temperature at any given length. The transfer function of the above system is

$$F(s) = \frac{\frac{2ka}{\delta x} - hA}{mcs + \left(\frac{2ka}{\delta x} + hA\right)}$$

In terms of Laplace variable 's', the system ca be approximately represented as:
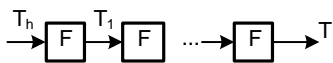
If the length l is divided into n sections for approximation, then $\delta x = l/n$ and the combined transfer function is,

$$F_n(s) = \left(\frac{2ka - hA\delta x}{mc\delta x \cdot s + 2ka + hA\delta x}\right)^n$$

This is the nth order model of the system.

Example 66

Assume that the pipe is made of copper and has the following specifications and properties:

Inside diameter = 1"; Pipe thickness = 0.125";
Thermal conductivity of copper $k = \frac{4W}{cm\,^\circ k}$,
convective heat transfer coefficient $h = \frac{10W}{m^2 {}^\circ K}$,
Density of copper is $\rho = 9 \frac{g}{cm^3}$, Specific heat capacity of copper $c = \frac{0.385J}{g^\circ K}$.

For l = 6 inch, plot the temperature T over time for T_h = 100°F above ambient. Assume that δx of 0.1" will be sufficient for this approximation.

For calculation, we will convert the numbers into consistent units.

$l = 6 * 2.54 = 15.24\ cm$

$\delta x = 0.1 * 2.54\ cm = 0.25\ cm$

Mean Dia $D = 1 + \frac{0.125}{2} = 1.0625\ in = 2.7\ cm$

$a = \pi D 0.125 = 0.417\ in^2 = 2.7\ cm^2$

$A = 2\pi D \delta x = 0.67 in^2 = 4.3\ cm^2$

$m = \rho a \delta x = \frac{9g}{cm^3} \cdot 2.7\ cm^2 \cdot 0.25cm = 6.1g$

$n = 6/0.1 = 60$

$2ka = 2\frac{4W}{cm\,^\circ k} 2.7\ cm^2 = 21.6 \frac{W.cm}{^\circ K}$

$hA\delta x = \frac{10W}{m^2 {}^\circ K} \cdot 4.3\ cm^2 \cdot 0.25cm = 0.001 \frac{W.cm}{^\circ K}$

$mc\delta x = 6.1g \cdot \frac{0.385J}{g\,^\circ K} \cdot 0.25 = \frac{0.59Ws}{^\circ K}$

$$F_n(s) = \left(\frac{21.6 - 0.001}{0.59s + 21.6 + 0.001}\right)^{60}$$

A thing to notice here is the relative amount of heat transfer through conduction vs. convention. The first term in the numerator is the heat loss from conduction and the second term is heat loss from convection.

$$F_n(s) = \left(\frac{21.599}{0.59s + 21.601}\right)^{60}$$

The following MATLAB code will plot the step response of this function

```
F = tf(21.599,[0.59 21.601]);
Fn = F^60;
step(100*Fn)
ylabel ('°F above ambient')
```

Step Response

°F above ambient

Time (seconds)
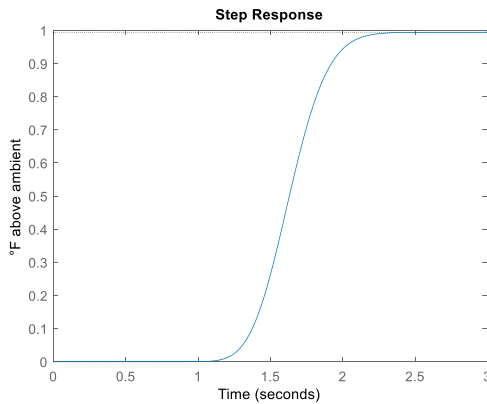
Example 67

This is an example of a coupled system, where two first order system interacts to create an oscillating higher model. The differential equation for glucose and insulin is given by:

$$\frac{dg}{dt} = -Ag - Bh$$

The negative sign of A indicates that the glucose level decreases over time in proportion to both glucose and insulin level.

The steady state level of glucose is not zero but some level that has to be maintained during the long fasting period. This is called a base level of glucose. The base level is usually the steady state level of blood glucose after about 12 hours of fasting.

When insulin is used in glucose metabolism, the level decreases. The rate of change of insulin is also proportional to blood glucose level which uses insulin. The differential equation for insulin and glucose is given by:

$$\frac{dh}{dt} = Cg - Dh$$

The positive value of C indicates that the body produces insulin in proportion to glucose level.

Combine two first order system to derive the complete system model.

