APPENDIX C

Thermal Conditions Associated with Uniform Energy Generation in One-Dimensional, Steady-State Systems

In Section 3.5 the problem of conduction with thermal energy generation is considered for one-dimensional, steady-state conditions. The form of the heat equation differs, according to whether the system is a plane wall, a cylindrical shell, or a spherical shell (Figure C.1). In each case, there are several options for the boundary condition at each surface, and hence a greater number of possibilities for specific forms of the temperature distribution and heat rate (or heat flux).

An alternative to solving the heat equation for each possible combination of boundary conditions involves obtaining a solution by prescribing *boundary conditions of the first kind,* Equation 2.24, at both surfaces and then applying an energy balance to each surface at which the temperature is unknown. For the geometries of Figure C.1, with uniform temperatures $T_{s,1}$ and $T_{s,2}$ prescribed at each surface, solutions to appropriate forms of the heat equation are readily obtained and are summarized in Table C.1. The temperature distributions may be used with Fourier's law to obtain corresponding distributions for the heat flux and heat rate. If $T_{s,1}$ and $T_{s,2}$ are both known for a particular problem, the expressions of Table C.1 provide all that is needed to completely determine related thermal conditions. If $T_{s,1}$ and/or $T_{s,2}$ are not known, the results may still be used with surface energy balances to determine the desired thermal conditions.

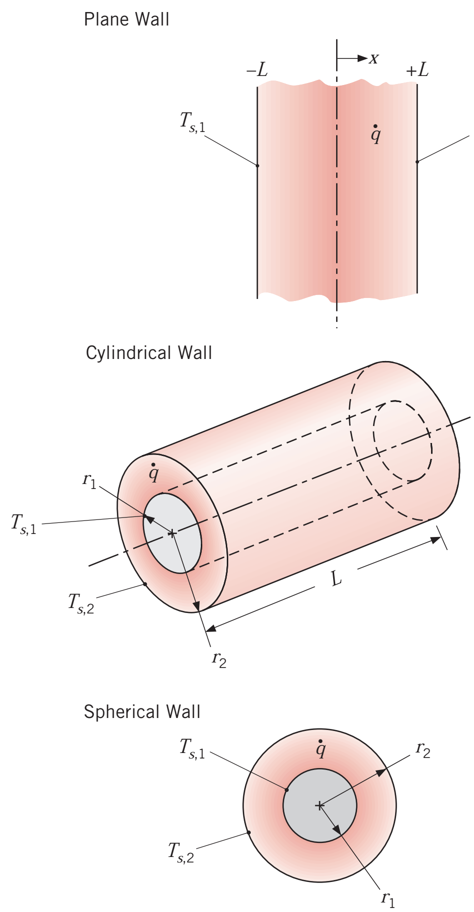

FIGURE C.1
One-dimensional conduction systems with uniform thermal energy generation: a plane wall with asymmetric surface conditions, a cylindrical shell, and a spherical shell.

TABLE C.1 One-Dimensional, Steady-State Solutions to the Heat Equation for Plane, Cylindrical, and Spherical Walls with Uniform Generation and Asymmetrical Surface Conditions

Temperature Distribution

Plane Wall
$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,1} + T_{s,2}}{2} \qquad \text{(C.1)}$$

Cylindrical Wall
$$T(r) = T_{s,2} + \frac{\dot{q}r_2^2}{4k}\left(1 - \frac{r^2}{r_2^2}\right) - \left[\frac{\dot{q}r_2^2}{4k}\left(1 - \frac{r_1^2}{r_2^2}\right) + (T_{s,2} - T_{s,1})\right]\frac{\ln(r_2/r)}{\ln(r_2/r_1)} \qquad \text{(C.2)}$$

