Chapter 10

Mathematical Structures

§ 1. The General Idea of "Structure"

In chapter 8, the idea of something called a "mathematical structure" was tossed down without further elaboration. What is a mathematical structure? For that matter, what is a "structure" in general? The dictionary lists no fewer than five definitions for this noun, all of which have some connection in one way or another to a Latin verb that means, "to heap together." What we are after is a technical explanation that can serve us in building the theory of computational neuroscience.

Although we will avoid it as much as is practical, the topic we are about to discuss has a tendency to become very abstract and very technical very quickly. Some of the scientific disciplines, such as biology or chemistry, have neither a present direct use for nor any particularly burning desire to acquire knowledge-in-depth of this topic. If this sounds like it describes your field, no matter. Read this chapter for qualitative appreciation and if allergies set in when it comes to the math, skim over the quantitative details and don't worry about it. This text is an introductory text aimed at an interdisciplinary audience, and not everyone is going to be engaged in-depth in everything we discuss. Set your sights on the goal of just being able to talk to your mathematically-inclined colleagues. If you should happen to learn one or two "cool" things you didn't know before in the process, so much the better.

Most of the different disciplines involved with neuroscience have their own usages for the word "structure" but few have one official technical definition in the way physics has and uses for the word "work." This is perhaps understandable in the sense that if a "structure" is more or less "something heaped together" with some rule, property, or convention that sees to it the things in the heap are "attached" in some specific way, pretty much anyone can probably recognize a "structure" when he sees one. For our purposes, we require a definition capable of moving from one discipline to the next without alteration. The definition we will use is the one stated by the renowned 20th century psychologist, Jean Piaget[1]:

> First of all, when we compare the use of the term "structure" in the various natural and human

[1] Piaget is generally recognized as "the father of developmental psychology" and is regarded by a good many people as the greatest psychologist of the 20th century. When one considers that psychology as a science only dates back to the mid-19th century, this is high praise, indeed. We'll be hearing more from Piaget as we go on.

> sciences, we find the following characteristics. Structure is, in the first place, a system of transformations having its laws, as a system, these therefore being distinct from the properties of the elements. In the second place, these transformations have a self-regulating device in the sense that no new element engendered by their operation breaks the boundaries of the system (the addition of two numbers still gives a number, etc.) and that the transformations of the system do not involve elements outside it. In the third place, the system may have sub-systems by differentiation from the total system (for example, by a limitation of the transformations making it possible to leave this or that characteristic constant, etc.) and there may be some transformations from one sub-system to another [PIAG1: 15].

We do, of course, have to clarify the central idea of "transformations" involved in this definition of a "structure." Elsewhere, Piaget elaborated on this a bit:

> First of all, a structure is a totality; this is, it is a system governed by laws that apply to the system as such, and not only to one or another element in the system. The system of whole numbers is an example of a structure, since there are laws that apply to the series [of whole numbers] as such. Many different mathematical structures can be discovered in the series of whole numbers. . . A second characteristic of these laws is that they are laws of transformation; they are not static characteristics. In the case of addition of whole numbers, we can transform one number into another by adding something to it. The third characteristic of a structure is that a structure is self-regulating; that is, in order to carry out these laws of transformation, we need not go outside the system to find some external element. Similarly, once a law of transformation has been applied, the result does not end up outside the system. Referring to the additive group again, when we add one whole number to another, we do not have to go outside the series of whole numbers in search of any element that is not in the series. And once we have added the two whole numbers together, our result still remains within the series [PIAG2: 23].

One attractive feature of this explanation of "structure" is that it covers open systems (e.g. systems that grow, as biological systems do) as easily as closed systems. Under the definition of "system" we are using in this text, it is perhaps clear that "transformations" and their "laws" belong to the "model" side of the system rather than the "object" side.

Piaget's definition makes use of the idea of "elements" of the system and "elements" outside the system. His use of this word is casual rather than technical and merely means "basic part." Used in this sense, a neuron is an "element" of a neural network, and a neural network is an "element" of a map. Somewhat more abstractly, signals are "elements" of the system because they are a "basic part" of our models. Here, however, if we are not careful we can start running into some issues due to Piaget's casual use of the word "element."

I hope you won't, but suppose you stub your toe on something when you're walking around barefoot. A pressure impulse (which is a signal under our definition) is transmitted to pain-sensing nerve endings in your toe, and the first thing you know a host of neuronal signals has you hopping around on one foot and saying, "Ouch! Shoot, oh dear!" (or something like that). Clearly, a large number of "transformations" have taken place involving signals inherent to the system that is your body. But what about the impulsive signal that started it all? Unless you are impressively clumsy, you did not stub your toe on yourself, and so whatever causal agent it was that interacted with you to produce the pressure signal, it is not part of the system that is your body. Therefore, in some way and at some point, would not the pressure signal be an element "outside the system"? If so, would this not contradict Piaget's third characteristic of a structure?

"Oh," someone might say at this point, "this is just a matter of semantics or philosophical nitpicking," and to some degree this is true. But it does give us the opportunity to say a few things about the difference between closed systems and open systems. Closed systems are the simplest kind of systems because everything about them is contained in them. Of course, unless the system we are speaking of is the entire universe, everyone knows that the system will interact with its environment. This is why we introduce the idea of "input" signals into our modeling. Input signals are sometimes called "aliments" of the system, meaning they are something that "feeds into" the system in some way. Whether a signal is an "element of the system" or an "aliment" depends on where one draws the dividing line between "the system" and "not-the-system." Not too many people have a philosophical problem with this.

In the case of open systems (what Piaget calls "structures in the formative stage"), some find the situation a little bit less clear cut. Piaget tells us,

> [In] the case of structures in the formative stage, the self-regulating system can no longer be reduced to a set of rules or norms characterizing the completed structure: it consists of a system of regulation or self-regulation . . . [In] the case of structures in the process of constitution or continued reconstitution (as with biological structures), exchange is no longer limited to internal reciprocities, as is the case between the sub-structures of a completed structure, but involves a considerable proportion of exchange with the outside, to enable these structures to obtain the supplies necessary for their functioning. . . This is especially so with biological structures, which are elaborated solely by constant exchanges with the environment, by means of those mechanisms of assimilation of the environment to the organism and adjustment of the latter to the former which constitutes the transition from organic life to behavior and even mental life.
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> A living structure . . . constitutes an 'open' system in the sense that it is preserved through a continual flow of exchanges with the outside world. Nevertheless, the system does have a cycle closing in on itself in that its components are maintained by the interaction while being fed from the outside [PIAG1: 16].

Two Piagetian ideas are pertinent here, namely "assimilation of the environment to the organism" and "accommodation of the organism to the environment." The first means something from the environment is "taken into" the organism, as when you eat dinner or stub your toe. The second means something changes about the organism as a consequence of the environment by means of a self-regulating transformation of the organism's structure. (If you broke your toe when you stubbed it, that's a change but it is not an accommodation).

This puts Piaget's definition of "structure" in a bit clearer light. The key subtlety is merely this: No element outside the system (an "aliment") is needed *in order to carry out the transformation*. An aliment might be required to *stimulate* the act of transformation, but it is not needed in order for the transformation itself to be possible. The transformation is self-regulating.

§ 2. Mathematicians and Their Systems

What do mathematicians do? In other words, what is the professional practice of mathematics all about? It sometimes seems to those of us who are not mathematicians that the mathematics community does a pretty fair job of keeping the answer to this question to itself. Oftentimes, if we ask, we are told, "Mathematicians prove theorems." That's rather like saying, "Carpenters saw boards in two." As an explanation it lags a little behind what a musician friend of mine said once when asked what he does when he plays one of his killer improvised guitar riffs: "I play what sounds good." To say, "Mathematicians do mathematics" is utterly meaningless unless we know precisely what "mathematics" is *to the eye of the mathematician*. Like pornography, most of us think we "know mathematics when we see it." But what *is* "mathematics"?

If one examines practically any mathematics paper published in a reputable mathematics journal since the mid-twentieth century, one invariably finds a standard format. There is a string of definitions. This is usually followed by several lemmas. Then there is the statement of a theorem. This is followed by a proof, which typically boils down to "apply the lemmas to the definitions." This "definition-theorem-proof" format is universal in modern mathematics papers. A "lemma" is an antecedent theorem, usually one that has an "easy" proof and is not too interesting in its own right. If it should be the case that an antecedent lemma is both new and non-obvious, it is called a "theorem" rather than a "lemma" and the paper will either give its proof or cite where one can find its proof. If it is a very famous or widely used theorem, the paper will just refer to it by name, e.g. "Gödel's second theorem," and expect the reader to know all about it. In addition, the paper will contain some definitions and lemmas *implicitly*, which is to say it will use ideas and reasoning steps regarded as so basic it can be taken for granted that anyone reading this particular journal will know these things without needing to be told. This is called "the standard argument." Words like "set" or "function" fall into this category. Even some kinds of notation, such as " $x \in \Re^n$ " (= "x is an n-dimensional vector of real numbers"), fall into this category. A mathematics student spends most of his or her time learning "the standard arguments."

When the paper is done and the new theorem is proved, what do we have? We have a new bit of knowledge (the new theorem) which is *true and certain to the extent that the lemmas and standard arguments are themselves true*. Note, however, that I did not say the lemmas and the standard argument are "true *and* certain"; I only said they are "true." The lemmas and standard arguments are themselves conditioned by some even earlier and more primitive statements, called the mathematical *axioms*, which are statements taken as primitive and for which the truth of these statements is to be regarded as taken for granted *for purpose of mathematical argument*. Thus, for example, one has "the axioms of Euclidean geometry," which are different from "the axioms of

Riemannian geometry." A theorem based on the former is "true and certain for Euclidean geometry." It might not be true (much less certain) for the system of Riemannian geometry. Truth is "the congruence of one's cognition (one's ideas and concepts) with the object," and mathematical axioms and definitions establish what the "object" is to be.

A professional mathematics paper, then, is just an erudite form of the same method used (I hope) when you were taught algebra in middle school. For example, consider the "trivial" problem of solving the equation $x^2 - 4 = 0$. If we omit (almost) no steps, we have

$x^2 - 4 = 0$;   (statement of the problem to be solved)
$(x^2 - 4) + 4 = (0) + 4$; ("the same thing added to two things that are equal gives two equal things")
$x^2 - 4 + 4 = 0 + 4$; (associative property of addition)
$x^2 - 4 + 4 = 4$; ("any number plus the additive identity equals that number")
$x^2 + -4 + 4 = 4$; (definition of the additive inverse of a number)
$x^2 + (-4 + 4) = 4$; (associative property of addition)
$x^2 + 0 = 4$; ("a number plus its additive inverse equals the additive identity")
$x^2 = 4$; ("any number plus the additive identity equals that number")
$x \cdot x = 4$; (definition of "the square of a number")
  $x \cdot x = 2 \cdot 2$; ("lemma" that 2 times 2 equals 4)
  $x \cdot x = -2 \cdot -2$; ("lemma" that –2 times –2 equals 4)
$x = 2$ and also $x = -2$; (definition of the "square root" of a number)
$x = \pm 2$; (solutions)

It is not uncommon for children to detest this kind of "show all your steps" exercise, but the purpose of doing it, whether the teacher mentions it or not, is to bring into the light all the definitions and lemmas that go into a rigorous "proof" of a mathematical proposition. Without this illumination, one cannot be a successful mathematician.

The example here is, of course, quite *ad hoc*. We wanted to solve *this particular* equation. A mathematician would not be satisfied until he or she can *generalize* the solution in the form of a "theorem": "$x^2 - a = 0 \Rightarrow x = \pm\sqrt{a}$ for every a." This is generally why a college mathematics professor rarely shows any enthusiasm if an engineer or a physicist brings one specific *ad hoc* equation to him and asks for help in solving it. An isolated *ad hoc* problem is not "of sufficient mathematical interest" for him to "waste his time" on it. Henri Poincaré, one of the most widely respected mathematicians of his day, put it this way in 1914:

> Mathematicians attach a great importance to the elegance of their methods and of their results, and this is not mere dilettantism. . . Briefly stated, the sentiment of mathematical elegance is nothing but the satisfaction due to some conformity between the solution we wish to discover and the necessities of our mind, and it is on account of this very conformity that the solution can be an instrument for us. This æsthetic satisfaction is consequently connected with the economy of thought. . .
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> It is for the same reason that, when a somewhat lengthy calculation has conducted us to some simple and striking result, we are not satisfied until we have shown that we might have foreseen, if not the whole result, at least its most characteristic features. Why is this? What is it that prevents our being contented with a calculation that has taught us apparently all that we wished to know? The reason is that, in analogous cases, the lengthy calculation might not be able to be

> used again, while this is not true of the reasoning, often semi-intuitive, which might have enabled us to foresee the result. This reasoning being short, we can see all the parts at a single glance, so that we may perceive immediately what must be changed to adapt it to all the problems of a similar nature that may be presented. And since it enables us to foresee whether the solution of these problems will be simple, it shows us at least whether the calculation is worth undertaking [POIN: 30-32].

