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Chapter 23

Mathematics and Mathematical Axioms

In every other science men prove their conclusions by their principles, and not their principles by the conclusions.

Berkeley

§ 1. Mathematics and Its Axioms

Kant once remarked that a doctrine was a science proper only insofar as it contained mathematics. This shocked no one in his day because at that time mathematics was still the last bastion of rationalism, proud of its history, confident in its traditions, and certain that the truths it pronounced were real truths applying to the real world. Newton was not a 'physicist'; he was a 'mathematician.' It sometimes seemed as if God Himself was a mathematician.

The next century and a quarter after Kant's death was a tough time for mathematics. The 'self evidence' of the truth of some of its basic axioms burned away in the fire of new mathematical discoveries. Euclid's geometry was *not* the only one possible; analysis was confronted with such mathematical 'monsters' as everywhere-continuous/nowhere-differentiable curves; set theory was confronted with the Russell paradox; and Hilbert's program of formalism was defeated by Gödel's 'incompleteness' theorems.

The years at the close of the nineteenth and opening of the twentieth centuries were the era of the 'crisis in the foundations' in mathematics. Davis and Hersch write:

> The textbook picture of the philosophy of mathematics is a strangely fragmented one. The reader gets the impression that the whole subject appeared for the first time in the late nineteenth century, in response to contradictions in Cantor's set theory. At that time there was talk of a "crisis in the foundations." To repair the foundations, three schools appeared on the scene, and spent some thirty or forty years quarreling with each other. It turned out that none of the three could really do much about the foundations, and the story ends in mid-air some forty years ago, with Whitehead and Russell having abandoned logicism, Hilbert's formalism defeated by Gödel's theorem, and Brouwer left to preach constructivism in Amsterdam, disregarded by all the rest of the mathematical world.
>
> This episode in the history of mathematics actually is a remarkable story. Certainly it was a critical period in the philosophy of mathematics. But by a striking shift in the meaning of words, the fact that foundationism was at a certain critical period the dominant trend in the philosophy of

> mathematics has led to the virtual identification of the philosophy of mathematics with the study of foundations. Once this identification is made, we are left with a peculiar impression: the philosophy of mathematics was an active field for only forty years, and after a while went back to sleep.
>
> In reality there has always been a philosophical background, more or less explicit, to mathematical thinking. The foundationist period was one in which leading mathematicians were overtly concerned with philosophical issues, and engaged in public controversy about them [DAVI: 322-323].

Well, yes. This is what we need to talk about.

§ 1.1 The Fall of the Axiom

The mathematical axiom has suffered a long fall from its ancient eyrie. Nearly 24 centuries ago it was held to be a self-evident truth, a statement that was absolutely beyond any suspicion that it could be false. Today mathematicians tell non-mathematicians that an axiom is only a premise or rule in a game, a starting point. In the more official language of Nelson's *Dictionary of Mathematics* (2nd ed.), an axiom is

> a well-formed formula that is stipulated rather than proved to be so through the application of rules of inference. The axioms and the rules of inference jointly provide a basis for proving all other theorems. As different sets of axioms may generate the same set of theorems, there may be many *alternative axiomatizations* of the formal system.

And, of course, different sets of axioms may also generate quite different theorems. Such is the case, for example, in the set of axioms for Riemannian geometry vs. Euclidean geometry.

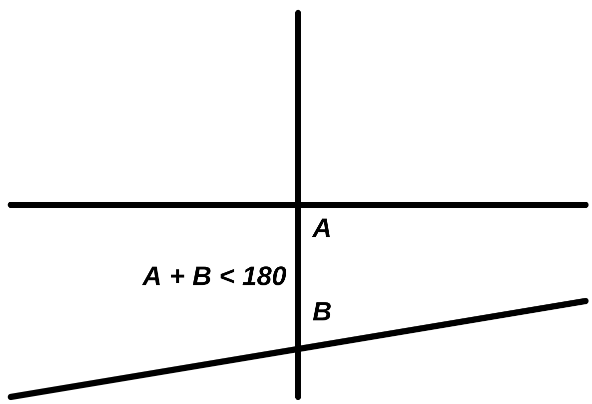
None of the mathematicians I have the pleasure of knowing is actually in his or her heart so cavalier about axioms as this 'rules of the game' official picture might lead the rest of us to think. They simply prefer not to spend their time in 'philosophical' arguments with non-mathematicians over whether or not the axioms they use are 'really true.' The mathematics community by and large has agreed upon a set of axioms (today, the axioms of 'axiomatic set theory') and they use these, implicitly or explicitly, in doing mathematics. There are a few 'bad boy' axioms that have historically caused trouble – and when one of these is used an adjective is usually applied to describe the particular brand of mathematics, e.g. 'Euclidean geometry' – but I think that they regard *most* of their axioms as 'really' being self-evident and descriptive of 'the way things obviously are.'

The problem is they cannot *prove* that the 'good boy' axioms are 'really' absolutely true. "Proof" to a mathematician is an almost holy concept. As they sometimes put it, "if you haven't got a proof, you have nothing." Mathematics is the art – and, I will say, the *science* – of proving theorems, and a theorem is a statement that, given the premises laid down by the axioms and certain agreed-upon rules of inference, is apodictically true.

For a long time it was thought that one could demonstrate an axiom was True. The idea was:

deny the axiom (assert its logical contradiction) and then show that the resulting formal system which followed from using that second axiom in place of the first was self-contradictory. Historically the first axiom on which this method was tried was Euclid's 'parallel axiom', which states "if two straight lines lying in a plane are met by another line, and if the sum of the internal angles on one side is less than two right angles, then the straight lines will meet if they are extended sufficiently on the side on which the sum of the angles is less than two right angles."

Now many of us, upon reading this statement, are likely to blink a couple of times. The statement looks complicated, and compared to Euclid's other axioms (e.g. "all right angles are equal") it is. This axiom is called 'the parallel axiom' because if the 'sum of the internal angles' is equal to 'two right angles' (180 degrees), the consequence held to follow is that the two lines will *never* intersect no matter how far they are extended. Indeed, this condition on the internal angles could be taken as the defining property of 'parallel lines' in Euclidean geometry. The axiom is phrased in this apparently 'complicated' way because it is easy to draw a picture and 'see' that the two lines will 'obviously' intersect. An example of such a picture is shown below.



Here it seems obvious that Euclid's parallel axiom is true. However, because the wording of this axiom is so complicated compared to the other axioms of Euclid, it was decided to 'prove' that this axiom was True by demonstrating its truth through the others. How was this to be done? The approach was to deny the parallel axiom and then show that when combined with the others contradiction would result. Girolamo Saccheri (1667-1733) and Kant's friend Johann Lambert (1728-1777) were two of the most notable early investigators of this approach. Both men thought they had succeeded in demonstrating the looked-for contradiction but, as it turns out, they were wrong. Nikolai Ivanovich Lobatschewsky (1793-1856) and Georg Friedrich Bernhard Riemann (1826-1866) both succeeded in coming up with perfectly self-consistent geometries using two different versions of the axiom that replaced Euclid's. As a result, we today have not one, not two, but *three* different and ‘perfectly good’ forms of geometry. And, as it turns out, they can be extended to more than just three ‘dimensions.’ Riemann, in fact, showed there were indefinitely many self-consistent geometries obtainable through playing with some of the other axioms, but it turned out that three were more than enough for nineteenth century mathematics. Two would have been more than enough.

Referring back to the previous figure, it probably strikes most of us as absurd that Euclid’s axiom (playfully known by some as “Euclid’s fifth”) could possibly be wrong. “Obviously” the lines will eventually intersect on the right. Equally “obvious” is that if we rotate the lower line until $A + B > 180$ degrees, the lines must eventually intersect on the left. “Clearly” then, at $A + B = 180$ degrees, the lines must either eventually intersect on *both* the left and right (which seems absurd) or else never intersect at all (which does not seem absurd). This is, of course, how “Euclid’s fifth” became an axiom in the first place. However, there are a couple presuppositions hiding in this line of reasoning. The first is that we ‘know’ what constitutes a ‘straight line’. The second is that we must suppose it ‘makes sense’ that we actually could ‘extend a straight line indefinitely’.

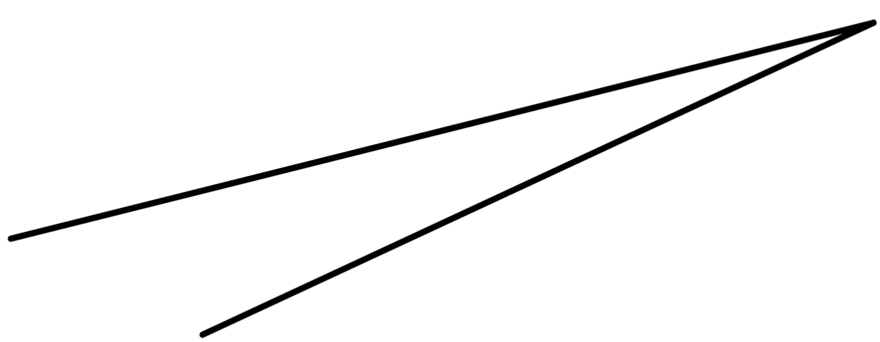
Both these presuppositions seem to be pretty safe, at least for those of us who have not over-trained in mathematics or philosophy. Some might even say they are “intuitively correct.” But why is this? Piaget, through a careful set of experiments and observations, was able to show that as children we develop geometric concepts and ideas *through sensorimotor capabilities* that end up being precisely the same ideas that underlie Euclidean geometry [PIAG9]. In other words, through schemes of construction we come to put together concepts that are at root those embedded in Euclid’s axioms and definitions. Indeed, these developed *and empirical* concepts ‘make more sense’ to many of us than do Euclid’s actual statements of his definitions and axioms. For example, Euclid’s first four definitions are:

1. A *point* is that which has no part.
2. A *line* is a breadthless length.
3. The extremities of a line are points.
4. A *straight line* is a line which lies evenly with the points on itself.

Until one is acclimatized to this way of talking through practice in geometry and intercourse with other people who constantly talk like this (mathematicians), definitions 1, 2, and 4 at least do not ‘make much sense’ to many of us. The reason is that these mathematical objects are idealizations of things we can actually construct using pencil, paper, and a good ruler. Even then definition 4 is something of a challenge for many of us. What exactly does it mean to say a line “lies evenly with the points on itself”? Most of us who bother to try to understand this statement come to understand it 'in reverse.' That is to say, we take some sensible objects we use as examples of 'things that are straight' and then tell ourselves, "okay, to 'lie evenly with the points on itself' means to look like *this*."

One thing you might find to be interesting is that neuroscience has identified within the primary visual cortex (Brodmann's area 17) groups of neurons that respond only to stimulations of specific areas of the retina that have properties we all agree to call 'linear' properties. Such neuron groups have what is officially called a 'rectilinear receptive field' of vision.[1] They are somatic representations of what we can justly call very small little 'straight line segments' but, of course, these representations are neither breadthless nor indefinitely extended. It also appears it might be the case that the 'wiring' of these neuron groups, or perhaps of the afferent neural pathways delivering signals to them, develops after birth and is not 'innate.'

Another factor pertinent to this discussion is the perceptual phenomenon called *perspective*. Suppose you are traveling on a very 'straight, flat' road with an unobstructed view for many miles ahead (say, for example, Interstate 80 west of Salt Lake City, Utah). If you look at the road ahead, it will appear as if the 'parallel lines' at the edge of the road converge in the distance, something like what is illustrated in the following figure. Compare this with what you can imagine the previous figure to look like if we made $A + B = 180$ degrees. I think you will find that the 'obviousness' that the lines in the previous figure would not intersect depends on your keeping a 'local perspective' on the lines, e.g. 'traveling along as we extend them' and noticing that the distance between the lines neither increases nor decreases. The point that I'm making here is this: When we actually encounter 'very long, straight, parallel lines' our experience is that of the perspective shown below. We base the 'obviousness' that parallel lines would not intersect 'just from looking' at the first figure on what is really an elaborate mental construction process that, in part, constrains our two parallel straight lines to maintain the appearance of equal distance between them 'as we go along with them.' Of course we all 'know' that if we drive down the road



[1] See Carol Mason and Eric R. Kandel, "Central visual pathways," in [KAND: 420-439].

the sides of the road really do not squeeze in on us, despite what the perspective looks like. But this, too, depends on our having this experience. Suppose you were Kenge the pygmy, whom we met in Chapter 16. Then what do you suppose you would think of taking a drive down this road?

To make matters more interesting, suppose we turn around and look back down the road from where we've come. We'll see the same perspective effect. But suppose we've not actually traveled the road – let's say we were born and raised at one place along the road and never left that spot for more than a mile or so in either direction.[2] How sure would we be that the straight edges of the road, clearly 'parallel' where we've lived, don't 'really' eventually intersect in *both* directions?

The point to all this is that the 'obviousness' of Euclid's fifth is based on a mental process of construction backed up by the experience we've had. In point of fact, four of Euclid's five axioms are expressed in terms of construction processes:

> Let the following be postulated:
> 1. To draw a straight line from any point to any point.
> 2. To produce a finite straight line continuously in a straight line.
> 3. To describe a circle with any center and distance.
> 4. That all right angles are equal to one another.
> 5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles [EUCL: 2].

Backing up these axioms were five 'common notions':

> 1. Things which are equal to the same thing are also equal to one another.
> 2. If equals be added to equals, the wholes are equal.
> 3. If equals be subtracted from equals, the remainders are equal.
> 4. Things which coincide with one another are equal to one another.
> 5. The whole is greater than the part [EUCL: 2].