Spherical Wall
$$T(r) = T_{s,2} + \frac{\dot{q}r_2^2}{6k}\left(1 - \frac{r^2}{r_2^2}\right) - \left[\frac{\dot{q}r_2^2}{6k}\left(1 - \frac{r_1^2}{r_2^2}\right) + (T_{s,2} - T_{s,1})\right]\frac{(1/r) - (1/r_2)}{(1/r_1) - (1/r_2)} \qquad \text{(C.3)}$$

Heat Flux

Plane Wall
$$q''(x) = \dot{q}x - \frac{k}{2L}(T_{s,2} - T_{s,1}) \qquad \text{(C.4)}$$

Cylindrical Wall
$$q''(r) = \frac{\dot{q}r}{2} - \frac{k\left[\frac{\dot{q}r_2^2}{4k}\left(1 - \frac{r_1^2}{r_2^2}\right) + (T_{s,2} - T_{s,1})\right]}{r\ln(r_2/r_1)} \qquad \text{(C.5)}$$

Spherical Wall
$$q''(r) = \frac{\dot{q}r}{3} - \frac{k\left[\frac{\dot{q}r_2^2}{6k}\left(1 - \frac{r_1^2}{r_2^2}\right) + (T_{s,2} - T_{s,1})\right]}{r^2[(1/r_1) - (1/r_2)]} \qquad \text{(C.6)}$$

Heat Rate

Plane Wall
$$q(x) = \left[\dot{q}x - \frac{k}{2L}(T_{s,2} - T_{s,1})\right]A_x \qquad \text{(C.7)}$$

Cylindrical Wall
$$q(r) = \dot{q}\pi Lr^2 - \frac{2\pi Lk}{\ln(r_2/r_1)}\cdot\left[\frac{\dot{q}r_2^2}{4k}\left(1 - \frac{r_1^2}{r_2^2}\right) + (T_{s,2} - T_{s,1})\right] \qquad \text{(C.8)}$$

Spherical Wall
$$q(r) = \frac{\dot{q}4\pi r^3}{3} - \frac{4\pi k\left[\frac{\dot{q}r_2^2}{6k}\left(1 - \frac{r_1^2}{r_2^2}\right) + (T_{s,2} - T_{s,1})\right]}{(1/r_1) - (1/r_2)} \qquad \text{(C.9)}$$

Alternative surface conditions could involve specification of a uniform surface heat flux (*boundary condition of the second kind,* Equation 2.25 or 2.26) or a convection condition (*boundary condition of the third kind,* Equation 2.27). In each case, the surface temperature would not be known but could be determined by applying a surface energy balance. The forms that such balances may take are summarized in Table C.2. Note that, to accommodate situations for which a surface of interest may adjoin a composite wall in which there is no generation, the boundary condition of the third kind has been applied by using the overall heat transfer coefficient U in lieu of the convection coefficient h.

TABLE C.2 Alternative Surface Conditions and Energy Balances for One-Dimensional, Steady-State Solutions to the Heat Equation for Plane, Cylindrical, and Spherical Walls with Uniform Generation

Plane Wall

Uniform Surface Heat Flux

$$x = -L: \qquad q''_{s,1} = -\dot{q}L - \frac{k}{2L}(T_{s,2} - T_{s,1}) \tag{C.10}$$

$$x = +L: \qquad q''_{s,2} = \dot{q}L - \frac{k}{2L}(T_{s,2} - T_{s,1}) \tag{C.11}$$

Prescribed Transport Coefficient and Ambient Temperature

$$x = -L: \qquad U_1(T_{\infty,1} - T_{s,1}) = -\dot{q}L - \frac{k}{2L}(T_{s,2} - T_{s,1}) \tag{C.12}$$

$$x = +L: \qquad U_2(T_{s,2} - T_{\infty,2}) = \dot{q}L - \frac{k}{2L}(T_{s,2} - T_{s,1}) \tag{C.13}$$