The Greek root of the word "mathematics" is *mathēma*, which means "what is learned." The ancient Egyptians knew a great deal about arithmetic and what we today call geometry. But all this knowledge was *ad hoc*. There is an ancient manuscript, dating back earlier than 1000 B.C., by an Egyptian scribe named Ahmes entitled "directions for knowing all dark things." This manuscript is thought to be a copy with emendations of a treatise more than 1000 years older still. It is a collection of problems in arithmetic and geometry for which the answers are given but not the processes by which these answers were obtained. It fell to the ancient Greeks to turn mathematics into a discipline by which "things are learned" using the definition-theorem-proof method.

The difference between mathematics papers since the mid-twentieth century and those of earlier days does not lie with the definition-theorem-proof format. It lies with the *unwavering* adherence to this in the format and style in which modern mathematics papers are written. Among the examples found in older mathematics, Euclid's *Elements* (circa 300 B.C.) could pretty much pass muster as writing in the modern style. But other ancient works, such as *Introduction to Arithmetic* by Nichomacus of Gerasa (circa A.D. 100), do not conform to this style. Many later professional works of mathematics likewise depart from the rigor used today, even if they give the appearance of conforming to the same style. As an example, here is a "proof" – today regarded as specious – by the very famous mathematician Richard Dedekind (1831-1916). It is found in his *Essays on the Theory of Numbers:*

> Theorem: There exist infinite systems.
>
> Proof: My own realm of thoughts, i.e., the totality S of all things which can be objects of my thought, is infinite. For if s signifies an element of S, then is the thought s', that s can be object of my thought, itself is an element of S. If we regard this as transform $\phi(s)$ of the element s then the transformation ϕ of S has thus determined the property that the transform S' is part of S; and S' is certainly proper part of S, because there are elements in S (e.g. my own ego) which are different from such thought s' and are therefore not contained in S'. Finally it is clear that if a, b are different elements of S, their transformations a', b' are also different, that therefore the transformation ϕ is a distinct (similar) transformation [here he cites his "definition 26"]. Hence S is infinite, which was to be proved.

The problem with this "proof" lies in its reliance upon things, such as "objects of my thought," that cannot be rigorously defined and, consequently, admit to no formal proof capable of establishing that implied statements such as "my ego is an element of S but not of S'" are true. Issues like this contributed to the discovery of a very famous paradox, the Russell Paradox, which was a discovery that eventually played a key role leading to the formal style used in mathematics today.

This agreed-upon style of doing and presenting mathematics is called "formalism." It is largely due to mathematician David Hilbert, in the early decades of the twentieth century, and to an interesting group of young, mostly French, mathematicians in the middle years of the twentieth century who are known collectively as "the Bourbaki mathematicians." (More will be said about the Bourbaki a bit later). Many of the world's leading mathematicians in the late nineteenth and early twentieth centuries were embroiled in what was known at the time as "the crisis in the foundations." Since the days of the ancient Greeks, the axioms of mathematics had been held to be universal and self-evident truths not about systems of mathematics but about nature itself. Mathematics was seen by everyone as the preeminent example of humankind's ability to know and understand nature through the sheer power of pure thought. (This belief was later called "rationalism"). But in the nineteenth centuries, several "catastrophes" in mathematics had been discovered (Riemann's geometry was one of them), and the fundamental axioms of mathematics teetered and fell in the sense that they could no longer be regarded as self-evident truths about nature. Heroic efforts were underway to find new "self-evident truths of nature." Formalism was developed precisely with this purpose in mind and to "restore the foundations."

But these efforts failed. The final nail in the coffin was driven in by a young Austrian mathematician named Kurt Gödel, who proved – using Hilbert's own methods – that the goals of the formalists' program could not be achieved. (These are "Gödel's first and second theorems"). Some contemporary mathematicians regard the last shovelful of dirt over the grave as being applied in 1963 by Paul J. Cohen in regard to something called "the continuum hypothesis" (what this hypothesis says is not important for our purposes here, at least for those of us who are not mathematicians; suffice it to say mathematicians see it as important). Among Gödel's achievements is a theorem proving the continuum hypothesis cannot be disproved; to this, Cohen added a theorem proving it could also not be proved. The final result of all this was a stunning fall from grace for the axioms of mathematics. They are no longer seen as self-evident truths about nature; today, when the philosophers troop in to pester the mathematicians, mathematicians adopt the public attitude that their axioms are "merely rules of a game; change the rules and you have a different game."

Today mathematics does not have merely one set of axioms. It has several, and which set one is working with determines which "mathematical system" is being worked on. Thus we have Euclidean and non-Euclidean geometry (actually, there is more than one system of non-Euclidean geometry), Cantorian and non-Cantorian set theory; standard and non-standard analysis, etc. And this brings us to the Bourbaki.

§ 3. The Mother Structures

§ 3.1 The Bourbaki Movement

World War I was a calamity without parallel in human history, and especially so for the major European belligerents. An entire generation of young men was decimated in the fighting, and many of the survivors experienced emotional scars that lasted for the rest of their lives. In France by the 1930s, this had led to what may be the largest "generation gap" ever experienced in Western universities since their establishment in A.D. 1200.

University students today encounter what almost amounts to a continuum of teachers, running from young new assistant professors, only a few years older than the students themselves, to grandfatherly old graybeards, who remember some of the students' parents from when they sat in those very same seats. But in 1930s France it was a very different story. The students were mostly young, as they are today, and the professors were mostly old (at least, in the eyes of the students). Like young people in most generations, the students were idealistic, full of confidence and vigor, and not just a little suspicious that they knew more about the world than their elders. Almost all of them had either been too young to be soldiers during the war or had not yet been born.

The Bourbaki movement started with a small group of French mathematics students who felt their professors were, as we would say today, very out of touch with things. The majority of French mathematics professors at the time regarded mathematics in much the same way Poincaré had, which is to say they largely eschewed formalism and adopted the "semi-intuitive" view Poincaré had championed during "the crisis in the foundations." The Bourbaki, on the other hand, embraced the "new" formalism and opposed the "Poincaré-ism" of their day. Although they began as a more or less light-hearted and semi-secret club, after a short time the light-heartedness vanished and they set about to change mathematics fundamentally and once and for all.

The Bourbaki mathematicians got their name from their practice of publishing mathematics papers anonymously under the pen name "Nicholas Bourbaki." Theirs was a youth movement, albeit not a large one by most standards. Legend has it that members were expelled when they turned 50 years of age. The Bourbaki produced a series of graduate-level mathematics textbooks in set theory, algebra, and analysis that came to have great influence in the 1950s and 1960s all over the world. Although most people – the great majority of us who are not mathematicians – find the Bourbaki textbooks less comprehensible than the writings of the Mayans, the Bourbaki movement spread far beyond the relatively small world of pure mathematics. The justly infamous "new math" that swept through primary education in America in the 1960s – which by now is the only brand of mathematics instruction most of today's young people know – came directly from the soul of "Nicholas Bourbaki." Most of what most people dislike the most about mathematics can be laid at their doorstep. Some people, mostly people old enough to remember "the old math," go so far as to blame "the new math" for leading to the drop in mathematical literacy well documented in the United States.

§ 3.2 The Structures of Mathematics

If formalism – and the Bourbaki were formalists, one-and-all – had seen the necessity of abandoning the view that the axioms of mathematics were "universal and self-evident truths" of nature, what was there left for the Bourbaki to do? Hilbert, Bertrand Russell, and others had already set up the methods, notations, and practices of mathematical formalism long before the first Bourbaki ever entered college. What did the Bourbaki accomplish that they succeeded in sweeping away the "Poincaré-ism" of the older generation?

If "true universality" is not to be found in the axioms of mathematics, this does not necessarily mean some kind of "universality" does not attach to mathematics itself. The Bourbaki set out to discover the "roots" of mathematics – something that was true of mathematics in general. They found it – or so they tell us – subsisting in three basic "mother structures" upon which all of mathematics depends. These structures are not reducible one to another. This just means no one of them can be derived from the other two. But by making specifications within one structure, or by combining two or more of these structures, everything else in mathematics can be generated. These three basic structures are called ***algebraic structure***, ***topological structure***, and ***order structure***.

The mother structures all do have at least two things in common. They all require something called a ***set***. They all require something called a ***relation***. Put as simply as can be, a set is an aggregate of things, called "the elements of a set," that defines the composition of a mathematical object. A relation is a transformation that acts on one or more sets to produce another set. A set plus a relation does not provide us with enough to say we have a structure. For example, we could juxtapose "the set of all birds in the latest issue of *National Geographic* magazine" with the relation "is the grandson of." Whatever we might want to call this, we would not call it a system, much less "a system of self-regulating transformations" etc., as we require in order to have a structure. To have a structure, we need a set, a relation, and rules establishing how we will put them together. The different mathematical structures have different kinds of rules for how they are to be put together. The Bourbaki's contribution is in discovering there are three distinct and irreducible ways to *categorize* the possible rules by which a structure can be put together in mathematics. Algebraic structure uses rules that fall into one category; the rules of topological structure fall into a second category; those of order structure fall into a third category. No distinct and fundamentally different category of rules of putting a structure together is required in mathematics. Everything else can be done by specific compositions of rules belonging to the three basic categories.

As perhaps you can appreciate, this is a pretty powerful generalization. Some might say it is perhaps the most powerful generalization yet devised by the human mind. Like all great generalizations, the description you have just read probably feels very, very fuzzy (unless you happen to be a mathematician). But that is basically "the nature" of all great generalizations. They generalize to such an extent and with such a degree of abstraction that one very oftentimes find it hard to even visualize an *example* of what sort of thing the generalization describes. (This has something to do with why so many papers written in one arena of mathematics are incomprehensible even to mathematicians who work in a different arena of mathematics; a mathematician one time said, somewhat in jest, that a "pure mathematics" paper has a target audience of about twelve people in the whole world). We all know the feeling. Somebody tells us something abstract and we say, "I don't see it," or "That doesn't make sense." Some more concrete examples of the mother structures will be presented a bit later, but first we need to talk about why this subject has been brought up in this textbook.

There are two great mysteries attending the phenomenon of mathematics. Anthropologists tell us that every human culture we know anything about, even the most primitive, has developed at least some amount of mathematics. Naturally, the mathematical abilities of a Kalahari bushman are dwarfed to insignificance by the awesome and magnificent edifice that is Russian mathematics. Still, "one and one is two" the world around. The rudimentary mathematics of the bushmen are also true for Andrei Kolmogorov. Second, mathematicians are not the only people interested in mathematics. Physicists, engineers, and, indeed, all scientists in every discipline use mathematics in some form or another to describe the world because, to a degree that would be utterly astonishing were it not so familiar to us, *it works*.

Now that even the mathematicians have conceded that mathematics is a pure invention of the human mind, that the axioms are not self-evident truths of nature but merely contingent statements to be "taken for granted" as the rules of a game, how are either of these mysteries possible? Are they both nothing more than fantastic coincidences? Or is something else at work here?

§ 3.3 The Piagetian Structures

Over the course of his 60 years of research on the development of intelligence in children, Jean Piaget discovered and documented the fact that this development takes place through slow, progressive structuring intimately tied to the child's practical activities. He studied children from birth to age 15 years and found both a remarkable continuity in the child's evolution of abilities arising from structuring activities, and a remarkably small number of forms upon which these structurings were based. Not being a mathematician, Piaget developed his own terminology for these structuring forms, using such terms as classification, seriation, and perceptual clusterings. It was his habit during the summertime to write up his findings in books documenting what he had discovered during the past year's research work. In the preface of his very first book [PIAG3], the notable psychologist Edouard Claparède wrote,

> The importance of this remarkable work deserves to be doubly emphasized, for its novelty consists both in the results obtained and in the method by which they have been reached. . .
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> Our author has a special talent for letting the material speak for itself, or rather for hearing it speak for itself. What strikes one in this first book of his is the natural way in which the general ideas have been suggested by the facts; the latter have not been forced to fit ready-made hypotheses.
> It is in this sense that the book before us may be said to be the work of a naturalist. And this is all the more remarkable considering that M. Piaget is among the best informed men on all philosophical questions. He knows every nook and cranny and is familiar with every pitfall of the old logic – the logic of the textbooks; he shares the hopes of the new logic, and is acquainted with the delicate problems of epistemology. But this thorough mastery of other spheres of knowledge, far from luring him into doubtful speculation, has on the contrary enabled him to draw the line very clearly between psychology and philosophy, and to remain rigorously on the side of the first. His work is purely scientific [PIAG3: ix-xvi].

Piaget maintained this steady, patient approach of "a naturalist" throughout his long life, never allowing speculation to run ahead of the facts. This is, of course, the best and proper course for a scientist to follow, even though it also guaranteed that the theory he developed would be presented to us in historical rather than topical order. Piaget presented his findings on the structuring of childish intelligence in the order he discovered it. Then one day he met Bourbaki mathematician Jean Dieudonné.

> A number of years ago I attended a conference outside Paris entitled "Mental structures and Mathematical Structures." This conference brought together psychologists and mathematicians for a discussion of these problems. For my part, my ignorance of mathematics then was even greater than what I admit to today. On the other hand, the mathematician Dieudonné, who was representing the Bourbaki mathematicians, totally mistrusted anything that had to do with psychology. Dieudonné gave a talk in which he described the three mother structures. Then I gave a talk in which I described the structures that I had found in children's thinking, and to the great astonishment of us both we saw that there was a very direct relationship between these three mathematical structures and the three structures of children's operational thinking. We were, of course, impressed with each other, and Dieudonné went so far as to say to me: "This is the first time that I have taken psychology seriously. It may also be the last, but at any rate it's the first" [PIAG2: 26].