The idea of 'parallel lines' in Euclid was definition 23:

> (Def.) 23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction [EUCL: 2].

For centuries every educated person looked at these and, more or less, said, "yes, I'll buy into that." It was just 'intuitive' that the non-ideal lines, points, etc. could in principle be made finer and finer until they 'reached the ideals' described in Euclidean geometry. I suspect that even 'extending a line indefinitely' posed no great problem for the classical Greeks since most of them, with the glaring exception of Plato, did not think anything could be 'infinite.' (Dividing by zero

[2] Interstate 80 in the great salt flat west of Salt Lake City runs about 90 miles more or less 'straight as an arrow.' About 45 miles in there's a filling station (or at least there was in 1975), and nothing else made by man along the whole stretch except the highway itself (or, at least, that was so in 1975).

was not an issue either, since in those days only a lunatic would think he could divide something by nothing). 'Indefinitely' in this sense more or less means 'as far as you can', and for parallel lines this implies 'extend them as far as you can and still they won't meet.'

But by the 18th century 'infinity' was much more popular (although it was still a 'becoming' and not yet what Cantor would later make it out to be), and mathematicians despise 'mere intuition' as an argument for the truth of something. If we can imagine that two lines never, ever will meet, we can equally well take a look at our road example and imagine that they will. Of course, in the case of geometry every mathematician was sure the latter was going to prove out to be just an illusion, and as soon as Euclid's fifth was vindicated everyone who was not stupid would realize that. Hume and the empiricists had assailed Euclid's fifth without success and geometry remained the bastion of mathematics and the redoubt of rationalism.

It is not at all clear when, where, why, or by whom work on 'vindicating' Euclid's fifth first began. According to Professor Ball, attempts to prove it (rather than take it as an axiom) date back to "from early times."[3] Bertrand Russell credits Karl Friedrich Gauss (1777-1855) with launching the research that led to non-Euclidean geometries [RUSS2: 19], but this is clearly wrong since it is known that Saccheri and Lambert both worked on it before Gauss was born. It is also clear that the 18th century work was aimed at vindicating it rather than challenging it. For example, in a 1766 letter to Kant, Lambert wrote

> Our knowledge of the form, as in logic, is as incontestable and right as in geometry. . . Axioms and postulates actually contain only simple concepts [KANT20a: 85 (10: 64-65)].

No one has ever been able to derive Euclid's fifth from the other axioms, and mathematics has by now concluded that it is not possible to do so.

But 'not being false' and 'being necessary to geometry' are not the same thing. By 1826 Lobatschewsky had denied the fifth and come up with a 'hyperbolic' geometry, and by 1854 Riemann had done the same and published his resulting 'elliptic' geometry. So now there were three: Euclidean (or 'parabolic'), hyperbolic, and elliptic. In Riemann's geometry there are no 'parallel' lines, and our speculation above that maybe the lines intersect on *both* sides is *true*. And if you find *that* unsettling, imagine how the mathematicians of the nineteenth century felt.

§ 1.2 The Axioms of Set Theory

Although we call them the 'axioms of Euclid,' it is very unlikely that Euclid originally formulated them. It was generally held in ancient times that Euclid had drawn upon the works of many

[3] W.W. Rouse Ball, *A Short Account of the History of Mathematics*, NY: Dover Publications, 1960, pg. 486. (Prof. Ball first published this work in 1888, and its 4th, and last, edition was published in 1908.)

predecessors, among them the Pythagoreans, Eudoxus and Theatetus. Almost nothing is known of Euclid's life. He lived around 300 B.C. and it is thought that he received his education in Athens from Plato's former pupils. Almost certainly Euclid put into an orderly form what had probably been a loose and haphazard tradition of geometric propositions that had been accumulating for probably at least two centuries, judging from Ball's *History*. Of Euclid's *Elements* Ball tells us,

> The geometrical part is to a large extent a compilation from the works of previous writers. Thus the substance of books I and II (except perhaps the treatment of parallels) is probably due to Pythagoras . . . But this material was rearranged, obvious deductions were omitted . . . and in some cases new proofs substituted . . . The whole was arranged as a complete and consistent body of theorems.
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> The form in which the propositions are presented, consisting of enunciation, statement, construction, proof, and conclusion, is due to Euclid; so also is the synthetical character of the work, each proof being written out as a logically correct train of reasoning but without any clue to the method by which it was obtained.
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> The defects of Euclid's *Elements* as a text-book of geometry have been often stated; the most prominent of these: (i) The definitions and axioms contain many assumptions which are not obvious, and in particular the postulate or axiom about parallel lines is not self-evident. (ii) No explanation is given as to the reason why the proofs take the form in which they are presented . . . (iii) There is no attempt made to generalize the results arrived at . . .[4]

That the definitions, axioms, and 'common notions' were compiled gradually by many authors over a long period of time, and that Euclid probably did not modify them beyond arranging them and perhaps standardizing the manner of expression, might lead one to think that perhaps these foundations were built on sandy ground. On the other hand, Euclidean geometry lasted without challenge or modification for 21 centuries. It will be quite some time before the axioms of set theory will have a chance to make a similar claim.

For set theory is where mathematics eventually turned following the disaster of the non-Euclidean geometries. By using the word 'disaster' I do not imply that it was somehow a bad thing that the impressive discoveries of Riemann, Lobatschewsky, and others were made. The 'disaster' to mathematics was the loss of its long-held conviction that mathematical truths were certain and that through them mankind could know the world solely by the raw power of logic and reason. Davis and Hersch write,

> Until well into the nineteenth century, the Euclid myth[5] was as well established among mathematicians as it was among philosophers. Geometry was regarded by everyone . . . as the firmest, most reliable branch of knowledge. Mathematical analysis – calculus and its extensions and ramifications – derived its meaning and legitimacy from its link to geometry . . .
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> In the nineteenth century, several disasters took place. One disaster was the discovery of non-Euclidean geometries, which showed there was more than one thinkable geometry.
>
> A greater disaster was the development of analysis so that it overtook geometrical intuition, as in

[4] W.W.R. Ball, *ibid.*, pp. 54-55.

[5] By "Euclid myth" they mean the traditional belief that the books of Euclid contain truths about the universe which are clear and indubitable.

> the discovery of space-filling curves and continuous nowhere-differentiable curves. These stunning surprises exposed the vulnerability of the one solid foundation – geometrical intuition – on which mathematics had been thought to rest. The loss of certainty in geometry was philosophically intolerable, because it implied the loss of all certainty in human knowledge . . .
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> The theory of sets was developed by Cantor as a new and fundamental branch of mathematics in its own right. It seemed that the idea of a set – an arbitrary collection of distinct objects – was so simple and fundamental that it could be the building block out of which all of mathematics could be constructed . . .
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> Set theory at first seemed to be almost the same as logic . . . So it seemed possible that set-theory-logic could serve as the foundation for all of mathematics [DAVI: 330-331].

At first things looked pretty promising for this ‘new foundation’ of mathematics. However, Cantor did not work from a well-enunciated system of axioms, and it did not take long for the trouble to begin anew with the appearance of disturbing antinomies and paradoxes – the most famous of which was, of course, the Russell paradox.

It was in response to the growing number of problems cropping up in set theory that Ernst Zermelo (1871-1953) first set out to formalize set theory on the basis of a set of definitions and axioms of its own. This was the birth, in 1908, of axiomatic set theory. However, coming up with a set of axioms proved to be no easy thing – a somewhat disturbing fact when one considers that the original purpose behind the whole nineteenth-to-twentieth century undertakings of Peano, Russell, Whitehead, Hilbert, and others was aimed at recovering the character of absolute mathematical certainty based upon self-evident truths. Zermelo began the work of axiomatic set theory, but was soon joined – and sometimes fought – by other mathematicians. The roster of important contributors includes Abraham Fraenkel, Thoralf Skolem, Paul Bernays, John von Neumann, and Kurt Gödel.

One reason the task proved so difficult was that axiomatic set theory was to be *the* foundation for *all* of mathematics, and this was taken to mean *all the mathematics that was already in existence at that time*. It would seem to the impartial outsider that the attitude of the general mathematics community was a somewhat peculiar one. Certainty was to be recovered for all of the existing mathematical *corpus* after the *ground* upon which that certainty was originally laid had collapsed from under it. With the exception of Brouwer – the tolerated heretic from Amsterdam – it seems as if the mathematics community simply refused to consider the possibility that maybe *some* existing mathematics really was uncertain, and that perhaps it might be worthwhile to separate ‘speculative mathematics’ from ‘apodictic mathematics.’

Perhaps this is an unfair characterization; perhaps not. The principals in this drama are long dead and so, barring the publication of documentary evidence, we will never know the thinking and motives involved. What we do know is this: There is not a single, universally accepted set of axioms for axiomatic set theory to this day. There *is* a set of axioms that enjoys the widest usage by a commanding margin. This is the Zermelo-Fraenkel-Skolem (ZFS) set of axioms, and this is the one we will take a look at here. We also know one more thing: The quest to recover absolute certainty failed.

> On foundations we believe in the reality of mathematics, but of course when the philosophers attack us with their paradoxes we rush to hide behind formalism and say, "Mathematics is just a combination of meaningless symbols," and then we bring out Chapters 1 and 2 on set theory. Finally we are left in peace to go back to our mathematics and do it as we have always done, with the feeling that each mathematician has that he is working with something real. This sensation is probably an illusion, but is very convenient. That is Bourbaki's attitude toward foundations.[6]

How the mighty have fallen. Encrypted as they are by the awful hieroglyphics invented by Russell and Whitehead[7], "Chapters 1 and 2 on set theory" make a pretty effective shield. Most of us have more fun things to do – like setting our hair on fire – than trying to decipher the code. In the presentation here, we will look at the ZFS system in English.

> **Axiom 1 (Axiom of Extensionality).** Two sets are equal if and only if they have the same members.

There are a couple of comments to be made with regard to Axiom 1. The first is that it is natural to presume that the terms "sets" and "members" must have some explicit definitions given prior to the statement of the axiom. However, it turns out that trying to provide explicit definitions of either can lead to problems in the system, particularly problems having to do with the production of antinomies in the theory. For example, a 'member' of a 'set' might very well itself *be* a 'set' and this opens up the specter of either an infinite regress or a vicious circle in definition. Fraenkel remarked,

> The axiomatization of set-theory renounces a *definition* of the concept of set and of the relation between a set s and its elements. The latter, a dyadic relation (or predicate) is denoted by $\in$; $x \in s$ reads "x is contained in, is an element of, belongs to, the set s" . . . and its negation is $x \notin s$. $\in$ enters as an (undefined) *primitive* relation, *the membership relation* [BERN: 4-5].

It is certainly easy for us to see where a mathematician gets his formalism defense of mathematics being "just a combination of meaningless symbols" from this statement! The non-mathematician wonders how the mathematician can do anything at all with axioms that are loaded up with terms that are undefined. Mathematicians accomplish their work through the aid of a formal 'language'

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[6] J. Dieudonné, "The work of Nicolas Bourbaki," *American Mathematical Monthly*, **77**, 134-145, 1970.

[7] In the branch of information theory that deals with codes and coding, one very important class of code is known as the 'instantaneous code.' The word 'instantaneous' refers to being able to decode the message as it comes in. Sometimes special 'punctuation marks' are required in order to be able to do this. I have never been convinced that the cipher used by the mathematics community isn't missing some key punctuation marks.

that rather looks like a computer programming language. Although I'll try not to burden you too much with examples of this 'language', one example is in order here as a mere illustration. Axiom 1 is written *formally* as

$$\forall x, y\ (\forall z(z \in x \rightarrow z \in y) \rightarrow x = y$$

which at first, second, and third glance doesn't look at all like the English version of the axiom as stated above. Translated more or less literally into English, this hieroglyph reads "for every pair *x*, *y* if for every *z* if '*z* belongs to *x*' implies '*z* belongs to *y*' then *x* is equal to *y*." This formal language is full of run-on 'sentences' like this. *x*, *y*, and *z* in this 'language' are basically 'placeholders' ("variables") into which things can be plugged when one wishes to say something specific about some thing. The 'language' tries to express something that is to be held as having a 'truth value' of "true" regardless of whatever gets plugged in for the variables. More practically speaking, *x* and *y* are what get called the "sets" and *z* is what gets called an "element" or "member" of these sets. The axiom is 'phrased' in this stilted manner in an attempt to be *formally* "precise." In the next section we will look at some criticisms of this formalist approach.

Axiom 2 (Axiom of the Null Set). There exists a set with no members (the empty set).