Solution:

Combining above two equations,

$$\frac{d^2g}{dt^2} = -A\frac{dg}{dt} - B\frac{dh}{dt} \text{ or,}$$

$$\frac{1}{B} \times \frac{d^2g}{dt^2} + \frac{A}{B}\frac{dg}{dt} = -\frac{dh}{dt} = -Cg + Dh$$

But,

$$h = -\frac{1}{B}\frac{dg}{dt} - \frac{A}{B}g,$$

So,

$$\frac{1}{B} \times \frac{d^2g}{dt^2} + \frac{A}{B}\frac{dg}{dt} = -Cg + D\left\{-\frac{1}{B} \times \frac{dg}{dt} - \frac{A}{B}g\right\}$$

Or, $\frac{1}{B} \times \frac{d^2g}{dt^2} + \left\{\frac{A}{B} + \frac{D}{B}\right\}\frac{dg}{dt} + \left\{\frac{AD}{B} + C\right\}g = 0$

Or, $\frac{d^2g}{dt^2} + \{A + D\}\frac{dg}{dt} + \{AD + CB\}g = 0$

Similarly Solving for h will give the insulin model as:

$$\frac{d^2h}{dt^2} + \{A + D\}\frac{dh}{dt} + \{AD + CB\}h = 0$$

Example 68

For Insulin–Glucose model, let's say, A = 0.1, B=0.4; C=0.1 and D = 0.1. Assume that h = 0 and g = 100 at t = 0. Time is measured in hours. Plot glucose and insulin level for 72 hours with no additional food intake

Solution:

At t = 0,

$$g'(0) = \frac{dg}{dt}\bigg]_{t=0} = -Ag - Bh = -10$$

$$h'(0) = \frac{dh}{dt}\bigg]_{t=0} = Cg - Dh = 0.1 * 100 = 10$$

For Glucose:

$$\frac{d^2 g}{dt^2} + 0.2\frac{dg}{dt} + 0.05g = 0$$

Taking Laplace Transform both sides,

$$S^2 G - Sg(0) - g'(0) + 0.2SG - 0.2g(0) + 0.05G = 0$$

$$S^2 G - 100S + 10 + 0.2SG - 20 + 0.05G = 0$$

Or,

$$G(S^2 + 0.2S + 0.05) = 100S + 10$$

$$G = \frac{100S + 10}{S^2 + 0.2S + 0.05}$$

For Insulin:

$$\frac{d^2 h}{dt^2} + 0.2\frac{dh}{dt} + 0.05h = 0$$

Taking Laplace Transform both sides,

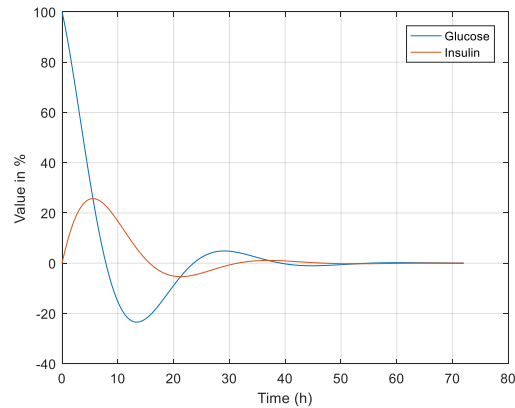$$S^2 H - Sh(0) - h'(0) + 0.2SG - 0.2g(0) + 0.5G = 0$$

Or,

$$H(S^2 + 0.2S + 0.05) = 10$$

$$H = \frac{10}{S^2 + 0.2S + 0.05}$$

The following MATLAB Code will make the plot

```
t = 0:.1:72;
G =tf([100 10],[1 .2 .05]);
H = tf(10,[1 .2 .05]);
g = impulse(G,t);
h = impulse(H,t);
plot(t,g,t,h);
legend('Glucose','Insulin')
xlabel('Time (h)')
ylabel('Value in %')
grid on
```


System stability

A stable system will converge over time. If no external force is applied, the PV value of a **stable system** approaches zero with time. An **unstable system**'s PV value grows towards infinite with time. If a system is marginally stable, its natural response will neither increase nor decrease over time.

A system can be checked for its stability from the natural response of the system.

Example 69

In a biological system, an initial number of bacteria was 100. The growth rate of bacteria is proportional to its number. The bacteria grow exponentially doubling every 3 hours. Find the differential equation expressing this phenomenon.

Solution:

$$\frac{dy}{dt} = ky$$

or $\frac{dy}{dt} - ky = 0$

Taking Laplace Transformation,

$$L\left\{\frac{dT}{dt} - ky\right\} = 0$$

$$sF(s) - 100 - kF(s) = 0$$

$$F(s) = \frac{100}{s-k}$$

Taking the inverse Laplace transform,

$$y = \mathcal{L}^{-1}(F(s)) = 100e^{kt}$$

Since the bacteria number doubles every three hours, we can use the number of bacteria at t = 0 and t = 3 hours to calculate k.

$$y(0) = 100 \text{ and } y(3) = 100e^{3k} \text{ but,}$$

$$y(3) = 2y(0) = 200,$$

Solving for k gives, $k = 0.23$. Therefore, the differential equaiton for this model is:

$$\frac{dy}{dt} - 0.23y = 0$$

Example 70

Check the system stability of the above model.