What Piaget calls "children's operational thinking" is a long time in developing. It does not make its appearance until around age seven to eight years, and it is not until then that the structures of childish thought are recognizably those of algebraic, order, and topological structure. Piaget does not claim that these sophisticated mathematical structures are innate in the newborn.

However, he *does* find that the pathway by which intelligence develops in the child leads inexorably to these structures (at least in all children who have not suffered severe brain damage; Piaget did not study the development of children with this sort of handicap). The seeds from which these structures grow are found in infants, and all non-pathological human beings follow this developmental pathway.

> Once the principal sensori-motor schemes have been developed and the semiotic function has been elaborated after the age of one and a half to two years, one might expect a swift and immediate internalization of actions into operations. The scheme of the permanent object and of the practical "group of displacements" does, in fact, prefigure reversibility and the operatory conservations, and seems to herald their imminent appearance. But it is not until the age of seven or eight that this stage is reached, and we must understand the reasons for this delay if we are to grasp the complex nature of the operations.
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> Actually, the existence of this delay proves that there are three levels between action and thought rather than two as some authorities believe. First there is a sensori-motor level of direct action upon reality. After seven or eight there is the level of the operations, which concern transformations of reality by means of internalized actions that are grouped into coherent, reversible systems (joining and separating, etc.). Between these two, that is, between the ages of two or three and six or seven, there is another level, which is not merely transitional. Although it obviously represents an advance over direct action, in that actions are internalized by means of the semiotic function, it is also characterized by new and serious obstacles [PIAG4: 92-93].

The idea of a "scheme" is central to Piaget's theory.

> We call a *scheme* of an action that which makes it repeatable, transposable, or generalizable, in other words, its structure or *form* as opposed to the objects which serve as its variable contents. But except in the case of hereditary behaviors (global spontaneous movements, reflexes or instincts), this form is not constituted prior to its content. It is developed through interactions with the objects to which the action being formed applies. This is truly a case of interaction for these objects are no longer simply associated among themselves through an action, but are integrated into a structure developed through it, at the same time that the structure being developed is accommodated to the objects. This dynamical process comprises two indissociable poles: the *assimilation* of the objects into the scheme, thus the integration of the former and the construction of the latter (this integration and construction forming a whole), and the *accommodation* to each particular situation [PIAG5: 171].

Piaget discovered that the structures of intelligence first develop as practical schemes of actions (sensori-motor schemes) and only later are "internalized" into mental representations by which the child can perform 'mental actions' without having to also perform the physical action from which the child's mental representations arise. (The child's *demonstration* of the ability to form and use mental representations is what Piaget refers to as "the semiotic function"). This was documented vividly in a series of experiments on cognizance [PIAG6]. Piaget calls the means by which all this takes place the "central process of equilibration" [PIAG7], equilibration being the balancing of assimilation and accommodation.

Piaget found there are precisely three general forms of assimilation. A careful examination of what most people regard as the "centerpieces" of Piaget's theory [PIAG8-15] shows that these three forms are most closely linked to the development of order structure, algebraic structure, and topological structure, respectively. As for the three forms of assimilation,

> Assimilation, which thus constitutes the formatory mechanism of schemes (in a very general biological sense, since organisms assimilate the environment to their structure or form, which can in turn vary by accommodating to the environment) appears in three forms. We will speak of functional assimilation (in the biological sense) or 'reproductory' assimilation to designate the process of simple repetition of an action, thus the exercise which consolidates the scheme. Secondly, the assimilation of objects to the scheme presupposes their discrimination, i.e. a 'recognitory' assimilation which at the time of the application of the scheme to the objects makes it possible to distinguish and identify them. Lastly, there is a 'generalizing' assimilation which permits the extension of this application of the scheme to new situations or to new objects which are judged equivalent to the preceding ones from this standpoint [PIAG5: 171-172].

After his eventful meeting with Dieudonné, Piaget began to more and more use terminology that a mathematician could recognize. Not being a mathematician himself, Piaget's descriptions of the Bourbaki mother structures form a nice bridge over which those of us who are likewise not mathematicians may more easily pass to reach the less familiar environs of the mathematician's world.

> The first is what the Bourbaki call the algebraic structure. The prototype of this structure is the mathematical notion of a group. There are all sorts of mathematical groups: the group of displacements, as found in geometry; the additive group that I have already referred to in the series of whole numbers; and any number of others. Algebraic structures are characterized by their form of reversibility, which is inversion in the sense I described above[2]. . .
>
> The second type of structure is the order structure. This structure applies to relationships, whereas the algebraic structure applies essentially to classes and numbers. The prototype of an order structure is the lattice, and the form of reversibility characteristic of order structures is reciprocity. We can find this reciprocity of the order relationship if we look at the logic of propositions, for example. In one structure within the logic of propositions, *P* and *Q* is the lower limit of a transformation, and *P* or *Q* is the upper limit[3]. *P* and *Q*, the conjunction, precedes *P* or *Q*, the disjunction. But this whole relationship can be expressed in the reverse way. We can say that *P* or *Q* follows *P* and *Q* just as easily as we can say that *P* and *Q* precedes *P* or *Q*. This is the form of reversibility that I have called reciprocity; it is not at all the same thing as inversion or negation. There is nothing negated here.
>
> The third type of structure is the topological structure based on notions such as neighborhood, borders, and approaching limits. This applies not only to geometry but also to many other areas of mathematics. Now these three types of structure appear to be highly abstract. Nonetheless, in the thinking of children as young as 6 or 7 years of age, we find structures resembling each of these three types [PIAG2: 25-26].

When Piaget refers to a "prototype" of a mother structure, what he means is "best example." A mathematical "group" is actually a quite advanced structure, and it is built up from simpler

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[2] Piaget's "inversion" refers to a negation, as when a number is added to its additive inverse (e.g., "subtraction") to return to the starting point of an action or scheme. In more mathematical language, an identify element in an algebraic structure ("0" in the case of addition, "1" in the case of multiplication) corresponds to an overall action that leaves the situation unchanged from its starting point. For example, a first action (opening one's mouth) negated by a second action (closing one's mouth) returns the subject to his initial situation, and this is a dynamical action-scheme equivalent to an identity element in a mathematical algebraic structure.

[3] It may be helpful to read this as "(*P and Q*)" and "(*P or Q*)". "And" and "or" in this quote designate the logic operations of conjunction and disjunction, respectively.

underlying structures by incorporating more and more properties into the transformations (relations). Everyone who knows how to add and subtract whole numbers is *practically* familiar with one type of mathematical group (called, naturally enough, "the additive group"), even if you might not have ever heard of the mathematician's technical explanation of how a "group in general" is defined. Fraction arithmetic, on the other hand, involves an even higher type of algebraic structure (it is called a "field"), and this is why primary school children have more trouble learning "fractions" than they do learning addition or multiplication. On the other hand, division using quotients and remainders (e.g. $4 \div 3 = 1$ with remainder 1) involves an algebraic structure intermediate to the group and the field (it is called a "ring")[4]. Primary school children typically have less trouble learning "quotients and remainders" than they do "fractions."

Piaget tended to trot out his "mother structure" discussions mainly when he was discussing the child at the "operations level" of development. It seems likely this might have been because the direct comparison of "operations" and "mother structures" is far less equivocal at this level than at the lower levels of simple sensori-motor schemes. As one descends from the mathematical "group" to lower level algebraic structures (monoids, semigroups, groupoids), one encounters less structure and, therefore, more difficulty in pointing to something and saying, "Look there! That is equivalent to a semigroup structure!" Besides, Piaget was not a mathematician and the "elegance" of putting his theory into these mathematical terms was likely not as obvious to him as it would have been to, say, Dieudonné. Whatever the case, it is nonetheless clear that Piaget's stages of child development all show increasing elaboration from lower, less constrained mathematical structures to more organized "higher" mathematical structures.

We cannot go into great descriptive depth for Piaget's findings in this textbook. After all, the subject properly belongs to psychology, and in 60 years Piaget compiled *a lot* of detail. Still, a few examples are appropriate here. One telltale sign of algebraic structure is found in the child's ability to make *classifications*.

> In children's thinking algebraic structures are to be found quite generally, but most readily in the logic of classes – in the logic of classifications. . . Children are able to classify operationally, in the sense in which I defined that term earlier, around 7 or 8 years of age. But there are all sorts of more primitive classifying attempts in the preoperational stage. If we give 4- or 5-year-olds various cutout shapes . . . they can put them into little collections on the basis of shape. . . They will think that the classification has been changed if the design is changed.
>
> Slightly older children will forego this figural aspect and be able to make little piles of similar shapes. But while the child can carry out classifications of this sort, he is not able to understand the relationship of class inclusion. . . A child of this age will agree that all ducks are birds and that not all birds are ducks. But then, if he is asked whether out in the woods there are more birds or more ducks, he will say, "I don't know; I've never counted them" [PIAG2: 27-28].

[4] For those readers who know more about abstract algebra, the net structure is called a "Euclidean domain."

Were we to follow up in more detail for this example, what we would see is that the child's classification scheme at this preoperational stage lacks what mathematicians call "the associative property" (as in "the associative property of addition"). This type of structure is an algebraic structure the mathematicians call a "groupoid." The transformations exhibit closure (ducks are birds; robins are birds; ducks and ducks are ducks; ducks and robins are birds) but not the associative property. For this child, $(A + A) - A = A - A = 0$, but $A + (A - A) = A + 0 = A$. Piaget called this structure a "grouping."

Although Piaget names "the lattice" as the prototype of order structure, a mathematical lattice is, like a mathematical group, a fairly advanced structure. The telltale sign of an order structure is a "partial ordering" sequence, e.g. $1 < 2$, $2 < 3$, etc. Reversibility in the case of order structure involves the discovery by the child of reciprocal relationships, e.g. if $1 < 2$ then $2 > 1$. Rudimentary instances of partial ordering appear at a very early age.

> A good example of this constructive process is seriation, which consists of arranging elements according to increasing or decreasing size. There are intimations of this operation on the sensori-motor level when the child of one and a half or two builds, for example, a tower of two or three blocks whose dimensional differences are immediately perceptible. Later, when the subject must seriate ten markers whose differences in length are so small that they must be compared two at a time, the following stages are observed: first the markers are separated into groups of two or three (one short, one long, etc.), each seriated with itself but incapable of being coordinated into a single series; next, a construction by empirical groping in which the child keeps rearranging the order until he finally recognizes he has it right; finally, a systematic method that consists in seeking first the smallest element, then the smallest of those left over, and so on. In this case the method is operatory, for a given element E is understood in advance to be simultaneously larger than the preceding element ($E > D$, C, B, A) and smaller than the following elements ($E < F$, G, etc.), which is a form of reversibility by reciprocity. But above all, at the moment when the structure arrives at completion, there immediately results a mode of deductive composition hitherto unknown: transitivity, i.e. if $A < B$ and $B < C$ then $A < C$ [PIAG4: 101-102].

It is noteworthy that very rudimentary partial orderings appear to be taking place in infants so young that the activities described above are still beyond their capability. Here are some examples of developed schemes involving feeding in infants only a few months old:

> Jacqueline, at 0;4 (27)[5] and the days following, opens her mouth as soon as she is shown the bottle. She only began mixed feeding at 0;4 (12). At 0;7 (13) I note that she opens her mouth differently according to whether she is offered a bottle or a spoon.
>
> Lucienne at 0;3 (12) stops crying when she sees her mother unfastening her dress for the meal.
>
> Laurent too, between 0;3 (15) and 0;4 reacts to visual signals. When, after being dressed as usual just before the meal, he is put in my arms in position for nursing, he looks at me and then searches all around, looks at me again, etc. – but he does not attempt to nurse. When I place him in his mother's arms without his touching the breast, he looks at her and immediately opens his mouth wide, cries, moves about, in short reacts in a completely different way. It is therefore sight and no longer only the position which henceforth is the signal [PIAG8: 60].

At birth and for the first days thereafter, the child's ability to feed depends strictly on innate

[5] 0 years; 4 months (27 days)

reflex, specifically the sucking reflex triggered by contact with the child's lips. Piaget documents how in the days and weeks which follow the infant begins to develop his "feeding scheme" through assimilation and accommodation [PIAG8]. By the ages in the preceding observations, the child is connecting various cues with initiation of sensori-motor schemes which have in his/her experience led to the satisfaction resulting from feeding. There are definite pronounced orders in the sequence of actions the infant goes through his or her various sensori-motor schemes, which is nothing else than an exhibition of the ability to put together *practical* sensori-motor partial orderings based on prior experience of relationships discovered by accident.

Evidence of rudimentary topological structure appears very, very early in life, and topological schemes are put in place by the child long before there is any evidence presented indicative of Euclidean conceptions of geometry:

> The third type of structure, according to the Bourbaki mathematicians, is the topological structure. The question of its presence in children's thinking is related to a very interesting problem. In the history of the development of geometry, the first formal type was the Euclidean metric geometry of the early Greeks. Next in the development was projective geometry, which was suggested by the Greeks but not fully developed until the seventeenth century. Much later still came topological geometry, developed in the nineteenth century. On the other hand, when we look at the theoretical relationships among these three types of geometry, we find that the most primitive type is topology and that both Euclidean and projective can be derived from topological geometry. In other words, topology is the common source for the other two types of geometry. It is an interesting question, then, whether in the development of thinking in children geometry follows the historic order or the theoretical order. More precisely, will we find that Euclidean intuitions and operations develop first, and topological intuitions and operations later? Or will we find that the relationship is the other way around? What we do find, in fact, is that the first intuitions are topological [PIAG2: 30-31].