If one tries to allow the 'set' variable to mean some *thing*, an *objective* interpretation of this axiom immediately plunges us into ontological deep weeds. This is because such an objective interpretation usually comes to the conclusion that the empty set (also called the null set) represents Nothing (*nihil negativum* or "negative nothing"). Thereupon the non-mathematician accuses the mathematician of speaking preposterous nonsense ("How can you say that Nothing exists when Nothing *means* non-existence?"). Naturally this is annoying, so the mathematician pulls out "Chapters 1 and 2 on set theory" and beats us with it until we go away. The literal translation of the formal statement of this axiom reads: "There is *x* such that for every *y* not-*y* is a member of *x*." Personally, I think this statement isn't free of problems either, but what the axiom is intended to enable is to let people make statements like "there are no dodos" *with mathematical precision*, e.g., "The set of living dodos is the empty set." Many binary relations in formal mathematics could not be 'general' without the artifice of the null set. For example, if *C* is the set of all cattle in Wyoming and *D* is the set of living dodos, we could not say $C \cup D = C$ [8] if we were not permitted to use the idea of the empty set. One theorem that comes out of the ZFS system is that the empty set (symbolize it as $\varnothing$) is *ex officio* a member of every other set. It thus plays the

[8] This is read, "The union of *C* and *D* is *C*".

role in 'the algebra of sets' homologous to that of the 'identity element' in group theory. For example, this theorem allows us to *formally express* the intersection of sets *C* and *D* from the example above as $C \cap D = \varnothing$. If we were not allowed to do this then the binary relation symbolized by $\cap$ could not be 'universal' because whether or not we could 'say it' in mathematical language would then depend on 'what' the two sets represented. This axiom therefore has a very *practical* use despite the fact that its historical origin grew out of the criticism that 'Aristotle's logic' had 'existential import.' Mathematics when it is *applied* in science is used in the role of a very, very, very precise language. If we're going to have a language, we'd like to be able to use it to express any idea that could occur to us, such as "the set of cattle-that-are-also-dogs." The *practical* interpretation of the null set is: $\varnothing$ signifies and means *contradiction*. It is a *modal* idea.

> **Axiom 3 (Axiom of Unordered Pairs).** If *x* and *y* are sets then the set of pairs (*x*, *y*) or the set of pairs (*y*, *x*) exists.

This axiom is used to permit us to *construct* new sets from other (given) sets. Earlier in this treatise we introduced the important mathematical idea of the Cartesian Product of two sets, i.e. the set of all ordered pairs of members of *A* and members of *B*, $A \times B$. This operation is essential for being able to define relations and functions, and axiom 3 formally allows us to construct such a set. However, the axiom operates 'at a more abstract level' than the Cartesian product because what it permits is to pair up *sets*, not merely the members *in* each set. Put another way, the axiom does not require us to be able to *enumerate* the members of either set. Suppose $\mathfrak{R}$ represents the set of 'real numbers' and $\mathfrak{I}$ represents the set of 'imaginary numbers' (numbers which when squared are negative real numbers). The axiom permits us to construct the set of 'complex numbers' $(\mathfrak{R},\mathfrak{I})$, which is needed for finding the roots of polynomial equations.

> **Axiom 4 (Axiom of the Sum Set or Union).** If *x* is a set of sets, the union of all its members is a set.

This one might make us blink a bit. Suppose the set *x* is given by $\{A, B\}$. Suppose further that *A* is a set given as $\{a, b, c\}$ and *B* is a set given as $\{d, e, f\}$. The axiom says that $\{a, b, c, d, e, f\}$ is also a set. Axiom 4 is another construction axiom. What it basically does is permit the analytic division of the sets that belong to *x* and then the re-collection of all the 'pieces' in another legitimate set.

> **Axiom 5 (Axiom of Infinity).** There is a set *x* that contains the empty set and that is such that if any *y* belongs to *x* then the union of *y* and $\{y\}$ also belongs to *x*.

This one is a very strange axiom when expressed in its formal language. I think I can guess what question is probably going through your mind right now: "What does that '{y}' symbol mean?" This symbol stands for "the singleton set", which is defined as a set with only one member. The axiom is intended to legitimize the mathematical 'existence' of sets that have infinitely many members by simply declaring that such a set can be constructed through induction. One way that Fraenkel used to describe the operation $y \cup \{y\}$ is the following. The empty set $\varnothing$ is a member of x. Therefore $\varnothing \cup \{\varnothing\} = \{\varnothing, \{\varnothing\}\}$ is a member of x. Therefore $\{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$ is a member of x. Therefore . . . [BERN: 22]. It's a good thing that our symbols 'don't mean anything' or right about now the registrar's office would be swamped with math students looking to change their major and civilization as we know it would be on the brink of collapse. The *cardinality* of a set means "the number of members in the set." The axiom is saying that it is not inconsistent to say there is a set made up of sets having, respectively, cardinality 0 ($\varnothing$), cardinality 1 (the set $\{\varnothing\}$), cardinality 2 (the set $\{\varnothing, \{\varnothing\}\}$), cardinality 3, etc. without limit. Not to put too fine a point on it, but the axiom basically is a definition in disguise for 'counting.' The set x is a "countably infinite set" and its cardinality was represented by Cantor using the symbol aleph nought ($\aleph_0$). This axiom is introduced specifically to permit the mathematical 'existence' of such a set. Without this axiom mathematics could not 'legally' work with the 'set of whole numbers' $\{0, 1, 2, 3, . . . \infty\}$. I will withhold further comment on axiom 5 until later.

> **Axiom 6 (Axiom of Replacement).** This is not actually an axiom but rather a schema for an unlimited bundle of axioms. Even professional mathematicians find it difficult to re-state the formal expression in English. Roughly, the axiom says "any 'reasonable' property that can be stated in formal language can be used to define the set *y* of things having the stated property."

The sticky point in this axiom is how to determine if a property is 'reasonable.' It turns out that axiom (schema) 6 has been a mathematical battleground for the pioneers named earlier. Zermelo originally presented a simpler version, roughly that which I have stated above, known as the "axiom of separation"; but it later turned out that in order to support some of the on-going practices in mathematics, as well as to satisfy certain objections to Zermelo's terminology, either more axioms or a re-work of Zermelo's axiom was deemed necessary [BERN: 11, 22-23]. Both approaches were put forward (and this accounts for some of why we do not have just one universally-embraced set of axioms for set theory). It is instructive to look at a 'two-axiom' version of axiom 6 made up of Zermelo's axiom plus another insisted upon by Fraenkel, Skolem, von Neumann and others:

> **Axiom 6a (Axiom of Separation).** For any set *s* and any predicate *P* which is definite for all members of *s*, there is a set *y* that contains just those members *x* of *s* which satisfy the

> predicate.
>
> **Axiom 6b (Axiom of Substitution).** For every set *s* and every single-valued function *f* of one argument which is defined for the members of *s*, there is a set that contains all *f*(*x*) with $x \in s$.

6a is a 'logician's expression' of axiom 6 and defines the subset of members for which the predicate is true. 6b is a more 'arithmetic' expression and defines the set of members that are the images *f*(*x*) of the members of the set. The ZFS axiom 6 in a sense tries to state both sub-axioms in one formal sentence and to allow multiple functions to be simultaneously expressed, which is one reason its formal expression is difficult to state verbally. Axiom 6 is called an 'axiom schema' because each set of predicates and/or functions defined in effect specifies an axiom unique to that specification. Note that all three axioms stated above specify that the predicate or the function must be 'definite' or 'definable' for each member in the set to which it is applied. A vague predicate, such as "is blessed of God," cannot be a predicate under Axiom 6.

It is noteworthy in regard to the axiom of replacement that it took the place of an older axiom that was implicit in Cantor's theory. This was the 'axiom of abstraction', which basically said "given any property, there is a set whose members are just those entities possessing that property." This simple and seemingly intuitive axiom was the one that allowed the Russell paradox. Axiom schema 6 is a complicated axiom because it puts in conditions and constraints on when the set *y* exists in order to avoid the Russell paradox.

> **Axiom 7 (Axiom of the Power Set).** For any set *x* the set *y* consisting of all the subsets of *x* exists.

The set of all subsets of *x* is called the 'power set' of *x*. Although *y* is defined by a property, namely that its members are subsets of *x*, this case is not covered by axiom 6 because the property is not defined as the range of a function. For example, axiom 6b specified that the function had to be unary (a function of one member of *x*), whereas in axiom 7 the property 'is a subset' is not a unary function. Axiom 7 is another construction axiom since *y* will have more members in it than did *x* if the number of members in *x* is finite.

> **Axiom 8 (Axiom of Choice).** If *a* implies non-empty set *x* is a function defined for every *a* belonging to *s* then there is another function *f*(*a*) for *a* belonging to *s* and *f*(*a*) belongs to *x*.

What this axiom tries to say is we can select one member from every non-empty subset of *s* and use our selections to define a set *x*. The axiom of choice differs from axiom 6 in that the 'choice function' *f* is not given any specified property that defines the 'choices'. The axiom lets us do an arbitrary 'choosing' even though we have no property that would define the choice function. Note that the axiom is an axiom proclaiming the ‘existence’ of choice functions *f* rather than the ‘existence’ of sets. The set *x* is called a ‘choice set.’ Some mathematicians regarded the axiom of choice as ‘counterintuitive’ because it asserts the ‘existence’ of choice functions *f* independently of any property the members of *s* satisfy. Its place in set theory can be seen as the set-theoretic counterpart to Euclid’s fifth in geometry. And, like Euclid’s fifth, it turned out to be possible to deny the axiom of choice and still obtain perfectly self-consistent “non-Cantorian” set theories. This was proved by Paul J. Cohen in 1963. Therefore, just as we can have more than one kind of geometry, today we can have more than one kind of set theory.

> **Axiom 9 (Axiom of Regularity).** Non-empty set *s* implies *x* exists such that *x* belongs to *s* and every *y* belonging to *x* implies *y* does not belong to *s*.

This axiom is meant to prohibit any set *s* from containing itself as a member. It was introduced by von Neumann in order to deal with certain highly technical issues involving unwanted circumstances that were shown to be consistent with the other axioms. One such issue was the possibility, under the system without axiom 9, of the existence of sets that did not contain some of the ‘primary constituents’ (such as the null set) of Zermelo’s set theory [BERN: 23-25].

Like the development of Euclid’s axioms and Aristotle’s ten ‘categories,’ the development of the axioms of set theory was historically an empirical and somewhat trial-and-error process. Commenting on ‘system **Z**’ (the modified Zermelo system, which is more or less like the one just described here), Fraenkel wrote:

> The rather arbitrary character of the processes which are chosen in the axioms of **Z** as the basis of the theory is justified by the historical development of set-theory rather than by logical arguments. The far-reaching aim of proving the consistency of **Z**, which would exclude contradictions of types as yet unknown, is not likely to be attained in the present stage, and in a well-defined sense cannot be attained at all, in accordance with Gödel’s incompletability theorem [BERN: 31].

He goes on to comment upon various weaknesses and disadvantages of ‘system **Z**’ and how additions and modifications to it made by von Neumann, Bernays and Gödel have addressed the specific issues he brings up. Unaddressed, of course, is the ‘existence’ of non-Cantorian set theories, which had not been proved to exist at the time Fraenkel wrote his ‘historical introduction’ to Bernays’ book (1958).

§ 2. Mathematics as an Empirical Science

We have just seen that the efforts of mathematicians to recover the apodictic certainty of the axioms as ‘self-evident truths’ during the ‘crisis in the foundations’ failed in its task. The last bastion of rationalism finally fell. All this took place during and after the rise of positivism in the nineteenth century and, in one of history’s ironies, while the ground was giving way beneath the rationalist position, positivism was declaring that mathematics was to be excluded from ‘the sciences’ *because it was not empirical*. We have just seen that the development of axiomatic set theory (or, if one prefers, *theories*) was carried out with at least one eye fixed on justifying the practices of *all* branches of mathematics. In the end what was achieved was not ‘certainty’ for all of them but rather for *none* of them. But does this not make the practice of mathematics ‘empirical’?

§ 2.1 Lakatos’ Criticisms of Formalism

Imre Lakatos thought so. Lakatos was an outspoken critic of formalism in mathematics. His main objection seems to have been that while formalism claims to be doing one thing, the actual practice of mathematics does something quite otherwise. It is not Lakatos’ position that mathematics ought to actually practice formalism. He recognizes quite well that the original goal which inspired the development of formalism proved to be unreachable. Rather, his position is that mathematics really is an empirical science (even though all its objects are supersensible), and that it ought to admit that fact and stop pretending to be something else. Lakatos is not known for being gentle in remonstration:

> Formalism disconnects the history of mathematics from the philosophy of mathematics, since, according to the formalist concept of mathematics, there is no history of mathematics proper. Any formalist would basically agree with Russell’s ‘romantically’ put but seriously meant remark, according to which Boole’s *Law of Thought* (1854) was ‘the first book ever written on mathematics.’ Formalism denies the status of mathematics to most of what has been commonly understood to be mathematics, and can say nothing about its growth. None of the ‘creative’ periods and hardly any of the ‘critical’ periods of mathematical theories would be admitted into the formalist heaven, where mathematical theories dwell like the seraphim, purged of all the impurities of earthly uncertainty. Formalists, though, usually leave open a small back door for fallen angels: if it turns out that for some ‘mixtures of mathematics and something else’ we can find formal systems ‘which include them in a certain sense’, then they too may be admitted. On those terms Newton had to wait four centuries until Peano, Russell, and Quine helped him into heaven by formalizing the Calculus. Dirac is more fortunate: Schwartz saved his soul during his lifetime. Perhaps we should mention here the paradoxical plight of the metamathematician: by formalist, or even by deductivist, standards, he is not an honest mathematician . . .
>
> Under the present dominance of formalism, one is tempted to paraphrase Kant: the history of mathematics, lacking the guidance of philosophy, has become *blind*, while the philosophy of mathematics, turning its back on the most intriguing phenomena in the history of mathematics, has become *empty* [LAKA1: 1-2].