Cylindrical Wall

Uniform Surface Heat Flux

$$r = r_1: \qquad q''_{s,1} = \frac{\dot{q}r_1}{2} - \frac{k\left[\frac{\dot{q}r_2^2}{4k}\left(1 - \frac{r_1^2}{r_2^2}\right) + (T_{s,2} - T_{s,1})\right]}{r_1 \ln(r_2/r_1)} \tag{C.14}$$

$$r = r_2: \qquad q''_{s,2} = \frac{\dot{q}r_2}{2} - \frac{k\left[\frac{\dot{q}r_2^2}{4k}\left(1 - \frac{r_1^2}{r_2^2}\right) + (T_{s,2} - T_{s,1})\right]}{r_2 \ln(r_2/r_1)} \tag{C.15}$$

Prescribed Transport Coefficient and Ambient Temperature

$$r = r_1: \qquad U_1(T_{\infty,1} - T_{s,1}) = \frac{\dot{q}r_1}{2} - \frac{k\left[\frac{\dot{q}r_2^2}{4k}\left(1 - \frac{r_1^2}{r_2^2}\right) + (T_{s,2} - T_{s,1})\right]}{r_1 \ln(r_2/r_1)} \tag{C.16}$$

$$r = r_2: \qquad U_2(T_{s,2} - T_{\infty,2}) = \frac{\dot{q}r_2}{2} - \frac{k\left[\frac{\dot{q}r_2^2}{4k}\left(1 - \frac{r_1^2}{r_2^2}\right) + (T_{s,2} - T_{s,1})\right]}{r_2 \ln(r_2/r_1)} \tag{C.17}$$

Spherical Wall

Uniform Surface Heat Flux

$$r = r_1: \qquad q''_{s,1} = \frac{\dot{q}r_1}{3} - \frac{k\left[\frac{\dot{q}r_2^2}{6k}\left(1 - \frac{r_1^2}{r_2^2}\right) + (T_{s,2} - T_{s,1})\right]}{r_1^2[(1/r_1) - (1/r_2)]} \tag{C.18}$$

$$r = r_2: \qquad q''_{s,2} = \frac{\dot{q}r_2}{3} - \frac{k\left[\frac{\dot{q}r_2^2}{6k}\left(1 - \frac{r_1^2}{r_2^2}\right) + (T_{s,2} - T_{s,1})\right]}{r_2^2[(1/r_1) - (1/r_2)]} \tag{C.19}$$

TABLE C.2 Continued

Prescribed Transport Coefficient and Ambient Temperature

$$r = r_1: \qquad U_1(T_{\infty,1} - T_{s,1}) = \frac{\dot{q}r_1}{3} - \frac{k\left[\dfrac{\dot{q}r_2^2}{6k}\left(1 - \dfrac{r_1^2}{r_2^2}\right) + (T_{s,2} - T_{s,1})\right]}{r_1^2[(1/r_1) - (1/r_2)]} \qquad \text{(C.20)}$$

$$r = r_2: \qquad U_2(T_{s,2} - T_{\infty,2}) = \frac{\dot{q}r_2}{3} - \frac{k\left[\dfrac{\dot{q}r_2^2}{6k}\left(1 - \dfrac{r_1^2}{r_2^2}\right) + (T_{s,2} - T_{s,1})\right]}{r_2^2[(1/r_1) - (1/r_2)]} \qquad \text{(C.21)}$$

As an example, consider a plane wall for which a uniform (known) surface temperature $T_{s,1}$ is prescribed at $x = -L$ and a uniform heat flux $q''_{s,2}$ is prescribed at $x = +L$. Equation C.11 may be used to evaluate $T_{s,2}$, and Equations C.1, C.4, and C.7 may then be used to determine the temperature, heat flux, and heat rate distributions, respectively.

Special cases of the foregoing configurations involve a plane wall with one adiabatic surface, a solid cylinder (a circular rod), and a sphere (Figure C.2). Subject to the requirements that $dT/dx|_{x=0} = 0$ and $dT/dr|_{r=0} = 0$, the corresponding forms of the heat equation may be solved to obtain Equations C.22 through C.24 of Table C.3.