Piaget goes on to illustrate this point through children's drawings and other illuminating behaviors. However, topological "intuitions" (as Piaget just called them) are evident even during the sensori-motor development stage of the infant.

> The most elementary spatial relationship which can be grasped by perception would seem to be that of 'proximity', corresponding to the simplest type of perceptual structurization, namely, the 'nearby-ness' of elements belonging to the same perceptual field. . . The younger the child, the greater the importance of proximity as compared with other factors of organization . . .
>
> A second elementary spatial relationship is that of *separation*. Two neighboring elements may be partly blended and confused. To introduce between them the relationship of separation has the effect of dissociating, or at least of providing the means of dissociating them. But once again, such a spatial relation corresponds to a very primitive function: one involved in the segregation of units, or in a general way, the analysis of elements making up a global or syncretic whole. . .
>
> A third essential relationship is established when two neighboring though separate elements are ranged one before another. This is the relation of *order* (or spatial *succession*). It undoubtedly appears very early on in the child's life . . . For example, the sight of a door opening, a figure appearing, and certain movements indicative of a forthcoming meal, form a series of perceptions organized in space and time, intimately related to the sucking habits. Inasmuch as the relations of order appear very early it is hardly necessary to point out that they are capable of considerable development in terms of the growing complexities of wholes. . .
>
> A fourth spatial relationship present in elementary perceptions is that of *enclosure* (or

> *surrounding*). In an organized series ABC, the element B is perceived as being 'between' A and C which form an enclosure along one dimension. On a surface one element may be perceived as surrounded by others, such as the nose framed by the rest of the face. . .
> Lastly, it is obvious that in the case of lines and surfaces there is right from the start a relationship of *continuity*. But it is a question of knowing in precisely what sense the whole of the perceptual field constitutes a continuous spatial field. For quite apart from the fact that the various initial qualitative spaces (such as the buccal, tactile, visual, etc.) are not for a long time coordinated among themselves, it has not been shown in any particular field, such as the visual, that perceptual continuity retains the same character at all levels of development. . .
> Generally speaking, it is also true that the basic perceptual relationships analyzed by Gestalt theory under the headings of proximity, segregation of elements, ordered regularity, etc. correspond to these equally elementary spatial relationships. And they are none other than those relations which the geometricians tell us are of a primitive character, forming that part of geometry called Topology, foreign to notions of rigid shapes, distances, and angles, or to mensuration and projective relations [PIAG10: 6-9].

To sum up: the selfsame structures claimed by the Bourbaki to be foundational for all of mathematics are also the most primitive structures observed in the development of intelligence in children. And this raises some fascinating questions.

§ 4. Is Neural Network Development Mathematical Structuring?

Two possible and opposing implications can be draw from the foregoing discussion. On the one hand, it may be that Piaget et al. saw mathematical structures in the phenomena of child development simply because mathematics is truly protean in its ability to precisely describe almost anything. If mathematics works in physics, in chemistry, in neuroscience, in economics, etc., etc., would it be all that surprising if it worked for the psychology of the development of intelligence as well? Most likely not. If this is the case, we might perhaps see the opening of a new field of application for mathematics, but probably nothing more profound than that.

On the other hand, let us recall the two aforementioned mysteries from the end of §3.2. Piaget did not find these structures in a few, in many, or even in most children studied. He found them in *all* children he studied. Furthermore, while different children pass through the different stages of development at different rates, *all* children pass through the *same* levels in exactly the *same* order[6]. The mental schemes, structures, ways of interpreting the world, etc. evolving in the process of developing intelligence from infancy to adulthood determine the way each and every one of us comes to think about and understand the world. At the most rudimentary levels of human intelligence, we are all much more the same than we are different. For all of us, up is up, the top of the Washington monument is the pointy end, carrots taste better than dirt, and so on. *Could it be that the structures of mathematics are what they are because human intelligence is structured the way it is?* If this hypothesis should turn out to be true, the two mysteries of

[6] I speak here of children with normal brain structure who are not impaired by brain damage of one kind or another. There are no equivalent facts on hand for children with severe brain pathologies.

mathematics cease to be mysteries at all. Epistemologically, things simply couldn't work out any other way.

Here is a place where it is worthwhile to recall how the definition of a "system" was described in chapter 1. You will remember that a "system" is jointly its object and our knowledge (model) of that object. Psychology long ago put to the test the "wax tablet" or "tabula rasa" notions of Aristotle, Locke, and the empiricist school of philosophy – namely that somehow or other the world "stamped" itself into our minds as some sort of copy-of-reality – and found this long-standing idea of these philosophers to be contrary to the psychological facts. Indeed, Piaget's research contains page after page of experimental results that refute this. At the same time, though, the happy notion of the rationalists – namely, that human beings possessed fully clothed and ready-to-go innate ideas by which we understand the world – is likewise refuted by the experimental findings of developmental psychology. There is nothing innate about our objective understanding of the world; as Kant said long ago, "all knowledge begins with experience." But what Piaget *does* find to be innate in the development of intelligence is the toolset of structuring, arising from a primitive, hereditary biological substratum and *extended by experience* to make the complex phenomenon we commonly call "intellect."

> Two conclusions seem to us to derive from the foregoing discussions. The first is that intelligence constitutes an organizing activity whose functioning extends that of the biological organization, while surpassing it due to the elaboration of new structures. The second is that, if the sequential structures due to intellectual activity differ among themselves qualitatively, they always obey the same functional laws. . .
>
> Now, whatever the explanatory hypotheses between which the main biological theories oscillate, everyone acknowledges a certain number of elementary truths which are those of which we speak here: that the living body presents an organized structure, that is to say, constitutes a system of interdependent relationships; that it works to conserve its definite structure and, to do this, incorporates in it the chemical and energetic aliments taken from the environment; that, consequently, it always reacts to the actions of the environment according to the variations of that particular structure and in the last analysis tends to impose on the whole universe a form of equilibrium dependent on that organization. In effect . . . it can be said that the living being assimilates to himself the whole universe at the same time that he accommodates himself to it . . .
>
> Such is the context of previous organization in which psychological life originates. Now, and this is our whole hypothesis, it seems that the development of intelligence extends that kind of mechanism instead of being inconsistent with it. In the first place, as early as the reflex behavior patterns and the acquired behavior patterns grafted on them, one sees processes of incorporation of things to the subject's schemes. This search for the functional aliment necessary to the development of behavior and this exercise stimulating growth constitute the most elementary forms of psychological assimilation. In effect, this assimilation of things to the scheme's activity . . . constitutes the first operations which, subsequently, will result in judgments properly so called: operations of reproduction, recognition and generalization. Those are the operations which, already involved in reflex assimilation, engender the first acquired behavior patterns, consequently the first nonhereditary schemes . . . Thus every realm of sensori-motor reflex organization is the scene of particular assimilations extending, on the functional plane, physio-chemical assimilation. In the second place, these behavior patterns, inasmuch as they are grafted on hereditary tendencies, from the very beginning find themselves inserted in the general

> framework of the individual organization; that is to say, before any acquisition of consciousness, they enter into the functional totality which the organism constitutes. . .
> In short, at its point of departure, intellectual organization merely extends biological organization [PIAG8: 407-409].

As convinced, and as convincing, as he was that the origin of intelligence lies in biological structure, and that intelligence – which he viewed as essentially a process of adaptation – merely extended biological organization, Piaget never speculated on what implications his findings and his theory might have for neural organization. To do so, given the state of knowledge in neuroscience throughout most of Piaget's long lifetime, would have been totally out of character with the patient, step-by-step, methodical way in which he conducted his scientific research. Piaget very rarely speculated, at least in print, about anything. In his landmark *The Origins of Intelligence in Children* and in many of his other numerous books, we find Piaget setting up all the likely alternative hypotheses that might explain the empirical findings. One by one, he would show where hypotheses fell short of the facts until he had just one left uncontradicted by the facts. To say Piaget was a patient man is a bit like saying the weather gets nippy in Siberia during the winter.

Today we know for a fact that synaptic connections in the brain are modified by experience. In computational neuroscience, we describe this by saying that the neural networking of the brain undergoes ***adaptation***. Now, Piaget wrote and spoke at length about adaptation, and he produced one of the best functional definitions for this term ever set into print: ***An adaptation is the equilibrium of assimilation and accommodation***. In computational neuroscience we have many different models of adaptation. Many are *ad hoc*, some are speculative, most are based on mathematical principles. It is pertinent to note here that one class, those used in ART maps, are designed to address what is known as the ***stability-plasticity dilemma***. Now, *stability* in a neural map or a neural network has in all its essentials the exact same connotation as *assimilation*. *Plasticity*, in contrast, has in all its essentials the exact same connotation as *accommodation*. What ART adaptation algorithms do is balance the two, which is precisely the role Piaget found for adaptation.

Let us dare to do what Piaget would not; let us make a speculative conjecture. Let us make the conjecture that synaptic adaptation and modulation processes in neural networks are such as to result in the development of *mathematical structures*. If so, and if the Bourbaki have not misled us, the structures coming out of neural adaptation would be those which constitute the makeup of the three mother structures and the differentiations and combinations among those structures that lead to more complicated (hybrid) mathematical structures.

What does such a conjecture, raised to the status of an hypothesis, do for us? We earlier raised the issue of the long-standing and unsolved mystery of the putative "neural code" every computational neuroscientist thinks must exist for the brain to process information. The history of the science has seen only a few ideas for how to begin to attack this question: firing rates, correlations, and the statistics of signal processing are the three main ideas that have pursued by the science. (In his very last work [NEUM3], John von Neumann advanced the statistical processing idea). None of these ideas have been all that widely successful to date, and all of them suffer from the same basic defect: *they lack a clear connection to psychological consequences*. Von Neumann long ago wrote,

> I suspect that a deeper mathematical study of the nervous system . . . will affect our understanding of the aspects of mathematics itself that are involved. In fact, it may alter the way in which we look on mathematics and logic proper [NEUM3: 2].

If the conjecture that "neural adaptation constructs mathematical structures" should one day prove to be successful, if the putative "neural code" should be found to subsist in the actions of this structuring process, then von Neumann's conjecture would have to be seen as one of the most prescient guesses in the history of science. It is a fundamental tenet of neuroscience that all behavior can ultimately be tied back to neural (and perhaps also glial) behavior. *The documented existence of mathematical structures in human behavior therefore implies neural networks might form corresponding mathematical structures in the development of the central nervous system*.

At the time of this writing, there is no great research program being carried out on the conjecture presented here, unless it be hiding in a room somewhere with the curtains drawn. If there is to be such a research program, one would need to know in more technical detail just what the object of the search looks like. In the eye of the mathematician, what are algebraic structures, order structures, and topological structures?

§ 5. The Mathematics of Structure

This is the point where the discussion must necessarily turn technical and somewhat abstract. Although the reader is encouraged to carry on here, in fairness it must be said that the larger contextual ideas and facts have already been covered. In what follows, the discussion devolves to the "nits and grits" of abstract mathematics.

It will be assumed the reader is already familiar with the ideas of a set and members of a set. For the purposes of this textbook it is sufficient to consider only finite sets, i.e. sets with a finite number of members. The number of members in a set, denoted $\#(A)$, is called the *cardinality* of the set.

At the risk but without the intent of offending some mathematicians, a great deal of set theoretic mathematics basically amounts to bookkeeping. There is a handy "bookkeeping device" used by mathematicians and called the ***Cartesian product*** of two sets. If A and B are sets, the Cartesian product of the two sets, denoted $A \times B$, is a set composed of the ***ordered pairs***, $\langle a, b \rangle$, of all members a of set A (denoted $a \in A$) and all members $b \in B$. The order is important. $\langle a, b \rangle$ is not the same as $\langle b, a \rangle$. For example, A might represent a set of possible speedometer readings for your car and B might represent a set of possible fuel gauge readings. Then $A \times B$ would be a set of instrument readings $\langle$speed, fuel$\rangle$. If $C = A \times B$, then $\#(C) = \#(A) \cdot \#(B)$.

The Cartesian product is a handy tool because we can use it to describe situations of arbitrary complexity. For instance, suppose we had to deal with three sets, A, B, and C. We can form the Cartesian product of all three, $A \times B \times C$, to obtain an ordered triplet, $\langle a, b, c \rangle$. *By definition* the Cartesian product is associative, that is, $A \times B \times C = (A \times B) \times C = A \times (B \times C) = D \times C = A \times E$.

A ***binary relation*** between two sets, A and B, is a rule specifying some subset of $A \times B$. It is permitted for the two sets to be the same, i.e. $A \times A$, but this is not required. Because either or both of the sets can be the result of another Cartesian product, e.g. $B = C \times D$, we need only work with binary relations even when the mathematics problem we are working on actually involves more than two sets. Likewise, there is a formal trick we can play that lets us use the idea of a binary relation even when we have only one actual set. A ***unary relation*** on a set A is a rule specifying some subset of A. Suppose we introduce a special set O that has exactly one member, and let us call this member "nothing." (We could call O the "no-set"). Then we can *define* the Cartesian product of A and O as $A \times \mathrm{O} = \mathrm{O} \times A = A$. This trick allows us to use the idea of the binary relation even for dealing with unary relations. You can probably see now why the Cartesian product is such a handy bookkeeping device.