Formalism is the bastion of logical positivism, that bankrupt child of the failed attempt to defend rationalism and preserve for mathematics the special status which it was believed to own for centuries. Lakatos makes it plain that he rejects logical positivism as well as rationalism. However, this does not make him an 'empiricist.' Some call him a 'neo-Popperian' since his views are an outgrowth of, yet differ from, those of Karl Raimund Popper (1902-1994). It might do to call Lakatos a 'methodologist' inasmuch as most of his work was concerned with proper methodology in mathematics and in science, and with the role of *refutation* (as opposed to 'confirmation') in both mathematics and science. His philosophy shares some factors in common with James-like pragmatism, but it is probably better to describe these factors as a unique brand of pragmatical skepticism, sometimes dubbed 'dubitability.' One theme that recurs in his writing is 'justification' – that is: upon what basis is it justified to keep using a theory or method and when should a theory or method be regarded as 'refuted'? Put another way: when is science (or mathematics) 'justified' in continuing to use a theory or method in the teeth of evidence that implies this theory or method is 'wrong'?

In Lakatos' view, formalism is long past that point. The actual practice of mathematics, he tells us, is quasi-empirical and mathematicians should admit it.

> The infinite regress in proofs and definitions in mathematics cannot be stopped by a Euclidean logic. Logic may *explain* mathematics but it cannot *prove* it. It leads to a sophisticated speculation which is anything but trivially true. The domain of triviality is limited to the uninteresting decidable kernel of arithmetic and of logic – but even this trivial kernel might some time be overthrown by some detrivializing skeptic criticism.
>
> The logical theory of mathematics is an exciting, sophisticated speculation like any scientific theory. It is an empiricist theory and thus if not shown to be false, will remain conjectural forever . . . Hilbert's theory was based on the idea of formal axiomatics. He claimed . . . that metamathematics, this new branch of mathematics set up to prove the consistency and completeness of formal systems, will be a particular brand of Euclidean theory: a 'finitary' theory, with trivially true axioms, containing only perfectly well known terms, and with trivially safe inferences . . .
>
> Gödel's *second theorem* was a decisive blow to this hope for a Euclidean metamathematics. The infinite regress in proofs cannot peter out in a 'finitarily' trivial meta-theory: consistency proofs have to contain enough sophistication to render the consistency of the theory in which they are carried out dubitable, and therefore they are bound to be fallible . . .
>
> Gödel's *first theorem* showed a second way in which a formal theory could misfire: if it has a model at all, it has more models than intended. In a consistent formal theory we can prove those and only those propositions which are true in all models; so we cannot formally prove those propositions which are, though true in the intended model, false in an unintended one. Gödel's first theorem showed that the selectivity of formal systems embracing arithmetic is irreparably bad, for in no consistent formalization of arithmetic can we 'tune out' unintended models which are essentially different from the intended one. Consequently in any consistent formalization there will be formally unprovable arithmetical truths [LAKA3: 19-21].

Like Popper, Lakatos held that 'inductivism' is not, as Bacon claimed, the actual method of science. It was Hume who first sharply pointed out that inductive inference is not a proof of anything since the only justification for trusting induction is that it has worked in the past – a position that uses induction to justify induction. When induction leads to an infinite regress that very fact shows there was no *foundation* for inductive inference. The Great Skeptic wrote:

> Our foregoing method of reasoning will easily convince us, that there can be no *demonstrative* argument to prove *that those instances, of which we have had no experience, resemble those of which we have had experience.* We can at least conceive a change in the course of nature; which sufficiently proves that such a change is not absolutely impossible. To form a clear idea of any thing is an undeniable argument for its possibility, and is alone a refutation of any pretended demonstration against it . . .
>
> The only connexion or relation of objects which can lead us beyond the immediate impressions of our memory and senses is that of cause and effect; and that because 'tis the only one on which we can found a just inference from one object to another. The idea of cause and effect is deriv'd from *experience*, which informs us that such particular objects, in all past instances, have been constantly conjoin'd with each other: And as an object similar to one of these is suppos'd to be immediately present in its impression, we thence presume on the existence of one similar to its usual attendant. . . The past production implies a power: The power implies a new production: And the new production is what we infer from the power and the past production . . .
>
> It shall therefore be allow'd for a moment, that the production of one object by another in any one instance implies a power; and that this power is connected with its effect. But it having been already prov'd that the power lies not in the sensible qualities of the cause; and there being nothing but the sensible qualities present to us; I ask, why in other instances you presume that the same power still exists, merely upon the appearance of these qualities? Your appeal to past experiences decides nothing in the present case; and at the utmost can only prove, that that very object which produc'd any other was at that very instant endow'd with such a power; but can never prove that the same power must continue in the same object or collection of sensible qualities; much less, that a like power is always conjoin'd with like sensible qualities. Shou'd it be said, that we have experience, that the same power continues united with the same object, and that like objects are endow'd with like powers, I wou'd renew my question, *why from this experience we form any conclusion beyond those past instances, of which we have had experience*. If you answer this question in the same manner as the preceding, your answer gives still occasion to a new question of the same kind, even *in infinitum*; which clearly proves that the foregoing reasoning had no just foundation [HUME1: 89-91].

The same argument applies as much to mathematical objects as to physical objects, as witnessed by the 'parallel' lines[9] arguments from earlier (where in one case the lines never intersect, but in the other they intersect both on the left-hand and on the right-hand sides). Merely because we can not see a reason why a regular process (e.g. 1+1=2; 2+1=3; 3+1=4 etc.) should ever change to produce an irregular one (e.g. $\infty+1=\infty$) is no guarantee that such a change cannot happen when the induction must proceed beyond the horizon of any possible experience. Arguments of induction *ad infinitum* favor only the skeptic.

Lakatos uses this to attack dogmatic adherence to the claim that inductivism is a valid method of science (including mathematics).

> The classical skeptical argument is based on the infinite regress. One can try to pin down the meaning of a term either by defining it in other terms – this leads to infinite regress – or by defining it in 'perfectly well-known terms'. But are the four terms of the expression 'perfectly well-known terms' really all perfectly well-known terms? One sees that the abyss of infinite regress opens up again. How then could mathematical philosophy still claim that in mathematics we have or we should have exact concepts? How does it hope to avoid the skeptics' strictures? How can it claim that it has offered foundations of mathematics – logicist, metamathematical or intuitionist? But even

[9] More accurately, the 'two straight lines cut by a third line making right angles to each" case.

> allowing for 'exact' concepts, how can we prove that a proposition is true? How can we avoid the infinite regress in definitions? Meaning and truth can only be transferred, but not established. But if so, how can we *know*? [LAKA3: 3].

Lakatos' conclusion, of course, is that this is not possible, and certainly not possible through induction.

> It is intriguing how mathematical logicians who are so squeamish about rigor, and who set out to achieve absolute certainty, can slip into the morass of inductivism. For instance, A. Fraenkel, the distinguished logician, dares to state that some axioms of logic receive their 'full weight' from 'the evidence of their consequences' [LAKA3: 17].

Axioms, however, are *principles* if they are supposed to ground the truth and certainty of the science that employs them. As Berkeley said (in the quote at the head of this chapter) in criticism of the Calculus, "men prove their conclusions by their principles, not their principles by the conclusions." Lakatos asks the mathematicians,

> Why not honestly admit mathematical fallibility, and try to defend the dignity of *fallible* knowledge from cynical skepticism, rather than delude ourselves that we shall be able to mend invisibly the latest tear in the fabric of our 'ultimate' intuitions? [LAKA3: 23].

§ 2.2 Confessions of the Formalists

The battle to save the foundations is over. The defenders of absolute certainty have long conceded defeat and left the field. Truth now in mathematics, as it has long been in the physical sciences, is contextual. Systems are not certain, they are "relatively consistent" – a term that means 'consistent within a framework.' J.B. Rosser, a follower of Russell, writes,

> We wish to make one point clear about our use of the word 'axiom'. Originally the word was used by Euclid to mean a 'self-evident truth'. This use of the word 'axiom' has long been completely obsolete in mathematical circles. For us, the axioms are a set of arbitrarily chosen statements which, together with the rule of *modus ponens*, suffice to derive all the statements which we wish to derive.[10]

Rosser is not an isolated voice. W.V.O. Quine, widely acknowledged as a leading figure in mathematical logic, wrote,

> We may more reasonably view set theory, and mathematics generally, in much the way in which we view the theoretical portions of the natural sciences themselves: as comprising truths or hypotheses which are to be vindicated less by the pure light of reason than by the indirect systematic contribution which they make to the organizing of empirical data in the natural sciences.[11]

Lakatos, naturally, loved to quote leading mathematicians who were willing to publicly admit that

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[10] J.B. Rosser, *Logic for Mathematicians*, NY: McGraw-Hill, 1953.
[11] W.V. Quine, "The philosophical bearing of modern logic," in R. Klibansky (ed.): *Philosophy in the Mid-Century*, vol. I, pp. 3-4, Firenze: La Nuova Italia.

the old rationalist dream was dead.

> Russell was probably the first modern logician to claim that the evidence for mathematics and logic may be 'inductive'. He, who in 1901 had claimed that the 'edifice of mathematical truths stands unshakable and inexpugnable to all the weapons of doubting cynicism', in 1924 thought that logic (and mathematics) is exactly like Maxwell's equations of electrodynamics: both 'are believed because of the observed truth of certain of their logical consequences'.
> Fraenkel claimed in 1927 that 'the intuitive or logical self-evidence of the principles chosen as axioms [of set theory] naturally plays a certain but not decisive role; some axioms receive their full weight rather from the self-evidence of the consequences which could not be derived without them'. And he compared the situation of set theory in 1927 with the situation of the infinitesimal calculus of the eighteenth century . . .
> Carnap, who at the 1930 conference in Königsberg still thought that 'any uncertainty in the foundations of the "most certain of all the sciences" is extremely disconcerting', [had decided by] 1958 that there is an analogy – if only a distant one – between physics and mathematics: 'the impossibility of absolute certainty' [LAKA3: 25].

Confession, it has been said, is good for the soul. That mathematics is 'fallible' insofar as that word is taken to mean 'not absolutely certain' does not in the least imply that the mathematician is careless or cavalier in establishing his theorems as perfectly true *within the framework of the axiom system in which he works*. A contextual truth this may be, but the word 'truth' has no objectively valid meaning other than congruence of the object with its concept. The mathematician's theorems are also contextually certain – that is, the theorem is apodictic *given* the acceptance of the axioms and the rules of inference that underlie its proof. None of the physical sciences has claim to a greater degree of contextual truth and certainty than mathematics. Indeed, all the 'exact' sciences in their theoretical departments depend fundamentally on the veracity of mathematics. The scientist in such a discipline must of course take the responsibility for determining if the *defined* objects of mathematics are good models for the *physical* objects he or she studies. But having made the determination, he or she cannot blame mathematics if it should turn out that Nature is not quite so neatly describable as is the mathematical object.

At bottom it is not really necessary that all mathematical knowledge be *absolutely* certain knowledge, and this is good because we have seen in this treatise that knowledge absolute, unlimited and certain is beyond the capacity of human reason. H.B. Curry wrote,

> The search for absolute certainty was evidently a principal motivation for both Brouwer and Hilbert. But does mathematics need absolute certainty for its justification? In particular, why do we need to be sure that a theory is consistent, or that it can be derived by an absolutely certain intuition of pure time, before we use it? In no other science do we make such demands. In physics all theorems are hypothetical; we adopt a theory so long as it makes useful predictions and modify or discard it as soon as it does not. This is what has happened to the mathematical theories of the past, where the discovery of contradictions has led to modifications of the mathematical doctrines accepted up to the time of that discovery. Why should we not do the same in the future? Using formalistic conceptions to explain what a theory is, we accept a theory as long as it is useful, satisfies such conditions of naturalness and simplicity as are reasonable at that time, and is not known to lead us into error. We must keep our theories under surveillance to see that these conditions are fulfilled and

> to get all the presumptive evidence of adequacy that we can. The Gödel theorem suggests that this is all we can do; an empirical philosophy of science suggests it is all we should do.[12]

Earlier in this chapter it was pointed out that the development of the axioms of axiomatic set theory was carried out under the umbrella of an agenda that had not so much to do with finding certainty in the axioms as it did with putting an axiomatic basis to the existing corpus of mathematics. Much earlier in this treatise it was pointed out that Russell and Whitehead had followed a similar agenda in their *Principia*. Unpleasant or not, embarrassing or not, it is not a mean-spirited criticism of mathematics to point out this fact; it is tough love. The Critical epistemology in any case teaches us that the search for absolute certainty during the ‘crisis in the foundations’ was doomed to fail. The ‘foundations’ research did in any case succeed in ‘rescuing’ a great deal of mathematics from the fires of disillusionment, and this, too, is a good thing and of so large a benefit as to go begging of description. Suppose that Feynman’s conjecture that “space is not really continuous” were to turn out to some day be a better description of affairs at the quantum and sub-nuclear level? This would not detract from or diminish the usefulness of Newton’s Calculus. The plain and simple fact is nearly all of engineering relies on the Calculus, and especially on its ease of application, for the foundation of engineering practice. Were some puritanical Authority to someday decree that because of non-continuity the Calculus could no longer be taught, in swift order we would have craftsmen but we would have no engineers.

Of course, it is also true, now that the Hilbert program is a known failure and efforts to defend rationalism in mathematics have ceased, there seems to be no reason to continue to blindly adhere to the puritans of logical positivism and cloak the very description of mathematics in opaque run-on symbolic sentences that hide from the larger community of science the full benefits of mathematical insight. Mathematics needs a better grammar. The present way in which mathematics is *taught* is, as Poincaré dryly observed so long ago, “contrary to any healthy psychology.” Indeed, this anachronism is part and parcel of a larger issue:

> History teaching still turns a blind eye towards science, the most exciting and noble of human ventures, and science and mathematics teaching is disfigured by the customary authoritarian presentation. Thus presented, knowledge appears in the form of infallible systems hinging on conceptual frameworks not subject to discussion. The problem-situational background is never stated and is sometimes already difficult to trace. Scientific *education* – atomized according to separate techniques – has degenerated into scientific *training*. No wonder that it dismays critical minds.
>
> Now history-cum-philosophy of science has to show up science in history on the one hand and history in science on the other, and by doing this to exert an important *therapeutic influence* on both. If we fail to achieve this, we shall soon face a situation where an abundance of separate courses in history-cum-philosophy of science will turn the present two uncultures (to paraphrase Sir Charles Snow) into three, instead of helping to debarbarize both [LAKA3: 255].