Solution:

$$y(t) = 100e^{0.23t}$$

According to this equation, the number of bacteria grown exponentially without a bound. Therefore, the system is unstable.

Checking for system stability

Let us consider the first order differential equation, $\tau\frac{dy}{dt} + y = 0$. Taking a Laplace transform of this equation yields

$$F(s) = \frac{f(0)}{s+\frac{1}{\tau}}$$

The inverse Laplace transform yields,

$$y = \mathcal{L}^{-1}(F(s)) = f(0)e^{-\frac{1}{\tau}t}$$

The $-\frac{1}{\tau}$ is the solution for s, if denominator of the transform function is equated to 0; i.e. $s +$

$\frac{1}{\tau} = 0$ or $s = -\frac{1}{\tau}$. The value of S that makes the denominator of the Laplace transform 0 is called a 'Pole'.

We can see that value of y asymptotically approaches 0 for a negative pole making a system stable. Therefore, a general statement can be made to test for a system stability that

"A system is stable only if all the poles of the system are negative. A system is marginally stable if any of the poles is 0, and the system is unstable if any of the poles is positive."

Exercise

1. List a three controls system you are familiar with in your everyday life. Identify their input and output variables.

   i. _____

   ii. _____

   iii. _____

2. A step response of a system is given below.
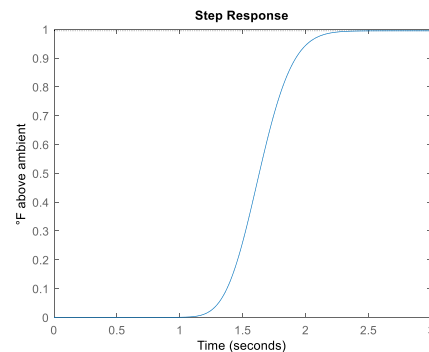
Use MATLAB `pidtool` to:

   a. Design a P controller with overshoot 2% or less. Record your k_p along with other system performances such as rise time, peak time and settling time, percent overshoot and steady state error.

b. Design a PI controller with overshoot 2% or less. Record your k_p along with other system performances such as rise time, peak time and settling time, percent overshoot and steady state error.

c. Design a PID controller with overshoot 2% or less. Record your k_p along with other system performances such as rise time, peak time and settling time, percent overshoot and steady state error.

3. Determine the approximate k_p, k_i and k_d values for P, PI and PID controller using Zeigler Nichols method in above question. Compare the the gain values from Zeigler Nichols method and from `pidtool`.

4. Find the step response and Impulse response of the following system.

$$2\frac{dy}{dt} + y = 0$$

5. For a glucose-insulin model, let's say, A = 0.1, B=0.4; C=0.1 and D = 0.1. Assume that h = 0.1 and g = 0.8 at t = 0. For the glucose-insulin model. Plot glucose and insulin over time.

6. For a copper pipe system with an impulse response $F_n(s) = \left(\frac{21.599}{0.59s+21.601}\right)^{10l}$, where, l is the distance from the heater in inches. Plot the temperature profile of the pipe at 0.1 interval from 0 to 6 inches at time = 2 seconds.

7. Create a temperature profile plot with a temperature distribution in y axis and time in y axis with space and time resolution of 0.1in and 0.1s.

8. For a healthy ecological system, there must be a balance between predator and prey population. In a typical predator prey system, the prey population grows exponentially if it were not for predators. In a specific environment, equation governing number of prey is given by:

$$\frac{du}{dt} = u - 5v$$

Where u represents prey count and v represents predator count. Similarly, the number of predator at any given time is given by:

$$\frac{dv}{dt} = 0.3u - 1.2v$$

a. Combine above two equations to get the second order differential equation showing how glucose level changes over time.

b. If the predator population was 10 and prey population was 500 at t = 0, take the Laplace transform of the equation to find U(s) which is the Laplace transform of u.

9. Is the system $\frac{dy}{dt} - y = x$, stable? Explain your finding.

10. Is the system $\frac{dy}{dt} = x$, stable? Explain your finding.

True or False?

11. Steady state error is unavoidable on a PI control system.

12. For a PID controller, the larger the time constant, the smaller should be the value of proportional gain.

13. For a PID controller, the larger the dead time, the larger should be the value of proportional gain.

14. Inclusion of derivative gain without changing proportional and integral coefficients will increases overshoot.

15. The steady state value of y for a system represented by differential equation $3\frac{dy}{dt} + 2y = 10$ is 5.