Special sets such as O are frequently defined by mathematicians as formal devices for keeping notations and ideas as simple as possible. Another such formal device is the "set with no members" $\varnothing$, variously called the ***null set*** or the ***empty set***. For people who are not mathematicians, the null set is often a very strange idea because it seems to be the same thing as nothing. After all, how and even why should one go to the trouble of defining a set that has no members in it? Isn't that a rather absurd idea? No, not really. Suppose we have two sets, A and B, and further suppose we have some binary relation ρ. Finally, let us suppose that A and B are *unrelated* so far as relation ρ is concerned. If we use the symbol R_ρ to denote the subset of $A \times B$ defined by relation ρ, the set R_ρ will have no members because A and B are unrelated by ρ. Put in formal notation, we would say $R_\rho = \varnothing$. Thus, one use for the empty set is to allow us to *formally* say things like "there is no relation between A and B" in mathematical notation. Similarly, if we were to define a set A as, for example, "the set of living dodos," we would have to say $A = \varnothing$

because there are no living dodos. Thus, the idea "does not exist" is mathematically expressible through the formal use of the empty set.

It needs to be mentioned that mathematicians generally do not use the symbol O. Instead, they sometimes use $\{\varnothing\}$ to express the idea for which we are using O, and at other times they use $\varnothing$ to express the ideas of no-relation or "does not exist." In no small part this is because the gospel according to Hilbert exhorts them to deliberately not assign meanings to mathematical symbols and to focus instead on purely formal aspects of mathematics. This is part of the baggage left over from the failed attempts of formalism to deal with "the crisis in the foundations" that engaged the leading mathematicians in the early twentieth century. Your author sees no reason why the rest of us should be burdened by the pseudo-philosophy of formalism when this pseudo-philosophy gets in the way of our being able to *precisely* say what we mean when we use mathematics. The *formal* difference between $\varnothing$ and O is $\#(\varnothing) = 0$ and $\#(O) = 1$, and this is enough of a reason for us to distinguish them here.[7] But enough of this digression; let us get back to business.

Let the set $R_\rho \subseteq A \times B$ denote a binary relation ρ between sets A and B. A very important special case occurs when every member $a \in A$ appears in exactly *one* member $\langle a, b\rangle \in R_\rho$. It is important here to understand that this restriction does not forbid some other member $c \in A$ from appearing in another $\langle c, b\rangle \in R_\rho$ with the same $b \in B$. A relation R_ρ having this property is called a ***mapping*** or a ***transformation*** or, more typically, a ***function*** (the mathematics community uses all three terms; they are synonyms). For example, the "square root" is a binary relation but it is not a function because both +2 and –2 are square roots of 4. The "square" on the other hand *is* a function and it doesn't matter that $(+2)^2$ and $(-2)^2$ both equal 4. In the case of the square root, both $\langle 4, +2\rangle$ and $\langle 4, -2\rangle$ are members of R_ρ. In the case of the square, $\langle +2, 4\rangle$ and $\langle -2, 4\rangle$ are both members of R_ρ, but this does not violate the property of a function. You should note that the ordered pair convention being used here is ⟨number being related, number to which it is related⟩. The function is such an important special case of the binary relation that it is given its own special notation, $\rho\colon A \to B$. When we want to speak of a specific member $a \in A$, $\rho(a) \mapsto b$ is the most commonly used notation. Set A is called the *domain* of ρ and set B is called the *codomain* of ρ.

We are now ready to talk about the three types of mathematical structure. Structure is introduced into mathematics through specification of the binary relations involved and through

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[7] There will be some mathematicians – probably a lot of them, in fact – who would vigorously protest what your author has just said. One reason for this is that the O vs. $\varnothing$ distinction messes up some of the arguments that go into the axiom system most widely used in axiomatic set theory, namely the Zermelo-Fraenkel-Skolem axiom system. Your author does not tell anyone what set of axioms they must use, but he does claim the right to not be bound by their decision. Gödel granted that right. As a peace offering, he points out that O is pretty much the same thing as the "singleton" construct used in the ZFS system.

the definition of certain limiting or distinguishing properties to be imposed on the binary relations. Some binary relations with specific properties turn out to have very wide scopes of application, and these are used in the mathematics you are probably accustomed to using. However, there is no law or rule of mathematics dictating any obligation to use any particular set of binary relations or any particular set of properties relations must have. It sometimes turns out that there are scientific problems very difficult to handle under those mathematical structures one is most accustomed to using, but which are very simple to handle if one defines a different set of relations and properties. Mathematics as a whole is general enough to cover an enormous range of special applications, and no one is forbidden to "invent his own system" for a particular application. Structure theory is basically a doctrine for teaching one how to do this.

§ 5.1 Algebraic Structures

Algebraic structures make use of a further specialization on the idea of a function. A ***binary operation on a set*** is a function that maps the Cartesian product of a set with itself back into the original set. Symbolically, $\rho: A \times A \to A$. Suppose A is the set of natural numbers and ρ is the usual addition operation. If we use the abstract notation of set theory, $1 + 2 = 3$ would be written as $\langle 1, 2 \rangle \mapsto 3$ where it has to be understood that the mapping operation is addition. In many cases, an algebraic structure has more than one defined operation, e.g. addition *and* multiplication. For cases such as these, it is conventional to give each operation a specific symbol (e.g. "+" for addition) and to write a specific transformation in the "$1 + 2 \mapsto 3$" form so that it is more immediately apparent which one of the operations is being used.

The idea of a binary operation on a set adds an important property constraint to the idea of a function, namely the property of ***closure***. The product $A \times A$ contains every possible pair of members of A, and the binary operation is defined so that all possible pairs are matched up with a member of A. Because for finite sets there are more members in $A \times A$ than in A, an information theorist would say a binary operation is always information lossy.[8]

Let us denote a set with a binary operation by the symbol $[A, \rho]$. If no further properties are specified for ρ, i.e. if the only property defined for ρ is that it is a binary operation, $[A, \rho]$ is then the simplest species of algebraic structure, and it is called a ***groupoid***. Thus, a groupoid is an algebraic structure whose only defining property is that it has closure.

One can readily see that a groupoid does not have very much structure. For that reason most

[8] A *function*, in contrast to a binary *operation*, is not necessarily information lossy. This is because in general a function maps a set A to another set B. If no two members of A are assigned to the same member of B, then knowing a member b uniquely identifies the member a that mapped to it, and the function is said to be information *lossless*. Functions of this sort are said to be "one-to-one" functions.

mathematicians do not find it to be very interesting. However, there is at least one thing very interesting about groupoids. All the innate schemes and many of the early acquired schemes documented by Piaget in the sensori-motor phase of the infant's development form groupoids.

Suppose we take a groupoid $[A, \rho]$ and endow ρ with the property of being associative. The resulting structure is then called a ***semigroup***. Unlike groupoids, semigroups do not generally appear to be innate scheme structures in human development, although this doesn't rule out the possibility that there might be some innate scheme structures that are semigroups. However, it is the case that schemes which form semigroups do develop over time.

Next suppose A has a member e such that groupoid $[A, \rho]$ is found to possess the property that for every member $a \in A$ we have $\langle a, e \rangle \mapsto a$. Member e is then called a ***right identity*** of A. On the other hand, if $\langle e, a \rangle \mapsto a$ then e is called a ***left identity*** of A. If both are true, then member e is called a ***two-sided identity*** of the groupoid. In this last case, we call this structure a ***groupoid with the identity***. Note that it is quite meaningless to say of a set A that some member of the set is an identity. The identity is an idea that requires not only a set A but also a binary operation. For example, in regular, everyday arithmetic "0" is the "additive identity" but "1" is the "multiplicative identity." For the first two cases above, we can call the structures a ***groupoid with right identity***, and a ***groupoid with left identity***, respectively.

As simple and relatively unconstrained as these groupoid structures are, there is nonetheless enough structure here to produce an unexpected consequence: if a groupoid has a left identity and a right identity, then *there is only one unique identity and it is a two-sided identity*.

To see this, let us replace our $\langle a, b \rangle \mapsto c$ notation with the notation $a \circ b \mapsto c$ where $\circ$ denotes the binary operation ρ applied to members a and b. Now suppose member d is a left identity and member e is a right identity of A but that d is not the same as e. Then it must be true that $d \circ e \mapsto e$ because d is a left identity. But it must also be true $d \circ e \mapsto d$ because e is a right identity. The only way to avoid a contradiction is if d and e are the same member and this member is both a left and a right identity. Furthermore, if we say there is some other right identity e' not the same as e, the same argument will lead to the same contradiction (and similarly if we suppose there is some left identity d' not the same as d). On the other hand, if a groupoid has only a left (or a right) identity, it can have more than one.

A semigroup inherits the identity properties if its underlying groupoid contains any identity members. We can thus speak of a semigroup with left identity, a semigroup with right identity, and a semigroup with a two-sided identity. This last is also called a ***semigroup with the identity*** because we then know the identity is a unique member of A.

Suppose [A, ρ] is a groupoid with a left identity e. Further suppose A has some member b such that $b \circ a \mapsto e$ for some member a. Then b is called the ***left inverse of a***. A similar definition can be made for the right inverse of a member a. Now, an interesting consequence results if [A, ρ] is not merely a groupoid but rather is a semigroup. Then $b \circ e \circ a = b \circ (e \circ a) = (b \circ e) \circ a$ by the associative property of a semigroup. For the middle term, $e \circ a \mapsto a$ because e is a left identity. But for the third term we cannot say $b \circ e \mapsto b$ unless e is a two-sided identity, which must be the case if the associative property is not to be violated. A similar result holds if we say e is a right identity. Therefore, if [A, ρ] is a semigroup with any identity member and there is a left (or right) inverse for any member a of A, then [A, ρ] is a semigroup with the identity. A semigroup with the identity is called a ***monoid***. One can easily appreciate that a monoid has "more structure" than either a semigroup or a groupoid with only left or right identity because its binary operation has both associativity and a unique identity for A.

The point of bringing this up is to illustrate that structure implies constraint, and the more structure there is, the more constraints attend it. For a groupoid with left identity e, there can be some member b for which A contains a left inverse a, $a \circ b \mapsto e$, and another member c that is a right inverse of b, i.e. $b \circ c \mapsto e$. There is no contradiction here. But if [A, ρ] is a semigroup, then, as we have just seen, it is not merely a semigroup but, rather, a monoid. And this, in turn, has yet another consequence, namely that a and c must be one and the same member of A. To see this, we note that the properties of the structure imply $a = a \circ e = a \circ (b \circ c) = (a \circ b) \circ c = e \circ c = c$. This means, of course, that $a = c$ is both left- and right-inverse of b.

Suppose [A, ρ] is a monoid, and further suppose that for every member of A there is an inverse. We already know that every inverse is both left- and right-inverse and e is the two-sided identity. Because $e \circ e \mapsto e$, e is its own inverse. This structure is called a ***group***, and by the time we get to this level of structure, we have a very powerful structure indeed ("powerful" in the sense that a group has a great many properties attending it just by virtue of how a group is defined). Entire books have been written on the properties of groups. We will look at an example of a group in the next section. When one says of a mathematician, "He studies group theory," this means the work of the mathematician involves the discovery of new properties *implicit* in a mathematical group. Such discoveries come in the form of theorems, and this gives the rest of us a glimpse, incomplete to be sure but a glimpse nonetheless, of what a mathematician does. We have seen here a few very simple examples of such theorems, such as the one above concerning left- and right-inverses in a monoid.

For the structures we have seen so far, the property of commutativity has not been a required property of the structure. That is, it has not been a requirement that $a \circ b = b \circ a$ for every pair of members in A. Naturally, commutativity has not been forbidden either; it merely has not been one of the defining properties of the structures. If $[A, \rho]$ is a group and if, furthermore, the commutative property is found for every pair of members in A, then $[A, \rho]$ is called an ***Abelian group***, named after the Norwegian mathematician Niels Abel. Common, everyday addition is an example of Abelian group structure so long as we are dealing only with finite results.[9] Any one of the lower structures we have considered could also have commutative species tallied among them, but none of these have official names other than such obvious descriptive ones as "commutative monoid."

So far we have only looked at algebraic structures comprised of a set and one binary operation. There are also algebraic structures defined with two or more binary operations. We will confine ourselves to just a couple of examples that happen to be of great importance in mathematics, with the understanding that these are not the only ones possible. We will need a pair of binary operations, and we will designate these using the symbols $\circ$ and $*$.

Let $[A, \circ]$ be an Abelian group and let $[A, *]$ be a semigroup. If, for every triplet of members a, b, and c in A, one or the other of the following two ***distributive properties*** holds,

$$a * (b \circ c) = (a * b) \circ (a * c), \text{ or}$$
$$(b \circ c) * a = (b * a) \circ (c * a),$$

then the structure $[A, \circ, *]$ is called a ***ring***. The new factor in play with this structure is the idea of *properties that relate the outcomes of the individual binary operations to one another*. If the structure $[A, \circ, *]$ is a ring and if, in addition, semigroup $[A, *]$ is commutative, then *both* distributive properties hold and the structure $[A, \circ, *]$ is called a ***commutative ring***.