[12] H.B. Curry, *Foundations of Mathematical Logic*, NY: McGraw-Hill, 1963.

§ 3. Acroamatic Mathematics

One reason the axioms in §1 end up being expressed in such a complicated way is that a too-simple statement of an axiom can open the door to antinomies. Because 'set' and 'member of a set' are not defined, they can be 'anything.' Therefore what are usually called 'unintended models' are more likely the more the set of axioms does not specifically prohibit particular 'types' of 'sets.' One famous case of an 'unintended model' is "the set *s* which includes every possible set except any set that includes itself." Does *s* include itself? Obviously it cannot because *s* is not allowed to contain any set that includes itself. Does *s* not include itself? Obviously it cannot exclude itself because *s* includes every set that does not include itself. So *s* both *must* and *must not* include itself. The specification of *s* is self-contradictory. This is the Russell paradox.

But another reason is because mathematics really is "about something" and what this "something" is depends upon whatever the person using or 'doing' the mathematics needs it to be. As we have already seen more than once, a mathematician "feels that he is working with something real." Every 'something' *is real in some contexts and is unreal in others*. I disagree with Dieudonné's guess that the mathematician's feeling "is probably an illusion." One need not regard Plato's 'World of Truth' as a "World of that-which-is" in order to think that an *idea* is 'real.' It is only when non-Critical ontological presuppositions get tacked on to object concepts that the 'reality' of the objects of mathematics gets called into question. Kant wrote that mathematics is knowledge through the *construction* of concepts. The concept of a supersensible object is called an idea. To carry out its task the *systematic doctrine* of mathematics must *in principle* be capable of constructing *any idea whatsoever* that is possible for a human being to think. This is also what the axioms of mathematics try to accomplish, although the degree of success they have in this endeavor is clearly open to debate.

The mathematical knowledge we have today, obtained through the present doctrine of mathematics, is contingent knowledge. This is because mathematics' theorems are proved within the context of the system of axioms used and this system is itself contingent. This is demonstrated by the historical character of its development. However, the statement "absolutely all knowledge is contingent" is false due to overgeneralization. There are indeed some 'things' we can know and can know with absolute certainty. But to appreciate this, we must examine the *Realerklärung* of the idea "absolute certainty." Only then will we be able to appreciate whether or not 'certainty' is achievable for *axioms* of mathematics, what the limitations on the scope of this certainty might be, what the *Existenz* of this certainty 'looks like,' and how such certainty can be established *a* *priori*. We begin this examination by looking at the reason present-day mathematics is held to be ‘lacking certainty.’

§ 3.1 What Gödel’s Theorems Say

Hilbert’s program for re-establishing mathematical certainty was based on what he called a system of ‘metamathematics.’ Gödel’s theorems spelled doom for this program. But what was this program? and how was it done in by Gödel?

Hilbert’s metamathematics work was prompted by his alarm at a call from Luitzen Brouwer, a Dutch mathematician and neo-Kantian, in 1912 for a complete restructuring of mathematics from the ground floor up. In Hilbert’s view, to follow Brouwer’s radical proposal would mean the loss of the greatest parts of mathematics and the dismemberment of what was left. Hilbert’s program was an attempt to guarantee consistency in the sense that no contradictions could be derived within the structure of the formal mathematical system. (Recall that antinomies arising out of Cantor’s set theory constituted the main reason for regarding the ‘foundations’ as being in trouble).

Hilbert took the Kant-like position that at least the ‘finite’ part of mathematics was a solid foundation (and such also was Brouwer’s view). To this he added a second premise, namely that a securely-founded theory about formal sentences could validate all the activities carried out in doing mathematics (a position that Brouwer did not accept). Hilbert’s program had three steps: 1) introduce a formal language and formal rules of inference sufficient to represent every correct proof through a formal derivation starting from axioms and advancing in machine-like steps; 2) develop a theory of a sort of ‘symbolic calculus’ that defined valid symbolic manipulations that could be applied to formulae; and 3) prove by purely finite arguments that no contradictions could be derived within this system. For step (1) he largely adopted the formal language and rules of inference already put together by Russell and Whitehead. Step (2) is ‘metamathematics’ properly so-called; during the 1920s this part of the system was developed through the work of a number of ‘metamathematicians.’

The ‘symbolic calculus’ is interpreted as representing a deductive system in which a relatively few basic formulae are taken to be the ‘axioms’ of the system and the rules of symbolic manipulation are regarded as the logical rules of inference of the system. It is important to note that the symbolic calculus is to be developed independently of *any* interpretation of it. Put another way, the symbols in this symbolic calculus are “extra-mathematical.” Two ‘concepts’, called ‘sentences’ and ‘consequences’, are taken as ‘primitives’ within the deductive system and are defined in terms of some set of axioms. Within this set there are some regarded as ‘basic’ axioms (those of general set theory) and others regarded as 'special' axioms (also written in the formal language) that apply to specific deductive disciplines (e.g. 'arithmetic').[1]

The first thing we should note here is that Hilbert and the 'metamathematicians' were not in the business of coming up with 'fundamental axioms' of set theory; that task they left to others such as Zermelo. The second thing to note is that this formal system necessarily had to take some of its parts, namely those having to do with rules of inference and the representations of formulae, from set theory itself. Hilbert et al. were not trying to invent a new set theory; they were trying to make its deductions so airtight that even Brouwer would have to concede its correctness.

Now, what was it that Gödel did? Gödel applied Hilbert's method to the mathematical system known as *arithmetic*. This means what you think it means: the system of whole numbers and addition and multiplication. For his 'deductive system' Gödel took the portion of Russell's and Whitehead's *Principia* in which is found the theorems of whole-number arithmetic. Following the methods of metamathematics, Gödel *proved* two theorems, both of which came as a profound shock to everyone. The first is his 'Unprovability' Theorem. It says essentially that there exist possible 'propositions' within the system of arithmetic that cannot be proved within this system. The second theorem states that the 'consistency' of the arithmetic system likewise cannot be proved within this system. The second theorem was an even greater shock to everyone than was the first, especially since the results applied to the oldest – and in many ways the simplest – of all mathematical systems: basic whole-number arithmetic. Hilbert's entire program was based on the premise that 'correct' theorems could always be proved within the deductive system and that the consistency of the deductive system could likewise always be self-demonstrated by the 'formal' approach. Gödel's theorems, to put it simply, tell us: This is not so.

Although Gödel made the system of arithmetic the focus of his paper, the proofs of these two 'incompleteness' theorems called upon surprisingly few characteristics of this formal system. There are other mathematical systems that also meet the conditions of the theorem and therefore contain unprovable propositions and are incapable of proving their own consistency. Gödel called the first two of these conditions 'assumptions 1 and 2', which simply state that the class of axioms and rules of inference can be formally stated in a particular way and are definable in the system under consideration. The technical term for the base condition is that formal expressions are 'recursively definable' in the 'meta-language' of metamathematics, and this condition simply means that propositions can be stated as consequences of previous propositions given the axioms

[1] see A. Tarski, "On some fundamental concepts of metamathematics," in *Logic, Semantics, Meta-Mathematics*, 2nd ed., John Corcoran (ed.), J.H. Woodger (tr.), Indianapolis, IN: Hackett Publishing Co., 1983, pp. 30-37.

and rules of inference. The third condition is the property of being ‘ω-consistent’, which was a technical term he introduced that expresses a particular (provable) condition of ‘consistency’ met by the system. In a system that is not ‘ω-consistent’ a proposition *and its contradiction* can be ‘proved’ to be true. Every mathematical system that is ‘consistent’ is also ω-consistent, but not every ω-consistent system is necessarily consistent in the general sense. After proving the first of the two ‘incompleteness’ theorems, Gödel commented,

> Hence in every system that satisfies assumptions 1 and 2 and is ω-consistent, undecidable propositions exist . . . and so too in every extension of such a system made by adding a recursively definable ω-consistent class of axioms. As can be easily confirmed, the systems which satisfy assumptions 1 and 2 include the Zermelo-Fraenkel and the von Neumann axiom systems of set theory, and also the axiom system of number theory which consists of the Peano axioms, the operation of recursive definition . . . and the logical rules. Assumption 1 is in general satisfied by every system whose rules of inference are the usual ones and whose axioms . . . are derived by substitution from a finite number of schemata.[2]

Thus, it is not only arithmetic but the entire system of axiomatic set theory and the ‘logicism program’ of repairing the foundations that falls victim to the Gödel ‘incompleteness theorems.’

§ 3.2 The Constructionist Proposal

The Hilbert plan, in effect, identifies ‘mathematics’ with the formulae that are used to *express* mathematics. However, to execute this plan requires some set of axioms and some basic set of rules of inference as a starting point. Hilbert’s prescription does not say where these are to come from, and to obtain them mathematics used the material it already had on hand. But, as we have seen, these axioms (and the rules of inference contained in *Principia Mathematica*) were proposed empirically, as presuppositions mathematicians were already making in practice, rather than on the basis of any epistemological analysis of their ‘self-evidential character.’ If these ‘foundations’ had worked out under Hilbert’s program, then of course the rationalist bastion would have survived. But unfortunately for rationalism, they did not. Nor can this problem be attributed to our simply not being currently in possession of some ‘missing axiom’ which, if we had it, would solve the problem. Gödel’s proof showed that the problem cannot be solved by adding more axioms to the ‘baseline’ set up by, for example, the ZFS system.

Does this mean the ‘heretical’ approach proposed by L.E.J. Brouwer (in 1908) known as the ‘constructivist school’ is by default the ‘correct’ way to proceed in mathematics? On the face of it, the answer here appears to be ‘no’ as well.

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[2] Kurt Gödel, “On formally undecidable propositions of *Principia Mathematica* and related systems,” translated and reprinted as *On Formally Undecidable Propositions of Principia Mathematica and Related Systems*, B. Meltzer (tr.) and R.B. Braithwaite, NY: Dover Publications, 1992.

> Brouwer's position was that the natural numbers are given to us by a fundamental intuition, which is the starting point for all mathematics. He demanded that all mathematics should be based *constructively* on the natural numbers. That is to say, mathematical objects may not be considered meaningful, may not be said to exist, unless they are given by a construction, in finitely many steps, starting from the natural numbers. It is not sufficient to show that the assumption of nonexistence would lead to a contradiction.
>
> For the constructivists, many of the standard proofs in classical mathematics are invalid. In some cases they are able to supply a constructive proof. But in other cases they show that a constructive proof is impossible: theorems which are considered to be well-established in classical mathematics are actually declared to be false for constructivist mathematics [DAVI: 333-334].

Brouwer regarded Hilbert's program as misconceived at the outset because it identified mathematics with the formulas used to represent mathematics. However, there are two factors working against Brouwer that are pertinent here. First, Gödel's method did employ 'the natural numbers' in its method, albeit in a novel way sometimes called "Gödel numbers." This method has sometimes been described as the homologue in metamathematics to Descartes' invention of the Cartesian coordinate system in analytic geometry. It appears that Gödel's theorems apply with the same force to the sort of mathematical systems sanctioned by Brouwer.

Second, and more importantly, Brouwer's approach presupposes that the 'natural numbers' are 'given intuitively.' Is this so? The body of *psychological* evidence finds against this. We all know, of course, that children have to be *taught* their numbers and basic arithmetic. But is the ability to teach them founded upon an innate 'intuition' for numbers? Piaget studied this question and the answer is 'no':

> All the preceding experiments have shown that at a first level (usually at about the age 4½ - 5), the child evaluates discontinuous quantities as if they were continuous, i.e., extended, quantities. His quantitative judgments are thus based only on the general shape of the set and on global relationships such as 'more or less long', 'more or less wide', etc. . . . That the child should begin by considering non-analyzed wholes, without feeling the need to decompose them as long as experience does not compel him to do so, is perfectly consistent with what we know of the psychology of thought at this stage.
>
> Hence the child who is asked to pick out 'as many' counters as there are in a given set is in no way equipped by his intellectual structure to consider the set as being a union of units, i.e. 1+1+1 . . . etc., which would imply that he already possessed the notion of whole number. For the child, therefore, 'as many' merely means a set that is similar to the model with respect to its overall qualities. But if he does not feel the need for decomposition, does this mean that he is incapable of it? To our mind, the preceding reactions have given a decisive answer to this question.
>
> The only principle of synthesis at the disposal of the child at the first level is the intuitive global perception of the general configuration itself, in the absence of 'operations' which would enable him to put together again the various parts of the perceptual intuition if it were broken up. That is why, as soon as there is an alteration in the data, children of the first stage apparently base their evaluations on one criterion only: length of the rows, width of the figures, density, etc.
>
> But might it not be said that in the very action of copying there is co-ordination of the global qualities, and therefore at least a suggestion of decomposition, since the model is roughly reproduced? There precisely is the point: the copy is only roughly correct. Closed figures depending on a given number of elements are well reproduced, because they involve a 'good' configuration, but collections, rows, open figures, and even closed figures with an arbitrary number of elements,

> are not correctly copied. Linear series, in particular, are evaluated by their total length only, irrespective of their density.
>
> Although the method of global comparison allows of a rough comparison between two sets of the same form, covering the same area, and having approximately the same density, it becomes inadequate as soon as these properties differ substantially. Two sets which the child has stated to be identical by global comparison are no longer considered equivalent when the elements of one set are spaced out. He cannot understand that when there is a change in the shape, and therefore in the distribution of the parts, something remains invariant, namely, the number of elements. The reason is that he has not yet acquired the notion of number, but only of perceptual wholes. Hence, in our view, the notion of conservation of a set is lacking because the elementary relationships inherent in the global perceptions are merely juxtaposed instead of being coordinated [PIAG10: 86-87].