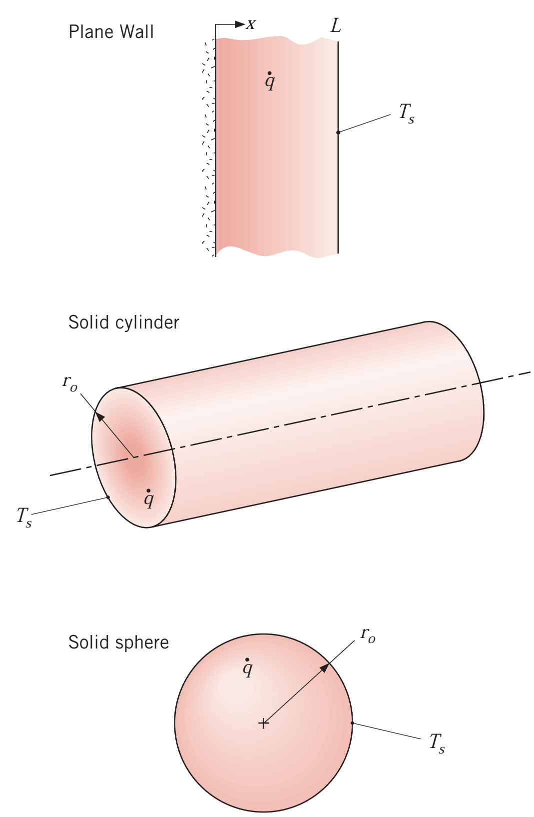

FIGURE C.2
One-dimensional conduction systems with uniform thermal energy generation: a plane wall with one adiabatic surface, a cylindrical rod, and a sphere.

TABLE C.3 One-Dimensional, Steady-State Solutions to the Heat Equation for Uniform Generation in a Plane Wall with One Adiabatic Surface, a Solid Cylinder, and a Solid Sphere

	Temperature Distribution	
Plane Wall	$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s$	(C.22)
Circular Rod	$T(r) = \frac{\dot{q}r_o^2}{4k}\left(1 - \frac{r^2}{r_o^2}\right) + T_s$	(C.23)
Sphere	$T(r) = \frac{\dot{q}r_o^2}{6k}\left(1 - \frac{r^2}{r_o^2}\right) + T_s$	(C.24)
	Heat Flux	
Plane Wall	$q''(x) = \dot{q}x$	(C.25)
Circular Rod	$q''(r) = \frac{\dot{q}r}{2}$	(C.26)
Sphere	$q''(r) = \frac{\dot{q}r}{3}$	(C.27)
	Heat Rate	
Plane Wall	$q(x) = \dot{q}xA_x$	(C.28)
Circular Rod	$q(r) = \dot{q}\pi L r^2$	(C.29)
Sphere	$q(r) = \frac{\dot{q}4\pi r^3}{3}$	(C.30)

TABLE C.4 Alternative Surface Conditions and Energy Balances for One-Dimensional, Steady-State Solutions to the Heat Equation for Uniform Generation in a Plane Wall with One Adiabatic Surface, a Solid Cylinder, and a Solid Sphere

Prescribed Transport Coefficient and Ambient Temperature

Plane Wall

$x = L$: $\quad \dot{q}L = U(T_s - T_\infty)$ (C.31)

Circular Rod

$r = r_o$: $\quad \frac{\dot{q}r_o}{2} = U(T_s - T_\infty)$ (C.32)

Sphere

$r = r_o$: $\quad \frac{\dot{q}r_o}{3} = U(T_s - T_\infty)$ (C.33)

The solutions are based on prescribing a uniform temperature T_s at $x = L$ and $r = r_o$. Using Fourier's law with the temperature distributions, the heat flux (Equations C.25 through C.27) and heat rate (Equations C.28 through C.30) distributions may also be obtained. If T_s is not known, it may be determined by applying a surface energy balance, appropriate forms of which are summarized in Table C.4.