We know $[A, \circ]$ has an identity e because it is a group. If $[A, *]$ is not merely a semigroup but rather a monoid, then $[A, *]$ likewise has an identity (generally different from e). If $[A, \circ, *]$ is a ring, then in this case (i.e. the case where $[A, *]$ is a monoid), it is called a ***ring with identity***. Finally, if $[A, \circ, *]$ is a ring with identity and also the monoid $[A, *]$ is commutative, then the ring is called a ***commutative ring with identity***.

When $[A, \circ]$ is a group with identity e and $[A, *]$ is a monoid with identity f, it will generally be the case that $[A, *]$ is not a group because A will not be found to contain any member a for which either $a * e = f$ or $e * a = f$. In other words, $[A, *]$ will not have an inverse for e. For

[9] If we allow ourselves to regard "infinity" – denoted by ∞ – as a member of A, the group structure is broken because ∞ has no unique inverse defined in $[A, +]$. "$\infty - \infty$" is not necessarily 0. Unpleasant facts like this plus the fact that no experimental quantity is ever "infinite" or "infinitesimal" are the reasons this book confines itself to only finite mathematics. We leave "the infinite" and "the infinitesimal" to the philosophers, the theologians, and those mathematicians whose work is called "mathematical analysis."

example, there is no multiplicative inverse for 0 in everyday arithmetic, and 0 is the additive identity in everyday arithmetic. However, it might be the case that if we form a subset B defined by simply excluding e from B while including all the other members of A, then $[B, *]$ might be a group. If $[A, \circ, *]$ is a ring *and* $[B, *]$ is an Abelian group, then $[A, \circ, *]$ is called a ***field***. Fields are among the most powerful structures in all of mathematics (in the sense that fields have the greatest number of properties accruing merely from the fact that they *are* fields). For the overwhelming majority of people (excluding mathematicians, mathematics students, and a few physicists and engineers), all the mathematics they know works *because* it is based on fields.

All the example algebraic structures we have looked at are examples based on binary operations, i.e. operations operating on two members of A. However, as mentioned earlier, by using the bookkeeping method of the Cartesian product, all of this is easily extended to include operations that operate on n members of A, where n is a non-negative integer. Such an operation is called an n-ary operation. The purposes of this textbook do not require that we belabor this point; it is enough to say this is so. If we have a structure $[A, \rho_1, \rho_2, \ldots, \rho_k]$ where each ρ_j, is an n_j-ary operation (n_j being a non-negative integer), $[A, \rho_1, \rho_2, \ldots, \rho_k]$ is called a ***universal algebra***. We can thus see why the structures we have been discussing are called "algebraic structures."

§ 5.2 Example: The Permutation Group

The training one receives in any scientific discipline has a tendency to produce modes of thinking and ways of looking at the world that we might call "the habits of the discipline." Habits of the discipline are not to be sneered at because these habits promote productivity in scientific research and are one key vehicle by means of which scientists are able to communicate with one another. Unfortunately, habit also has a tendency to narrow one's thinking and blunt the imagination when dealing with subjects outside one's own discipline, and this is something quite disadvantageous, especially when it comes to understanding the abstract ideas of pure mathematics. For this reason, it is worth our while to make a brief digression and examine a particular example of an algebraic structure. The one we will look at is called a ***permutation group***. This particular example is chosen because, on the one hand, it is a very simple instance of a group and, on the other hand, it will be unfamiliar to the majority of readers and for that reason more instructive.

If we are given a collection of objects, arranged in some particular order or pattern, and we alter this arrangement by swapping objects around but otherwise leaving the basic pattern unchanged, this swapping around is called a ***permutation***. In this sense of the word, a permutation is an *action* one performs on the collection of objects. The members of the set G in the permutation group are the swapping actions themselves, not the things being swapped around.

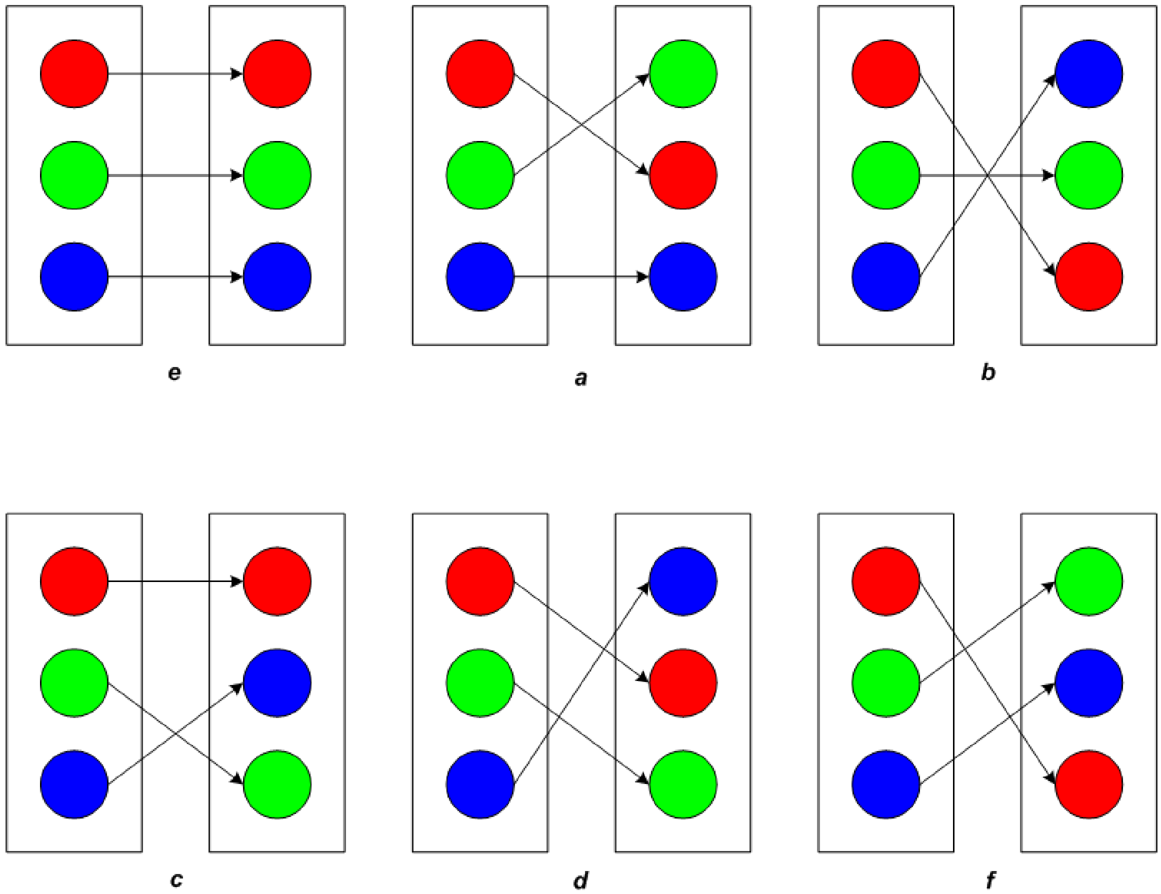

Figure 10.1: Illustration of a permutation group. Permutations *a* through *f* are denoted by the arrows illustrating the rearrangement of the balls from their initial to final positions. Permutation *e*, which leaves the positions of the balls unchanged, serves as the identity of the permutation group.

Figure 10.1 illustrates the setup for a simple permutation group. We have an arrangement of three colored balls, the positions of which can be swapped. The specific colors of the balls in no way matters in the permutation group. The only thing that matters is how the arrangement is changed. Although the balls are initially set up as red, green, blue from top to bottom, this is not relevant so far as the group structure is concerned. There are six different ways we can permute three balls, and these are shown in the figure. Each unique permutation is designated by a letter, and our set G is defined to be the set of these permutations, i.e. $G = \{e, a, b, c, d, f\}$.

A set all by itself does not constitute a structure, and so we also need a binary operation to go with G. This operation will be denoted by the symbol $\bullet$. In the permutation group this operation is ***concatenation***, i.e. a succession of permutations. The operation $a \bullet b$ thus denotes performing permutation a on the initial arrangement followed by performing permutation b on the results of the first permutation. Referring to Figure 10.1, a produces the arrangement (green, red, blue) from top to bottom; applying b to this new arrangement produces (blue, red, green). We now take note that the outcome of this concatenation of permutations is equivalent to what we would have gotten by applying permutation d to the initial arrangement; therefore $a \bullet b$ is equivalent to permutation d. Symbolically, $a \bullet b = d$. We may next note that permutation e leaves the arrangement unaltered. Therefore for any permutation x belonging to G we will get $x \bullet e = e \bullet x = x$. Permutation e therefore is the identity for $[G, \bullet]$. Next, we note that some concatenations end up with the permutations canceling each other, i.e. $f \bullet d = e$ and $d \bullet f = e$. Thus permutation d is called the inverse of f, and f is likewise the inverse of d. In some cases a permutation can cancel itself, i.e. $a \bullet a = e$, thus a is its own inverse.

By taking the permutations in pairs we can construct an *operation table* for operation $\bullet$. An operation table is similar to the addition table we all learned in elementary school. This operation table is shown in Figure 10.2 below. The table is arranged so that the order of the operations is from row to column, e.g. $a \bullet b = d$, etc. Referring to our earlier definition of a group, it is easily seen by inspecting the table that the closure property is satisfied. Inspection of the e row and the e column demonstrates that the identity property is likewise satisfied. Each row of the table has in it exactly one occurrence of e, and each column of the table has in it exactly one occurrence of e. This means that for each member in G there is exactly one member that acts as its inverse, and so the existence-of-inverses property of the group is satisfied. That the associative property is also satisfied is not so self-evident from the operation table, but this can be demonstrated by taking all triplets of permutations and applying the operation term by term across the triplet. For example,

$$(a \bullet b) \bullet c = d \bullet c = b; \qquad a \bullet (b \bullet c) = a \bullet d = b$$
$$\Rightarrow a \bullet b \bullet c = (a \bullet b) \bullet c = a \bullet (b \bullet c).$$

The same is found to be true for all triplets in G and so the associative property is established. This completes the proof that the permutation group is indeed a group.

$\bullet$	a	b	c	d	e	f
a	e	d	f	b	a	c
b	f	e	d	c	b	a
c	d	f	e	a	c	b
d	c	a	b	f	d	e
e	a	b	c	d	e	f
f	b	c	a	e	f	d

Figure 10.2: The operation table for the concatenation operation. The table is read from row to column such that, for example, $a \bullet b = d$ and $b \bullet a = f$.

The permutation group has another interesting property. If we extract from G the subsets

$$G_1 = \{a, e\};\ G_2 = \{b, e\};\ G_3 = \{c, e\}; \text{ and } G_4 = \{d, e, f\}$$

and apply Figure 10.2 to construct operation tables for these subsets, we find that each of these subsets by itself also constitutes a group (that is, each by itself satisfies all the properties that define a group). The reader is invited and encouraged to try it and to verify this statement. These four groups are called the ***subgroups*** of group $[G, \bullet]$. We may also note there are many cases where operations in group $[G, \bullet]$ do not commute. For example, $a \bullet b = d$ but $b \bullet a = f$. Thus, the order in which the operations are applied makes a difference. The permutation group is therefore an example of a non-Abelian group.

Note that the union of these four subsets gives us the set G. Note further that the only things each subgroup has in common with the others are the inclusion of the identity e and the operation $\bullet$. The group $[G, \bullet]$ is *not* obtained simply as the union of the four subgroups because this union leaves undefined the results of operations such as $a \bullet d$ (note that this operation is not defined by any of the operation tables of the four subgroups).

We can take advantage of the existence of the four subgroups to bring up an analogy the reader may find extremely interesting. The pure mathematics involved here stand on their own, but what might we use this group to represent outside the environs of pure math? Since the permutations are actions anyway, suppose we liken them to Piaget's sensori-motor schemes of action. The balls we can liken to sensations serving as aliments of the schemes. Now, for the infant in the first several months of life, different types of reflex and acquired sensori-motor schemes (e.g. haptic, visual, tactile, etc.) are initially uncoordinated. We can liken these to the four subgroups of $[G, \bullet]$. If we are given only the four subgroups and asked to find the group for the set G we would not be able to do so using only the operation tables of the four subgroups. Instead we would have to *construct* the entries missing from the operation table for $[G, \bullet]$ by going back to Figure 10.1 and finding out what happens when we concatenate all the missing combinations such as $a \bullet b$. In other words, we would have to *try it* and "see what happens." This is analogous to Piaget's idea of *structuring* a system by means of assimilation and accommodation. In the process of building up group $[G, \bullet]$ we would be said to "coordinate" the four original subgroups; this is analogous to Piaget's "coordination of sensori-motor schemes."

§ 5.3 Topological Structure

If we regard the color of the balls as analogous to different sensational matter in sensibility, the color-blind "nature" of the "movements" means that the "motor schemes" (to continue the analogy) do not intrinsically depend on the details of the sensational matter. In Figure 10.1 we used an initial order of (red, green, blue) to illustrate the group, but we could have used any other initial order involving these three colors equally as well. We used the sensational matter in order to figure out the equivalences in the operation table, but not to describe the members of *G* or the subgroups.