The proposition that we have an innate 'intuition' for natural numbers is, therefore, demonstrably false. When we discussed the transcendental schemata we saw that Kant called the transcendental schema of Quantity 'number.' However, as we discussed at that time, 'number' used here does not refer to the 'numbers' of counting and arithmetic, e.g. 0, 1, 2, etc. It referred to a *process* of successive aggregation of homogeneous parts which has three *modi*, corresponding to the categories of unity, plurality, and totality, respectively. It is erroneous to equate 'number' as transcendental schema to 'number' in the sense of the 'natural' numbers. Brouwer's program also miscarries at the first step.

§ 3.3 Certainty, Transcendental Acroams and Set Theory

Mathematics is "knowledge through the construction of concepts," but philosophy is "knowledge through concepts." Specifically, it is knowledge through transcendental concepts (objectively valid ideas) by which we understand what is necessary for the possibility of experience as human beings are said to 'have' it. Metaphysics analyzes the *form* of all our knowledge. Furthermore, empirical concepts that do not overrun the fundamental condition of being a representation of an object possible in an *actual* sensuous experience are concepts that are not transcendent.

When joined to the first principles of Critical epistemology, such concepts provide the basis of an objectively valid *Realdefinition* of 'certainty.'

> Holding-to-be-true is in general of two kinds, *certain* and *uncertain*. Certain holding-to-be-true, or *certainty*, is combined with consciousness of necessity, while the uncertain, or *uncertainty*, is combined with consciousness of contingency or the possibility of the opposite. The latter is again either *subjectively as well as objectively* insufficient, or *objectively insufficient* but *subjectively sufficient*. The *former* is called *opinion, the latter* must be called *belief* [KANT8a: 570-571 (9: 66)].

In terms of the form of logical function in determinant judgments, holding-to-be-true is Quality, i.e. the judgment is affirmative, negative, or infinite (negative-regarded-as-affirmative, e.g. 'Hans is not-French'). Thus the quality of holding-to-be-true also has a *degree*, which is what we refer to when we speak of 'absolute certainty.' But the term 'absolute' implies a kind of maximum, i.e.

a holding-to-be-true compared to which no other judgment is deemed to have any greater degree. In the Critical epistemology the standard for absolute certainty is the holding-to-be-true that each of us accords the real *Dasein* of one's *I* of transcendental apperception. As human beings, each of us is more certain of nothing compared to the certainty in which we hold our own *Dasein*. That which follows with 'unqualified' necessity from the holding-to-be-true of our own *Dasein* follows apodictically with *absolute* certainty. Of course, there are not many objects of judgment that meet this standard. But the transcendental first principles, as concepts, *do* fit this character owing to their unique status as concepts necessary for the possibility of experience *in the consciousness of* the transcendental Subject.

We call these transcendental first principles the **acroams** of Critical epistemology. If, then, we wish to establish a doctrine of formal mathematics upon axioms we can hold-to-be certain, these axioms must be *constructed* on the basis of *consequences* deduced from the transcendental acroams, for only in that way can they be linked to our standard of absolute certainty in epistemology. The deduction of such a set of axioms would then constitute *an applied metaphysic of mathematical doctrine*. The issue of 'proving the truth' of such axioms then does not arise in the science of *mathematics* because this establishment of axiomatic truth properly belongs to the *metaphysic* of mathematics, this being none other than its Critical applied metaphysic. Critical metaphysics proper has acroams but no axioms; mathematics has axioms but no acroams. The *possibility* of axioms in mathematics rests on the *acroam* of Axioms of Intuition (which is not itself an axiom but rather is a first principle for the construction of axioms).

Regarding axioms, Kant tells us

> These are synthetic *a priori* first principles so far as they are immediately certain. Now one concept cannot be synthetically yet immediately combined with another because for us to be able to go beyond a concept, a third mediating cognition is necessary. Now since philosophy is merely the knowledge of reason according to concepts, no first principle is to be encountered in it that deserves the name of an axiom. Mathematics, on the contrary, is capable of axioms because by means of the construction of concepts in an intuition of the object it can connect the predicates of the latter *a priori* and immediately, e.g. that three points always lie in a plane. A synthetic first principle, on the contrary, can never be immediately certain merely from concepts, e.g. the proposition that everything that happens has its cause, because I must always look around for a third part, namely the condition of time-determination in an experience, and could never directly know such a principle immediately from concepts alone. Discursive first principles are therefore something entirely different from intuitive ones, i.e. axioms [KANT1a: 640 (B: 760-761)].

Kant's distinction here is that mathematical axioms define objects (mathematical objects), and to do so requires construction of the concept of the object by means of a synthesis in sensibility. Mathematics can and does make abstraction from empirical intuition, i.e. it removes reference to the sensational matter of the intuition and keeps only its form of sensible representation. Because mathematics can (and always does) *define* its objects, the cognitions of these objects are *made* concepts, and therefore we are always immediately certain so far as knowing the mathematical *object* is concerned. Knowing, however, that two different defined objects are defined in such a manner that these definitions are reciprocally consistent is another matter, and this is why mathematics is a science and not merely a poetic exercise.

In this limited sense, Riemann geometry is as ‘certain’ as Euclidean geometry, and ZFS set theory is as ‘certain’ as non-Cantorian set theory. However, this certainty is of limited degree and is not ‘absolute’ because the objects of these topics are not defined in connection with the possibility of experience but merely in the context of a particular ‘type’ of experience called ‘mathematical experience.’ Regardless of whether or not one is willing to tag a mathematical object with a specific ontological meaning in some context of sensuous Nature, the concept of a mathematical object – for instance, the concept of a ‘set’ – nonetheless makes reference to an object and has some kind of definition (even if this definition is merely *operational* and is not *physically ontological*). If it did not it would be impossible for the mathematician to ‘feel’ that he or she is working with ‘something real.’ That mathematics is ‘intuitive’ does not mean that it is undisciplined speculation or mere daydreaming; it means the mathematician ‘knows what he’s talking about.’ Poincaré has shared an anecdote with us that might help to illustrate this point. His anecdote is actually a string of related anecdotes, so we will look at only the first few here. The rest of the story, for those who are curious, can be found in the citation:

> It is time to penetrate further, and to see what happens in the very soul of the mathematician. For this purpose I think I cannot do better than to recount my personal recollections. Only I am going to confine myself to relating how I wrote my first treatise on Fuchsian functions . . . What will be interesting for the psychologist is not the theorem but the circumstances.
>
> For a fortnight I had been attempting to prove that there could not be any function analogous to what I have since called Fuchsian functions. I was at that time very ignorant. Every day I sat down at my table and spent an hour or two trying a great number of combinations, and I arrived at no result. One night I took some black coffee, contrary to my custom, and was unable to sleep. A host of ideas kept surging in my head; I could almost feel them jostling one another, until two of them coalesced, so to speak, to form a stable combination. When morning came, I had established the existence of one class of Fuchsian functions, those that are derived from the hypergeometric series. I had only to verify the results, which only took a few hours.
>
> Then I wished to represent these functions by the quotient of two series. The idea was perfectly conscious and deliberate; I was guided by the analogy with elliptic functions. I asked myself what must be the properties of these series, if they existed, and I succeeded without difficulty in forming the series that I have called Theta-Fuchsian.
>
> At this moment I left Caen, where I was then living, to take part in a geological conference arranged by the School of Mines. The incidents of the journey made me forget my mathematical work. When we arrived at Coutances, we got into a break to go for a drive, and, just as I put my foot on the step, the idea came to me, though nothing in my former thoughts seemed to have prepared me for it, that the transformations I had used to define Fuchsian functions were identical with those of non-Euclidean geometry. I made no verification, and had no time to do so, since I took up the conversation again as soon as I had sat down in the break, but I felt absolute certainty at once. When I got back to Caen I verified the result at my leisure to satisfy my conscience [POIN2: 52-53].

This 'intuitive' process is a far cry from the robot-like drill that formalism uses as a myth to describe the doing of mathematics. I trust it is evident to the reader that Poincaré 'had an object in mind' (which most of us would find abstract beyond comprehension) as he was doing this work.

But 'absolute' certainty was the grail being sought after during the time of the 'crisis in the foundations,' and let us turn our discussion in that direction. Our present goal is a modest one. Mathematics today is built up from set theory (or, if you prefer, set *theories*), and so let us ask: is the idea of a 'set' an idea with an objective validity that can go beyond the limited and contingent objective validity of, for instance, Poincaré's 'Fuchsian functions'? If it is, what sort of limitations must there be for the context of this object to ensure it remains objectively valid and does not trespass beyond the horizon of any possible experience, and thereby risk introduction of transcendent speculations with the attendant *loss* of certainty in its concept?

We derive the idea of a set from the primitive idea of *representation*. Every representation is represented in terms of a matter and a form. The making of any specific representation is an act of combination (*conjunctio*). The matter of a combination is composition (*compositio*) and its form is connection (*nexus*). Composition is the synthesis of a manifold of what does not necessarily belong to each other. Such a synthesis Kant calls *mathematical* (synthesis of the homogeneous). Composition can itself be divided into a matter (Quality) and a form (Quantity). Quantity in the synthesis of a manifold is called *aggregation* and is directed at the production of an *extensive magnitude* of representation. An extensive magnitude is that in which the representation of parts makes possible the representation of the whole. An aggregate is that which is produced by aggregation, is therefore an extensive magnitude, and 'aggregate' was the term originally used in mathematics for that which we today call a mathematical *set*. **A set is the Quantity in a representation**.

Now, this definition has logical validity but this is not enough for our purposes. For this concept of a set to have *real* objective validity, we must find its necessary relationship to the *I* of transcendental apperception. We find this relationship in the acroam of the Axioms of Intuition, which from the theoretical Standpoint states: As regards their intuition, all appearances are extensive magnitudes; from the judicial Standpoint the acroam states: All intuitions are extensive magnitudes. All our knowledge of objects requires empirical intuition – the representation of which is made possible by the pure *a priori* intuitions of space and time as we discussed in Chapters 17 and 21 – and therefore the concept of a set is grounded in a transcendental principle and thereby acquires its basis for apodictic, real, and *objective* validity.

We may note here that the concept of a set is placed within the *matter* of representation. However, as Quantity in a representation the concept of a set is a concept of *form* of composition, and this is consistent with the *formal characteristic* of mathematics. But we must also take note that while we now have a *Realdefinition* for the concept of 'set,' this *Realdefinition* does not extend to the concept of the 'elements in a set.'

The idea of an 'element' of a set is taken from a wider and older tradition of thought. Although some people assume that the idea of an 'element' of a set is an analogy to the idea of 'elements' in chemistry, this is not the actual genesis of the term. To find a technical description of the origin of the concept of an 'element' as this term is broadly used in modern times, we must go back much farther, namely to Aristotle and his metaphysics.

> We call an element that which is the primary component immanent in a thing, and indivisible in kind into other kinds, e.g. the elements of speech are the parts of which speech consists and into which it is ultimately divided, while *they* are no longer divided into other forms of speech different in kind from them. If they *are* divided, their parts are of the same kind, as a part of water is water (while a part of the syllable is not a syllable). Similarly, those who speak of the elements of bodies mean the things into which bodies are ultimately divided; while *they* are no longer divided into things differing in kind; and whether the things of this sort are one or more, they call these elements. The elements of geometrical proofs, and in general the elements of demonstrations, each of which is implied in many demonstrations, are called elements of demonstrations; and the primary deductions, which have three terms and proceed by means of one middle, are of this nature.
>
> People also transfer the word 'element' from this meaning and apply it to that which, being one and small, is useful for many purposes; for which reason the small and simple and indivisible is called an element. Hence comes the facts that the most universal things are elements (because each of them being one and simple is present in a plurality of things, either in all or in as many as possible), and that unity and the point are thought by some to be first principles. Now, since the so-called genera are universal and indivisible (for there is no formula for them), some say the genera are elements, and more so than the differentia, because the genus is more universal; for where the differentia is present, the genus accompanies it, but where the genus is, the differentia is not always. It is common to all the meanings that the element of each thing is the first component immanent in each [ARIS7: 1601-1602 (1014^a26-1014^b15)].

Aristotle's metaphysics is not the metaphysics of the Critical Philosophy, and so we must see what to make out of this idea of 'element' from the Copernican perspective. The word 'immanent' means "living, remaining or operating within; inherent." With regard to composition, that which is 'immanent' in any composition is its matter, and this is nothing else than the Quality in the representation. In terms of the synthesis, this is *coalition* in the composition and is directed at the production of *intensive* magnitudes. An intensive magnitude is a unity that can be apprehended as plurality only through an approximation to negation. We do not speak here of analytic division (because this is division of an aggregate), but rather of negation as a real opposition (*Widerstreit*). Thus the Critical concept of intensive magnitude has that character of which Aristotle spoke when he said that if an 'element' of a thing *x* is divided "the parts are of the same kind" **but can no longer be called elements of** whatever *x* we started with. Intensive magnitude 'has no parts different in kind'. Euclid's idea of a 'point' is consistent with this idea of an 'element': A Euclidean point is "un-extended". Try to "divide" one and you must either *negate* it (annihilation) or turn it from an *element* into something else, e.g. a "monad" of non-standard analysis [ROBI: 57]. **An 'element' of a set is a representation of a coalition in intensive magnitude**. A **set-theoretic point** (***member*** of a set) is something else entirely. It is *a complete and singular composition employed as a unit in the process of aggregation*. It should not be called an element. A *member* 'belongs to' a set; an *element* does not.