From this point of view, the "motor schemes" (permutations) in and of themselves do not involve the specific perceptual details "in a sensation" and in this sense can be called "practical" representative factors in a scheme. Piaget found that in order to describe the behaviors of infants in the early sensori-motor stages of life, he had to do so in the context of what he called "practical sensorimotor spaces."

At the time when Piaget was first publishing his landmark studies, in the late 1940s and early 1950s, psychology subscribed to the idea of an innate perceptual geometry in explaining human perception. Simply put, Piaget's work overturned this viewpoint.

> According to currently accepted explanations of the perceptual process, every perceptual 'field', from the most elementary to the most highly developed, is organized in accordance with the same type of 'structure'. This organization is supposed to be of a geometrical character right from the start, quite apart from the effects of the laws of 'good gestalt', and to involve the immediate formation of perceptual constancies of shape and size. This would mean that at any age a baby could recognize the shape of an object independent of perspective, and its size apart from its distance. Thus there would be from the very outset a perception of relationships at once spatial and metric. If this hypothesis were correct it would only be necessary to call to mind the laws of spatial configurations in order to describe perceptual space.
>
> But we have already shown in the work referred to above[10] that the constancy of the shape of objects is far from being complete at the outset, since at 7 or 8 months a child has no idea of the permanence of objects, and does not dream of reversing a feeding bottle presented to him the wrong way round[11] . . . Since then we have shown, together with Lambercier, that as regards size constancy, great differences still persist between an 8-year-old child and an adult, whilst Brunswik and Cruikshank have demonstrated its absence during the first six months of existence. It is thus by no means absurd to suppose that perceptual relationships of a projective order (perspective) and of a metric order (estimation of size at varying distances) should appear later than these more elementary spatial relationships whose nature has first to be defined. It is also quite obvious that the perception of space involves a gradual construction and certainly does not exist ready made at the outset of mental development [PIAG10: 5-6].

Yet, although the most elementary constitution of projective or metric geometry is found to be absent in the child during the first two stages of sensorimotor intelligence, it also appears to be the case that the earliest perceptions are not entirely lacking in structural form. Piaget and Inhelder go on to say,

> The first two stages of development are marked by an absence of coordination between the various sensory spaces, and in particular by the lack of coordination between vision and grasping – visual and tactile-kinaesthetic space are not yet related to one another as a whole. . .
>
> It is therefore necessary . . . to try to reconstruct the spatial relations which arise in primitive or rudimentary perceptions (e.g. in the exercise of the reflexes of sucking, touching, seeing patches

[10] *The Construction of Reality in the Child.*
[11] Observation 78 in *The Construction of Reality in the Child.*

> of light, etc., and the earliest habits superimposed on these reflexes). But since these initial perceptions fail to attain constancy of size and shape, what sort of relations go to make up such a space? [PIAG10: 6].

Piaget and Inhelder were able to identify five relationships which appear to be elementary constituents of the baby's earliest perceptions. All five undergo extensive development and change as the baby ages. The five "elementary spatial relations" are *proximity*, *separation, continuity*, *enclosure*, and *relations of order*. These relations are those found in ***topological structure***.[12]

Now, what do the terms "topological" and "topology" denote when used as technical and mathematical terms? Mathematicians define "topological space" in the following way:

> **topological space –** A set together with sufficient extra structure to make sense of the notion of continuity when applied to functions between sets. More precisely, a set X is called a topological space if a collection T of subsets of X is specified satisfying the following three axioms:
> (1) the empty set and X itself belong to T;
> (2) the intersection of two sets in T is again in T;
> (3) the union of any collection of sets in T is again in T.

The collection T of subsets T_i is usually called "the topology of X." No doubt most readers will find this definition somewhat opaque since a great deal of training in mathematics is required in order to appreciate the implications of this definition. To quote *The Penguin Dictionary of Mathematics*, topology is

> the study of those properties of geometrical figures that are invariant under continuous deformations (sometimes known as "rubber sheet geometry"). Unlike the geometer, who is typically concerned with questions of congruence or similarity of triangles, the topologist is not at all interested in distances and angles, and will for example regard a circle and a square (of whatever size) as equivalent, since either can be continuously deformed into the other. Thus such topics as knot theory belong to topology rather than to geometry; for the distinction between, say, a granny knot and a reef knot cannot be measured in terms of angles and lengths, yet no amount of stretching or bending will transform one knot into the other.

Topology is concerned with defining such things as "what points are in some sense 'neighbors' of a specific point *x*," and with providing a rigorous means by which we can define "neighborhoods" of points. To the topologist the inside of your stomach, when your mouth is open, is "outside your body" because a route can then be traced from the outside world to the interior of your stomach without cutting through your body. A point three feet in front of your nose and a point located inside your stomach are, in this sense, "neighbors."

"Point" is the word topologists like to use in lieu of the term "member" of a set. For purposes of this textbook, we can regard "point" and "member of a set" as synonyms. By itself, neither the

[12] As you might guess, "relations of order" also have to do with order structure, which we will discuss in the next section. The relations of order involved with topology are fairly simple and specific, so although we find relations of order in both topological structure and order structure, the two *structures* are not reducible one to the other. A relation by itself is not a structure.

set X nor subsets T_i belonging to the *collection* T of these subsets is a topological structure. We can note from the definition above that to have a topological "space" the collection T of subsets T_i has to exhibit particular properties, namely those of the three axioms stated above. Topological structure refers to how the subsets T_i are put together such that $T = \{T_1, T_2, \ldots, T_n\}$ satisfies the three axioms.

Topologists do this by defining what is called a ***system of neighborhoods*** for every point x contained in X. Let us use the symbol N_x to denote the system of neighborhoods for some point x. N_x will itself be a set made up of other sets, which we denote by $n_1(x)$, $n_2(x)$, and so on. If there are k of these, then $N_x = \{ n_1(x), n_2(x), \ldots, n_k(x) \}$. An N_x must be set up for every x in X. The construction of the various $n_i(x)$ is subject to four constraints. First, x must be a member (that is, a "point") of every $n_i(x)$ in its N_x. Second, if some set V is a superset of some $n_i(x)$, then V must be one of the sets in N_x. Third, if any pair of sets $n_1(x)$ and $n_2(x)$ both belong to N_x, then their intersect $n_1(x) \cap n_2(x)$ must also belong to N_x. Finally, if an $n_1(x)$ belongs to N_x, then there must also be some $n_2(x)$ belonging to N_x such that if another point y is a member of $n_2(x)$, then $n_1(x)$ also belongs to the system of neighborhoods N_y for the point y. In regard to this fourth and final constraint on N_x, it is worth noting that the set $n_2(x)$ will be a subset of $n_1(x)$.

The collection of all the N_x sets for all the points x in X is called the ***topology*** of the system. If one has had no formal training in topology theory, it is not unlikely that just about everything that has just been said is as opaque as a piece of granite. Unlike algebra, which every reader of this text can be presumed to be familiar with, not many non-mathematicians have had any substantial prior exposure to the ideas of topology theory. This makes it rather difficult to "see" what is going on from the set theoretic expression of topological concepts. Topology theory is built, as we have seen, on the basis of sets of *sets*, rather than sets of *points*, and that makes this subject matter a bit more abstract than the abstract algebra we looked at earlier. Although not at all evident at a glance, the conditions and sets just described are formal descriptions of such ideas as *continuity*, *proximity*, and *connectedness*.

In this context, it is worth the reminder that one nickname for topology is ***rubber sheet geometry***. Two quite different looking geometrical objects are topologically identical if one can be transformed into the other by stretching and bending without actually tearing the figure, punching a hole in it, etc. Figure 10.3 illustrates three planar shapes where we can regard X as being the set of all points in the white background. The thick black lines are points not contained in X. (The dotted line merely denotes the border of each shape). Regions completely cut off from other regions of X can be regarded as neighborhoods in the technical sense defined above.

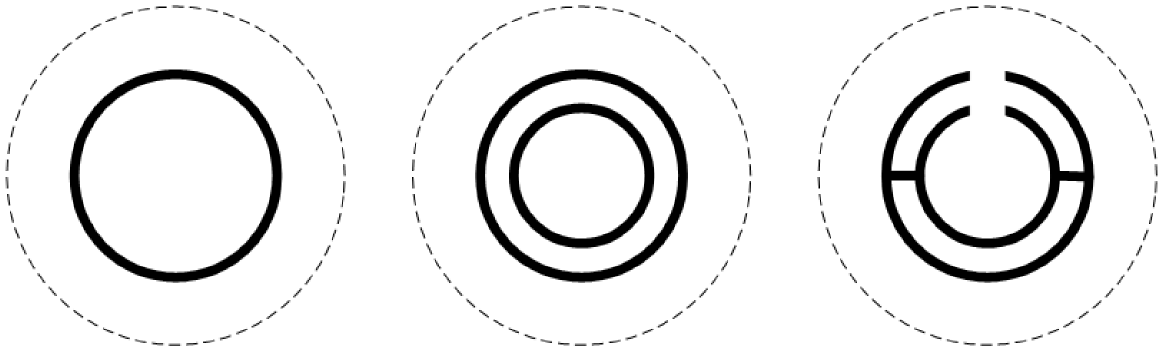

Figure 10.3: Some geometrical figures illustrating topological concepts and neighborhoods

Topologically speaking, the left-most and right-most shapes in figure 10.3 are identical. This is because either can be stretched and twisted until the other is obtained. A clue that this is so can be taken from the fact that both shapes have only two disconnected regions. Similarly, we might guess that the middle shape in the figure is different because it has three disconnected "neighborhoods." Furthermore, we could replace the circular black boundary lines in any of the three shapes by rectangular ones and the topology of each shape would remain unchanged. (A circle and a rectangle are topologically identical). Size and shape do not matter in topology; continuity and connectedness do. This is one reason why the technical description of topology is cast in terms of sets of sets rather than in terms of sets of points. If we think about how one would have to go about obtaining a *general* and precise way to describe these sorts of things, one can begin to appreciate why the formal mathematical expression of topologies is couched in the unfamiliar-to-most-of-us way that it is.

Still, we must not simply stop here and leave things, so to speak, tied in a knot. What possible link could there be between this esoteric topology theory and neuroscience? What in the nervous system, for example, could possibly correspond to a "point" x in a topological space? At the present time, no clear and generally accepted answer this question has been given. One point of view has been put forth by Malsburg in the context of his correlation theory of brain function. Malsburg envisions neural networks as "topological networks." In his topological network the individual neurons would presumably make up the members of the set X. The subsets in T would presumably be determined by the interconnections among the neurons. How exactly to go from this idea to a formal mathematical treatment is something that has not yet been communicated to the neuroscience community, if indeed anyone has even figured out how to do it yet.

Another idea is to regard neural signaling *activities*, rather than the neurons themselves, as possibly providing the proper biological model for the sets T_i in a topological structure. Most computational neuroscientists agree that at least some part of neuronal signaling activity in the brain functions in the role of ***binding codes***. There is not so much agreement on "what a binding code looks like" or even if one has ever actually been seen. One conjecture on what binding codes might be has been published by Damasio. In Damasio's hypothesis, objects and events are represented by the brain as correlated and time-locked activations of specific neural networks located in numerous different brain regions. He calls the individual networks "feature fragments," and the cooperation of numerous feature fragments makes up the whole of the specific information on entities and events being represented.

> Because feature-based fragments are recorded and reactivated in sensory and motor cortices, the reconstitution of an entity or event so that it resembles the original experience depends on the recording of the combinatorial arrangement that conjoined the fragments in perceptual or recalled experience. The record of each unique combinatorial arrangement is the binding code, and it is based on a device I call a convergence zone.
>
> Convergence zones exist as synaptic patterns within multi-layered neuron assemblies in association cortices, and satisfy the following conditions: (1) they have been convergently projected upon by multiple cortical regions according to a connectional principle that might be described as many-to-one; (2) they can reciprocate feed-forward projections with feedback projection (one-to-many); (3) they have additional, interlocking feed-forward/feedback relations with other cortical and subcortical neuron assemblies. The signals brought to convergence zones by cortico-cortical feed-forward projections represent temporal coincidences (co-occurrence) or temporal sequences of activation in the feeding cortices . . . I envision the binding code as a synaptic pattern of activity such that when one of the projections which feed-forward to it is reactivated, firing in the convergence zone leads to simultaneous firing in all or most of the feedback projections which reciprocated the feed-forward from the original set. . .
>
> I propose two types of convergence zones. In Type I, the zone fires back simultaneously and produces concomitant activations. Type I zones inscribe temporal coincidences and aim at replicating them. Type II convergence zones fire back in sequence, producing closely ordered activations in the target cortices. Such zones have inscribed temporal sequences and aim at replicating them. . .
>
> Type I convergence zones are located in sensory association cortices of low and high order, and are assisted in learning by the hippocampal system. Type II convergence zones are the hallmark of motor-related cortices, and are assisted in learning by basal ganglia and cerebellum [DAMA1].