Again, this has logical validity but we still require a transcendental ground for it to have real objective validity. We find this in the transcendental acroam of Anticipations of Perception. From the theoretical Standpoint this principle states: In all appearances the sensation, and the real which corresponds to it in an object, has intensive magnitude. From the judicial Standpoint the principle states: The intensive magnitude (degree) of sensation presents the complete condition for marking sensibility at a moment in time. By this principle we tie the concept of a set 'element' to the manner in which the consciousness of the Organized Being is affected, and thereby to the representation of an *Existenz* through the power of receptivity. Thus, 'element' obtains its ground for apodictic, real, and *objective* validity.

> The real in appearance always has a magnitude, which is not, however, encountered in apprehension, as this takes place by means of the mere sensation in a moment and not through successive synthesis of many sensations, and thus does not go from the parts to the whole; it therefore has a magnitude, but not extensive [KANT1a: 291 (B: 210)].

We have obtained the *Realdefinition* for the concept of 'set' and for 'element' of a set. The acroamatic grounds for these definitions come through the synthesis in sensibility, and therefore all further refinements of these concepts must likewise be able to trace a necessary relationship to this synthesis. Any characteristic hypothesized as a further characteristic of either of these concepts will have real objective validity provided this relationship to the synthesis in sensibility can be established as necessary for its possibility. Absent of this relationship, the hypothesized characteristic has no transcendental ground and its incorporation into the concepts of 'set' or 'element' or 'point' becomes transcendent and robs us of real objective validity in the concept.

Now, in addition to the concepts of 'set' and 'element' set theory also contains concepts of relations ('functions' and 'rules of inference'). These always involve transformations between sets. For example, the notation $f : A \to B$ or $f : A \to A$ denotes a rule of transformation from a 'domain set' (A) to another set. In other words, this notation is a shorthand for saying that f defines some subset of a Cartesian product and does not *directly* denote, for instance, taking A immediately back into itself (as the notation $f : A \to A$ merely seems to imply). We must not confuse the abbreviated notation for the logical train of concepts this notation is meant to convey.

The transcendental ground for real objective validity for any set-theoretic relation is always to be found in *judgmentation*, and for *objective* relations we must look to the process of determining judgment (in the combination of concepts) or reflective judgment (for inferences of judgment). However, inferences of judgment are lacking an objectively sufficient ground (although such inferences have subjective sufficiency), and so the *Realdefinition* of any set-theoretic relation must turn to determining judgment under the acroam of the Analogies of Experience and must look for its synthesis in the combination of concepts under the rule of the categories of understanding. Here we must remember that every determinant judgment is a combination involving all four heads of representation, i.e. a category of Quantity plus a category of Quality plus a category of Relation plus a category of Modality. No set-theoretic relation can have real objective validity if it lacks having any notion in one or more of these four titles of representation. There are 81 forms of determinant judgments for combining two concepts.

We can, of course, take a plurality of concepts back into the synthesis of sensibility (through the synthesis of reproductive imagination), but if such a plurality of concepts is to be regarded as composing the set on the left-hand side of the above notation, these concepts must belong to the same logical *sphere* in the manifold of concepts. Therefore they are interconnected by specific determinant judgments, and these judgments fix rules for the form of comparates in the three-fold synthesis of the *Verstandes Actus* of sensible synthesis. The resulting intuition is, of course, a representation of the set on the right-hand side of the above notations. Because the number of distinct set-theoretic relations possible is indeterminate *a priori*, the *Realdefinition* of any particular set-theoretic relation must proceed by means of an analysis of the specific class of synthesis in comprehension that the relation is intended to represent. Its definition therefore will always involve consideration of the *Verstandes Actus* in the *Gestaltung* of sensibility. Because we are faced here with an *a priori* indeterminate number of possible functions, we must leave our discussion of the proper rules for the real definition of specific set-theoretic relations for a future work, namely to a doctrine of an applied metaphysic.

§ 3.4 Critique of the Axioms of ZFS Set Theory

Finally, let us take a look at how the axioms of the Zermelo-Fraenkel-Skolem set theory compare with the Critical requirements we have just discussed. To preview the outcome of this critique, we are going to find that there are axioms in the ZFS system that do not meet up with the conditions required for apodictic and real objective validity in a mathematical doctrine. This means just what you think it does: a mathematical system founded upon these axioms is incapable of representing mathematically a doctrine of Nature and serves only to produce a mathematical doctrine of a transcendent naturalism. A physicist, for example, cannot make a valid claim that his mathematical physics is an objectively valid description of Nature by saying "this is what the mathematics tells us." The mathematics he is using contains *transcendent* ideas, and therefore at some point *will* lose its objective validity for explaining Nature by disappearing over the horizon of possible experience. A physics theory given form by such mathematics might, nonetheless, maintain a limited *pragmatic* validity, but only if this theory makes novel predictions *that can be experimentally or observationally put to the test*. If such is the case then the theory can stand as a *pragmatic approximation* of Nature and be fecund in its application. However, if the theory makes no novel predictions that can be subjected to the test of actual experience then the theory *says nothing at all* as a doctrine of Nature. It is instead mere speculation and has no better *real* standing in science than Plato's *Timaeus*. This is to say *it can claim no standing at all*. It is and remains mere Platonism *until and unless* new *instruments* are found that *push out the horizon of possible experience* and thereby enable us to put the theory to the test.

Upon this point follows another comment which, I suspect, mathematicians may find a bit unpalatable. The larger part of the mathematics community reacted with alarm to Brouwer's radical call for abolishing traditional mathematics. They felt, and quite rightly, that Brouwer's 'constructivist' program would lead, in Hilbert's words, to "losing a great part of our most valuable treasure." A puritanical reaction to the critique we are about to undertake is a mistake because it would deprive the other sciences of their most indispensable tool in searching for an understanding of Nature. In other words, *there is value in a theory even if this theory's objective validity is merely pragmatic*. A pragmatic theory – in the sense described above – provides a 'model of a model', i.e. an *approximation* of Nature, and this has its uses in the growth and development of all sciences *including mathematics*. But the employment of a pragmatic theory is the employment of a two-edged sword, and it is important that we be able to distinguish between what has real objective validity vs. what has merely logical validity, and that we know where a logical approximation to *real objective character* begins and ends.

Mathematics is an empirical science, but it is also one within which some of its constructs are capable of maintaining apodictic, real, and objective validity. This part of mathematics is a doctrine we may justly call ***Critical mathematics***. The rest I call ***hypothetical mathematics***, and this part has great value for science and even more so for engineering. Let us not, therefore, look to a future where we obtain the former but throw away the latter. Let us instead approach the discipline of mathematics with an eye to distinguishing what within it we can hold objectively valid, because it is based upon axioms grounded in the Critical acroams, versus that which it teaches us is a consequence of useful but nonetheless contingent premises. Let there be a part of mathematics that is Critical and a part that is Pragmatic, and let this distinction replace the present-day distinction drawn between ‘pure’ mathematics and ‘applied’ mathematics.

With that, we proceed with our critique of the ZFS system. We will take each of its nine axioms in turn.

<u>1. The axiom of extensionality</u> (two sets are equal if and only if they have the same elements). This axiom is introduced in order to provide a definition of the idea of ‘equal.’ In Zermelo’s original system his expression of the earliest version of this axiom was expressed in terms of logical identity: two sets *x* and *y* are equal “if they denote the same object.” As *x* and *y* are to be sets, extensionality is a statement that a set is determined by its members (not its elements). Other mathematicians raised some objections to Zermelo’s axiom, leading to two other ‘attitudes’ (as Fraenkel put it) regarding how the relation of ‘equality’ should be treated. Denoting Zermelo’s ‘attitude’ as (1), the others were:

> 2) Equality as a (second) *primitive relation* within **Z**. Then the usual properties of any equivalence (equality) relation must be guaranteed axiomatically, in particular substitutivity with regard to ∈ in the two-fold sense: of extensionality as stated above, and of equal objects being elements of the same set.
> 3) Equality as a *mathematically defined relation*. We may define $x=y$ by “if every set that contains *x* contains also *y* and vice versa”, or by “if *x* and *y* contain the same elements”. The second way is possible only if, as assumed in the following, every object is a set (including the null set). In the former sense, extensionality must be postulated axiomatically; in the latter sense, an axiom has to guarantee the former property . . .
> Taking into account the admissibility of a set which has no element (“null set”), three positions about individuals are tenable: that the domain contains one null set and also other individuals, individuals but no null set, one null set but no other individuals. (A domain without null set and individuals would be impractical.) The first position was taken by Zermelo and, for instance, by Ackermann [1937]; the second by Quine from 1936 on; the third, first proposed by Fraenkel [1921/22] and later accepted by von Neumann, Bernays, and others, is adopted in the following [BERN: 6-7].

We can see what the mathematicians were aiming to accomplish with axiom 1, although to the non-mathematician it might seem surprising that there could be learned disagreement over how to go about accomplishing the aim. The axiom of extensionality contains two concepts that mathematicians tend to regard as ‘primitive.’ The first, of course, is the concept of ‘equals’ and is the binary relation the axiom is intended to define. The second is the concept of ‘belongs to’; if “*z* belongs to *x*” implies “*z* belongs to *y*” and vice versa, *x* and *y* are said to be ‘equal.’

This of course presumes that we know what it is for ‘*z* to belong to *x*’ and etc. for *y*. But what does this presuppose? This involves the idea of ‘comparison’ – which we have seen comes in two flavors in the synthesis of apprehension. There is ‘comparison’ as *Comparation* and there is ‘comparison’ as reflexion in the *Verstandes Actus*. But these acts are not acts of comparing two representations *with each other* to determine ‘likeness’ and ‘unlikeness.’ General comparison in the *Verstandes Actus* refers to expedience in representation and always involves a third factor, namely a *context of purpose*. We know there is a third factor present here because we are dealing with a synthesis and a synthesis always involves three factors.

Furthermore, the axiom of extensionality involves the making of an analytic division. That is to say, it is presumed we take 'take *x* and *y* apart' and look at the constituents of their extensive magnitude. But comparates in the synthesis of apprehension are not intuitions even if they come to the synthesis via the synthesis of reproductive imagination, and therefore they are no longer *sets*. This is because as representations they are now constituents of the *materia ex qua* of the synthesis. The outcome (an intuition), while it provides a set, is not presupposed to be *the same* as the comparates. We can imagine that the synthesis of reproduction of concept *x* and that of concept *y* produces indistinguishable *materia in qua* in an intuition, but in this case all we can say is that each concept (as a rule for the reproduction of an intuition) contains the same substance of information (in the context of the data of the senses), but this does not implicate absolute identity between *x* and *y* as concepts. All we can legitimately say is that there is an implication of the reproduction of an identical spatial form in an empirical intuition.

The problem with the axiom as stated is that it is attempting to compare 'elements', but an element has in its objectively valid context only intensive magnitude in a composition. The axiom attempts to compare elements in terms of extensive magnitude (decomposition first into compositional parts followed by an act of comparison that is not valid for the synthesis of the *Verstandes Actus*). This is the sort of real subreption against which Bergson argued. Axiom 1 *is not an objectively valid primitive definition as it stated*.

The mathematical concept of an equivalence relation is that it is a binary relation on a set in which the reflexive, symmetric, and transitive properties hold. The *Realerklärung* of these properties is grounded in the *a priori* synthesis of subjective time. But here, likewise, we look for the transitive in the form of a temporal chain, the reflexive in the form of a join, and the symmetric in the synthesis of concepts from which we can obtain a temporal reversal. Achieving all three properties, particularly the symmetric property, involves a series in judgmentation. Furthermore the child's *a priori* representation of equivalence classification is not, as Piaget's work shows, *quantitative* but rather *qualitative* in a topological sense of that word.

If we seek a primitive *Realdefinition* of identity in the sense of 'equivalence' as mathematics wishes to employ that concept, the proper starting point is with a relation of *real opposition* (*Entgegensetzung*) that produces a real negation (*Widerstreit*). In spirit this approach to obtaining a fundamental 'axiom of extension' is more in line with the concept of 'set difference' and, in particular, with an idea that the composition of a set difference *x-y* and of a set difference *y-x* produces *no set* (i.e. zero degree of intensive magnitude in composition). We must, of course, regard 'set difference' in this context in terms of the synthesis of a real *Entgegensetzung*. Put in non-technical language, "*x* is equal to *y* if and only if I can't see any difference." However, we must also bear in mind that such real comparison is always a synthesis, therefore always implies a context, and so equivalence between two sets can only be *relative to a context* and is not absolute (equivalence in *all* possible contexts). Context *delimits* equivalence.

2. Axiom of the null set (there exists a set with no members). To be a real set requires an objective representation, which means the composition has extensive and intensive magnitude. Therefore, 'set with no members' implies 'representation with no magnitude' and this is a contradiction in terms. This *ontological* axiom of the null set has no objective validity.