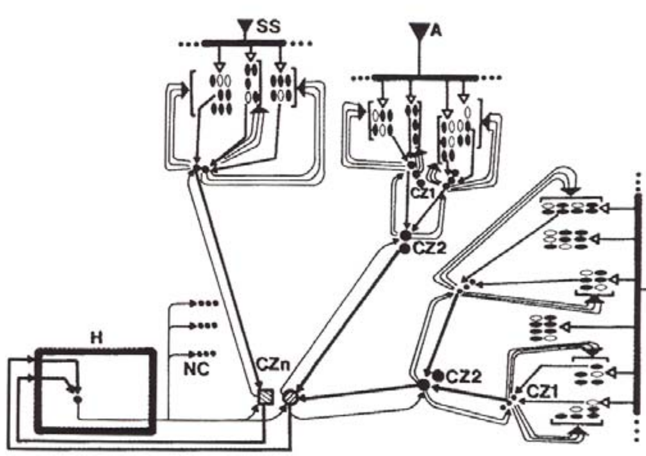

Figure 10.4: Simplified diagram of Damasio's convergence zone hypothesis

Figure 10.4 is a simplified illustration of Damasio's convergence zone hypothesis. The figure is taken from [DAMA2]. V, A, and SS denote early and intermediate sensory cortices in visual, auditory, and somatosensory modalities. CZ denotes convergence zone cell assemblies. H denotes the hippocampal system. NC denotes non-cortical neural stations in basal forebrain, brain stem, and neurotransmitter nuclei. Convergence zones are organized as a layered system of several orders (1, 2, ..., n). Dark lines depict feedforward connection paths, light lines depict feedback pathways. The figure is not a complete depiction, e.g. it omits the details of connections made to the output section of the motor control networks.

Damasio's hypothesis makes much use of the ideas of "maps" (in the cartographer's sense of the word rather than the mathematician's), "images" and other analogies that bespeak of topological arrangements. The brain structures mentioned here by Damasio are those which neuroscience regards as neural substrates of both cognitive and affective phenomena. It is a reasonable conjecture, but only a speculation at the present state of scientific knowledge, to posit that the neural activity patterns to which Damasio refers are the neural correlates of the "points" x in X. "Enhancement," i.e. binding which produces equilibrium in some of these patterns and brings about the extinguishing of others, could be seen in this context as the neural equivalent of forming subsets T_i within the set of all possible ways to carve up X.

But perhaps neither of these ideas is at all on track. Both Malsburg's and Damasio's models tend to make one think in terms of the "points" in X. The reader will note how the discussion above moves directly from a possible neural correlate to the question, "What are the points?" But topology is far less concerned with the points in X than with subsets of X. The historical roots of topology lie with geometry, of course, and for two millennia the mathematics of geometry has begun by considering points, then lines, then surfaces, etc. It has become "natural" (actually, habitual) to think of geometry and anything connected with geometry in these terms.

But if we take another look at the technical definitions given above, what one sees is that the idea of "the point x" seems to play no other definitional role than to identify or categorize subsets, e.g. N_x is the neighborhood system *of* x. But if we look at shapes such as those in figure 10.3, the individual x plays next to no role. What role it does have lies with such questions as, "Can I start from a point somewhere in *this* white region and trace a path to a point somewhere in *that* white region without ever crossing the black region?" A moment's reflection will show that in asking this sort of question, we never really care about "points" very precisely at all. This is just as well since one can never put one's pencil down on a geometric "point" anyway. (Points have neither length, width, or thickness; they are an abstract idea). Any sufficiently described "blob" of points will do adequately well for a "starting point"; any other sufficiently described "blob" will do equally well for an "ending point."

So perhaps the right level at which to look at things, in regard to some future "topological neuromathematics," might be at the level of "aggregates" rather than "points" – in a manner of speaking, "New Jersey" rather than "Trenton"; "Trenton" rather than "John Smith's house"; "outside the circle vs. inside the circle," etc. One way to look at Damasio's hypothesis is, "networks N_1, N_2, and N_6 are firing; networks N_3, N_4, and N_5 are not." It does not seem unlikely that this could be a way to first identify "neighborhoods" if not "neighborhood systems." The neural populations involved, their synaptic interconnections, and so on would then speak more to the topological ideas of connection and disconnection, continuity, closeness (proximity), and so on. If one is given "a topology" to start with, these are ideas that can be seen as coming *out of* that topology. Before one can examine the conjecture that neural networks develop to form topological structures, it is necessary to have a more concrete idea of how to recognize the existence of "topology" in the central nervous system. This is still an open question at this time.

§ 5.4 Order Structure

Ideas of relationships involving ordination permeate our day-to-day lives. The relationship "son of" is one such example: Edward son of Henry VI son of Henry V son of Henry IV son of Blanche of Lancaster. Another is the numerical relationship "less than": $1 < 2 < 3 < 4 <$ etc. Mathematics deals with order relationships through the use of what are called "partial order relations." Let us look at some of the mathematical ideas that make up this concept.

As an illustration let us take as a set S the people in the first example above and write S = {Edward, Henry VI, Henry V, Henry IV, Blanche}. All order structures are based on ordered pairs – which seems a very logical thing indeed! – and so we will need to set up the Cartesian product $S \times S$. In the case of this example, the Cartesian product will have 25 members. An ***order relation*** will pluck from this set only those ordered pairs that satisfy the specific order relation. Thus, ⟨Edward, Henry VI⟩ will be "plucked" by the order relation "son of"; ⟨Edward, Henry V⟩ and ⟨Henry VI, Edward⟩ will not. We will let R denote the subset of $S \times S$ specified by the order relation. Thus,

R = {⟨Edward, Henry VI⟩, ⟨Henry VI, Henry V⟩, ⟨Henry V, Henry IV⟩, ⟨Henry IV, Blanche⟩}.

Binary relations on a set can have different important definable structural properties. There are four such properties in particular that are important to our discussion. The first of these is called the ***reflexive property***. A binary relation is reflexive if for every member s in S, the ordered pair $\langle s, s\rangle$ is a member of R. Otherwise the binary relation is ***irreflexive***. Our example R from above is an irreflexive binary relation. An example of a reflexive binary relation is given by the set of integers {1, 2, 3} and the relationship "less than or equal to." The binary relation for this is

$$R = \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 2\rangle, \langle 2, 3\rangle, \langle 3, 3\rangle\}.$$

A third important property is the ***antisymmetric property***. Let a and b be any two distinct members of S and suppose $\langle a, b\rangle$ is a member of R. The binary relation is antisymmetric if for any pair $a \neq b$ the inclusion of element $\langle a, b\rangle$ in R implies $\langle b, a\rangle$ is not a member of R. The example we just gave using the three integers is an antisymmetric binary relation, and so is our "royal" binary relation example involving Edward, Henry VI, etc.

Finally, a binary relation might have the ***transitive property***. Let a, b, and c be members of a set S and let R be a binary relation on S. R is said to be a transitive relation if, for any three members a, b, and c, the inclusion of both $\langle a, b\rangle$ and $\langle b, c\rangle$ in R implies that $\langle a, c\rangle$ is also a member of R. Our previous example involving the three integers is a transitive relation. The example involving Edward, Henry VI, etc. is not a transitive relation, e.g. ⟨Edward, Henry V⟩ is not a member of R but both ⟨Edward, Henry VI⟩ and ⟨Henry VI, Henry V⟩ do belong to it. Note, however, that if our binary relation had been "descendent of" rather than "son of" then the pair ⟨Edward, Henry V⟩ *would* be a member of this binary relation and in fact this relation would be transitive.[13]

This brings us to the important mathematical idea of ***partial orders***. There are two types of mathematical partial orders of interest to us here. The first is called a ***strict partial order***. A binary relation on a set is a strict partial order if the relation has the irreflexive, antisymmetric, and transitive properties. Our 'descendent of' example is a strict partial order. The second type of partial order is called the ***weak partial order***. A binary relation is a weak partial order if it has the reflexive, antisymmetric, and transitive properties. Our "less than or equal to" example above is a binary relation that is a weak partial order. A weak partial order relation is sometimes called a "poset" ("partially ordered set") by mathematicians. Partial order relations in mathematics are important enough to be given special symbols for designating relationships. Symbol usages vary from mathematician to mathematician, but one very common pair of symbols employed is ' $<$ ' for a strict partial order and ' $\leq$ ' for a weak partial order. For our 'descendant of' example we can use this symbolic convention and write 'Edward < Blanche'; for the 'less than or equal to' example, we would write '$1 \leq 3$'.

As another example of a weak partial order let us use the set of integers {1, 2, 3, 4, 5, 6} and the binary relation 'divides'. Integer a is said 'to divide' integer c if there is an integer b in the set

[13] Edward was the grandson of Henry V. The example we have been using is one branch in the family tree of England's King Edward III (A.D. 1312-1377).

such that $a \cdot b$ is equal to c. If a divides c we write this as $a \leq c$. Our binary relation R for this example specifies the following:

for 1: $1 \leq 1, 1 \leq 2, 1 \leq 3, 1 \leq 4, 1 \leq 5, 1 \leq 6$;
for 2: $2 \leq 2, 2 \leq 4, 2 \leq 6$;
for 3: $3 \leq 3, 3 \leq 6$;
and: $4 \leq 4, 5 \leq 5, 6 \leq 6$.

The set R thus contains the 14 ordered pairs specified above and no others (e.g. $2 \leq 3$ is not in it). We can see that the idea of partial orders is indeed an idea with a very great scope.

Advancing these ideas yet another step, strict and weak partial orders are closely related. Suppose R is a weak partial order. Then R contains ordered pairs $\langle s, s\rangle$ for every member of the set S upon which the partial order R is defined. The subset $I_A \subset R$ consisting of all these $\langle s, s\rangle$ pairs is called the ***identity relation on R***. Now let P be the strict partial order on the same set S. The set P turns out to be a subset of R; it contains all the members of R except those $\langle s, s\rangle$ members belonging to I_A. R is therefore called the ***reflexive closure of P***. In general, if some set A has a binary order relation that lacks the reflexive property, the reflexive closure of A is the smallest reflexive relation C that contains A as a subset. Here 'smallest' means "having the least possible number of members in the set necessary to contain A as a subset and still satisfy the reflexive property." Because C is reflexive, it will contain a subset I_A (as we defined this symbol above) and C will simply be the union $A \cup I_A$.

Sets of ordered pairs specified by an order relation constitute the starting point for an order structure. One consequence of this is that the a and the b in an ordered pair $\langle a, b\rangle$ do not actually matter in and of themselves in an order structure. Rather, it is the ordered pair that matters. As an example, consider a time sequence of signal events $a \rightarrow b \rightarrow c$. This sequence is regarded in terms of order structure by inclusion of the ordered pairs $\langle a, b\rangle$ and $\langle b, c\rangle$ in our set R. Here a is said to ***directly cover*** b, and b is said to directly cover c.

How the overall order structure is put together depends very fundamentally on what sort of order relation is involved. For example, if the order relation is "immediately precedes" then the two ordered pairs above are in R but the ordered pair $\langle a, c\rangle$ is not because a does not immediately precede c in our temporal sequence. But if the order relation is "is earlier than" then $\langle a, c\rangle$ would be included in R.

Two members, call them x and y, in the set S are said to be ***comparable*** if one or the other of the orderings $x < y$ or $y < x$ are contained in R. In our "divides" example, 2 and 4 are comparable, 2 and 3 are not. If the order relation R is such that every x and y in S are comparable, then R is called a ***total order*** or ***chain***. For working the mathematics of order structure, mathematics provides us with a very handy tool called ***graph theory***. A mathematical graph consists of a set of "points" called ***vertices*** and a set of "lines" called ***arcs*** that define connections between vertices. A ***directed graph*** is a graph in which the arcs define ordered pairs of vertices. (The arcs are said to "have a direction"). An arc in a directed graph "going from" vertex a "to" vertex b thus defines the ordered pair $\langle a, b\rangle$. As you can no doubt appreciate, directed graphs can be very useful for describing an order structure. Entire books have been written on the subject of graph theory, which gives us an idea there is a great deal to say about the topic. Accordingly, we will not digress into it here but leave it for those readers who wish to explore it in more depth.

There are other important ideas that go into the theory of order structure. Like graph theory, whole books have been written on the topic of order theory, and it is not our purpose here to write another one. However, some of the more important of these ideas might have a familiar ring to them. They include the ideas of ***lower bounds***, ***upper bounds***, ***greatest lower bound***, ***least upper bound***, and some handy related ideas called the ***join*** and the ***meet***. Much earlier we saw Piaget refer to something called a "lattice" when he was talking about the Bourbaki structures. A lattice is an order structure for which a join and a meet exist for every pair of "points" (a and b) in the order structure. What this basically means is we can find a greatest lower bound and a least upper bound for every member of S.

As was the case for the other two types of structure, the challenge to theoretical neuroscience lies in identifying what it is in the general nature of the central nervous system that would correspond to the various mathematical beasties we have been talking about. For instance, Damasio's "Type II binding code" – if it really exists – would seem to be a prime candidate for consideration in describing order structure construction by neural networks – if neural networks really do this; remember, we are talking about a conjecture, not a fact.

If there is truth in the "structure conjecture" raised earlier, it is also important to keep in mind that we are unlikely to find pristine Bourbaki structures sitting around in pure form. Much more likely is that we would find – if we find anything at all – structures of a "mixed" nature. Remember, the Bourbaki said mathematics is built out of the mother structures by combining and differentiating different pieces of the three mother structures. The Bourbaki structures are said to constitute what is, in a manner of speaking, "atoms" for mathematics in the sense that "atoms" are said to be the constituents of all matter.

In short, "structure research" in pursuit of the conjecture raised in this chapter is unlikely to be either simple or obvious. But, of course, science is rarely simple or obvious except in hindsight. Were that not true, science would be boring.