The role of the "null set" in mathematics is essentially a formal role and it is used to allow the expression of such statements as "set *x* and set *y* are disjoint" ($x \cap y = \varnothing$). It is also quite handy in that the employment of the symbol ∅ facilitates the simpler expression of some mathematical formulas. In this *use* its homologue is the number 0 (the identity element in an additive group) and ∅ is *made to play* a similar role in set operations (e.g. $x \cup \varnothing = x$ for any *x*). But ∅ is, in real objective terms, only a signifier that a synthesis in sensibility contains a real contradiction (complete negation of intensive magnitude). Contradiction (*Widerstreit* between the set-defining properties of each set) *is possible* and this is all that ∅ can signify. ∅ cannot be a *set*. *The objectively valid axiom would be: Some compositions are objectively impossible*.

3. Axiom of unordered pairs (if *x* and *y* are sets then the set of pairs (*x, y*) or the set of pairs (*y, x*) exists). The concept of an *ordered* pair has objective validity tied to the synthesis of subjective time. If we presume that an (*x, y*) pair is not reflexive then the actuality of (*x, y*) at a moment in time necessarily requires the *Nichtsein* of ordered pair (*y, x*) at that same moment. Pairs (*x, y*) and (*y, x*) are, in other words, co-determining and therefore members of a disjunction. The *concept of the disjunction* is the unity of (*x, y*) and (*y, x*), and this unity is the *Realerklärung* of the concept of an 'unordered pair.' *Axiom 3 is objectively valid*.

4. Axiom of the Sum Set (if *x* is a set of sets, the union of all its members is a set). The first consideration to be dealt with in this axiom is the idea of a 'set of sets'. The concept of a 'set' is a concept of Quantity in a representation; therefore, taken literally, 'set of sets' implies extensive magnitude composed of extensive magnitudes. Now, an intuition is a singular representation and so we cannot *immediately* infer an aggregation *of extensive magnitudes* in an intuition. However, all that is required for the possibility of analytic division of an *object* of intuition into a aggregation of representations, each of which contains an extensive magnitude, is the possibility

of understanding the concept corresponding to that intuition by means of combinations with other concepts (coordinate concepts and remote concepts connected in a series). This is nothing else than the representation of a phenomenon by means of concepts connected in a common sphere.

Consequently, the idea of a 'set of sets' has objective validity through reference to the manifold of concepts. A set which is said to be composed of sets is, in terms of representation in cognition, the *mediate* concept of the Quantity of the manifold in an intuition of an appearance made by combining intuitions from multiple moments in time. The *distinct concept* of a set of sets is a representation possible with respect to the manifold of concepts. The extensive magnitude of this set is thereby represented through the sphere of a concept, and because all concepts contained in this sphere can be reproduced in sensibility through the act of reproductive imagination, the resulting intuitions satisfy the intent of the axiom. *Provided that this context is maintained for the term 'set of sets', axiom 4 is objectively valid*.

<u>5. Axiom of Infinity</u>. Recall that this axiom stipulates a synthesis of a mathematically infinite number of terms. It is introduced in order to provide an axiomatic justification for the idea of mathematical infinity. Now, the concept of an inference of induction (in reflective judgment) is objectively valid, as is the concept of a series of combinations of concepts in the manifold of concepts. However, an *actual* mathematically-infinite series is not an object of any possible experience. The concept of a series with no condition of termination that can be known *a priori* is objectively valid. However, to regard this merely problematic concept of mathematical infinity as the concept of an *actual* object is utterly transcendent. At every moment in time the connection in a series of concepts in the manifold of concepts must be regarded as having finite extensive magnitude. *The axiom of infinity has no real objective validity beyond mere 'becoming'*.

The concept of mathematical infinity is an idea of great pragmatic fecundity in mathematics. It is also quite often an idea that requires the additional construction of rules of correspondence in order to ward off antinomies in mathematics. (The branch of mathematics that deals with this is called 'analysis'). Under the usual rules of arithmetic, if we form a finite sequence of $n+1$ terms of the form $1 + ½ + ¼ + . . . + (½)^n$, it is easily verified that each additional term in the series contributes less and less to the sum. If we regard this successive addition of terms as *approaching a limit* as n is incremented without bound, the rule established by mathematical analysis tells us that this series of additions approaches the numerical value of 2. Calculus is completely dependent upon the supposition of the legitimacy of such limiting processes involving sums of 'infinitely' many terms. However, for all of this the axiom of infinity can never have ontological validity (has no real object), and as a consequence all mathematics that posits such a *set* belongs to hypothetical (Pragmatic) mathematics and not to Critical mathematics.

6. Axiom of Replacement. This is an axiom schema. During the development of axiomatic set theory this axiom became increasingly more complex to the point where it is difficult to even state it in non-symbolic language. Its *intent*, however, is to state that any ‘definite’ property that can be stated in a formal language can be used to define a set. The complications that have attended axiom 6 arise from the need to avoid the Russell paradox. It is in effect a formula system for trying to define what constitutes a ‘definite’ property. In its formal ‘language’ the axiom is a prescription of requirements for relationships among multiple ‘property sets’. A full critique of this axiom therefore requires a critique of each of its various parts, a task better suited for the job of developing an applied metaphysic. Here what can be said is: ‘definite property’ cannot be ‘definitely defined’ without reference to the *objects* that are to be members of the set *and* to the objective meaning of the ‘functions’ used to define the *predicate set*. To be objectively valid we must restrict these objects and also the predicate set *to be neither transcendent nor contradictory*. *The axiom does not ensure the former*. It therefore does not guarantee us an objectively valid set.

7. Axiom of the Power Set (for any set x the set y consisting of all the subsets of x exists). The concept of a ‘subset’ is a concept of a possible analytic division of a set. Now, a concept re-introduced into the synthesis of apprehension as *materia ex qua* of the synthesis is subject to an analytic judgment and does not necessarily have the extensive magnitude it contains completely included in the intuition resulting from the synthesis. This is because the *Verstandes Actus* of abstraction removes those factors of representation in which two concepts undergoing combination in a singular intuition differ (as to purpose). The homologue in set theory for abstraction is the intersect of two sets. Therefore, the concept of a ‘subset’ has objective validity.

In this context, all the concepts contained in the sphere of a concept can be regarded as able to represent ‘subsets’ of that concept. However, it is not possible for every aggregation of concepts in the entire sphere of every concept to be reproduced in sensibility under the synthesis of reproductive imagination (e.g. sphere of a disjunction). Therefore the possibility of a ‘set of all the subsets’ does not hold for every concept, and consequently *axiom 7 is not universally true*.

8. Axiom of Choice (for any set s there is a function f such that for any non-empty subset x of s, $f(x)$ is an element of x). As noted earlier, the axiom of choice has a somewhat checkered history in mathematics. A somewhat more literal translation of the symbolic statement of this axiom reads “if a implies non-empty set x is a function defined for every $a \in s$ then there is another function $f(a)$ for $a \in s$ and $f(a) \in x$.” The original intent of this axiom was based on the idea that any arbitrary objective property could be used as the basis for defining a set. In its

modern form, the axiom asserts: given any collection of sets, one may form another set by selecting precisely one element (chosen however one wishes) from each set in the given collection.

Now, the crucial factor in this axiom is the idea that such choices can be made *arbitrarily*. However, complete arbitrariness of choice lacks objective validity on three accounts. First, ‘element’ is a concept of Quality, not Quantity, and so ‘selection of an element’ implies specification of some representation of sensuous *materia in qua* from some concept contained in the sphere of the concept regarded as containing the representation of one of the ‘sets’ in the ‘collection of sets.’ But it is not objectively valid to regard the synthesis of reproductive imagination as reproducing the matter of a representation without also reproducing its form. The literal intent of the axiom on this account alone lacks objective validity. Second, two comparates in sensibility can be in real opposition (real *Entgegensetzung*) and consequently result in a real negation. The axiom attempts to avoid this case by specifying that the selection must involve only non-empty sets, but whether or not two comparates are going to negate each other cannot be known *a priori*. Finally, the synthesis of sensibility is always subject to determination by rules (of the pure synthesis of subjective space, of the pure synthesis of subjective time, of the *Verstandes Actus*, and of the regulation of the employment of determining judgment). Because the representation of an intuition must first of all serve the principle of formal expedience in reflective judgment, not every arbitrary collection of contributions from an aggregation of ‘sets’ can represent an objectively valid *context*. The axiom of choice presupposes the possibility of constructions without constraint, and this presupposition contradicts the transcendental character of the processes of synthesis. *Axiom 8 is not objectively valid*.

9. Axiom of Regularity (non-empty set s implies x exists such that $x \in s$ such that $y \in x$ implies $y \notin s$). This axiom specifies that a set cannot contain itself as a member. From our *Realdefinitions* of ‘set’ and ‘element’ we know this is true. *This interpretation of axiom 9 is objectively valid*.

§ 3.5 Acroamatic Set Theory

The point of the foregoing critique of the ZFS system is to illustrate the extent to which the formalist defense that the system of mathematics based upon it “isn’t about anything” is true if by ‘anything’ we *mean* all objects of Nature. Certainly mathematics based on ZFS is “about something,” and we can hardly do better than to call this ‘something’ by the name ‘hypothetical mathematics.’ *Given* the ZFS axioms and particular ‘rules of inference’ *as the hypothetical premise*, the theorems that follow are true in the sense of being in agreement with their object (namely, the ZFS-based mathematical *system*).

However, the ideal of mathematics from ancient times has been that mathematics provides a means of knowing the world through the pure power of rational thinking alone. Mathematics has come to admit that this ideal is not realized. But this goal was never possible to achieve because it rested upon a number of false ontological premises, e.g. the copy-of-reality hypothesis or the mistaking of appearances for things-regarded-as-they-are-in-themselves. *Critical* mathematics, on the other hand, would be mathematics based upon an applied metaphysic in accordance with the acroams of the Critical Philosophy. To bring Critical mathematics into being is a task that will require the development of a new system of axioms *deduced from acroamatic principles* so that this new axiom system involves no transcendent ideas and whose rules are connected to the capacity for objective representation of an Organized Being. This new system must be a *structure* (in the Piagetian sense of that word), which is to say that it must contain a doctrine for the construction of self-organizing rules of transformations. We will speak to this in Chapter 24.

The task before us is laborious, but the rewards for success are great. Such a Critical mathematics will reveal to science where its mathematical theories enjoy real objective validity, and point out where they do not. The development of Critical mathematics is crucial to making a science *proper* of mental physics, a doctrine in which neuroscience and psychology can find their unity. The lessons learned during the development of Critical mathematics can hardly help but have collateral benefits for mathematics education. A rich undiscovered country lies just beyond where we now stand. It awaits the coming of those who will pioneer this new frontier.

§ 4. Mind and Mental Physics

We began this treatise with the question: What is mind? Although this question is ontological, the characteristics of the phenomenon of mind are such that we were led to examine it from the grounds of Kant's epistemology. We have therefore undertaken a dual task: to understand Kant's Critical Philosophy on the one hand, and to apply this philosophy to understand the Nature of mind on the other.

Mind is one of the two principal phenomena that characterize the individual human being, the other being the phenomenon of body. Body is the sensible Nature of human *Existenz*, mind the supersensible. They are coordinate phenomena of being human, neither able to take precedence over the other. Rather, we have seen that these two aspects of the unity of human Nature are reciprocal phenomena, each determining and determined by the other. In the logical division of the phenomenon of being human we have expressed the organization of human

Existenz as the synthetical unity of three logical parts: *nous, soma* and *psyche*.

Throughout this treatise it has been our constant concern to ground the objective validity of the ideas of the theory. This has, naturally, required the examination of what it means for an idea to have objective validity and to undertake to understand the requirements for achieving it. Although the *Dasein* of mind is as self-evident as any phenomenon in Nature, the intelligible Nature of mind required that we examine its *Existenz* most carefully. Since the natural evidence of mind is apparent only through the Nature of one's experience, this has led us to the view that the proper description of the *phenomenon* of mind can be made only through a careful examination of mental capabilities and capacities. We have therefore given our efforts to the deduction of various representational processes and processes of knowledge synthesis. This has provided us with a very detailed account of the organization and processes of the phenomenon of mind, and these can be taken as *delimiting the phenomenon*.

Epistemological theories have implications for consequences in sensible Nature, and this treatise has therefore devoted considerable space to reviewing empirical evidence related to these consequences. This has not been because such evidence grounds the theory but, rather, because the theory has consequences and we needed to compare these against scientifically determined findings. The concordances we have seen in doing this helps us to better understand in detail the scope of our theoretical ideas and served as a tool for avoiding the error of transcendent over-generalization. This empirical evidence has provided us with insights that are lacking in Kant's great Critiques owing to the paucity of examples he was able to provide.

The great logical divisions of the capacities of *nous* are: the processes of synthesis of conscious representation in sensibility, the process of determining judgment, the process of reflective judgment, the process of practical judgment and the synthesis of appetition, and ratio-expression through the power of speculative Reason. Those of *psyche* include the theory of the data of the senses, the animating principles of *psyche*, the *Lust-Kraft* and *Lust*-organization of *psyche*, and the reciprocity of *nous* and *soma* through the sensorimotor capacities. These have been explained in terms of the primitive capacity of representation, which has formed the central core of the theory. This chapter has examined where mathematics-as-science fits in this theory.

Yet, for all that has been accomplished in this treatise, there is still much more to do. What has been done here is merely propaedeutic for a new science of mind, which I have called mental physics. This science does not yet exist. But the groundwork for its development has been laid here. How next does one proceed to make the transition from this work to that of an empirical science? To appreciate this question, we must deal with one more topic that goes to the core of the Critical *method*: the epistemology of **undecidability**. And so we come to our final Chapter.