Chapter 2 Slepian's Principle

§ 1. The Bandwidth Paradox

At the 1974 IEEE[1] International Symposium on Information Theory, mathematician David Slepian opened his Shannon Lecture[2] with the words

> Are signals really bandlimited? They seem to be, and yet they seem not to be.
>
> On the one hand, a pair of solid copper wires will not propagate electromagnetic waves at optical frequencies and so the signals I receive over such a pair must be bandlimited. . . . It would seem, then, that signals must be bandlimited.
>
> On the other hand, however, signals of limited bandwidth W are finite Fourier transforms,
>
> $$s(t) = \int_{-W}^{W} e^{2\pi i f t} S(f) df$$
>
> and irrefutable mathematical arguments show them to be extremely smooth. They possess derivatives of all orders. Indeed, such integrals are entire functions of *t*, completely predictable from any little piece, and they cannot vanish on any *t* interval unless they vanish everywhere. Such signals cannot start or stop, but must go on forever. Surely *real signals* start and stop, and so they cannot be bandlimited!
>
> Thus we have a dilemma: to assume that real signals must go on forever in time (a consequence of bandlimitedness) seems just as unreasonable as to assume that real signals have energy at arbitrarily high frequencies (no bandlimitation). Yet one of these alternatives must hold if we are to avoid mathematical contradiction, for either signals are bandlimited or they are not: there is no other choice. [Slepian (1976)]

This is the Bandwidth Paradox.

A signal is any physical phenomenon exhibiting variations (usually in time) that is said to carry information. When you speak to someone, variations in air pressure carry the sound from your mouth to the ear of the other person and these variations constitute an audible signal carrying information from you to the other person. Frequency, represented by the variable *f* in the equation above, is the rate of repetition of a regular event. For example, a violin string vibrating 1000 times per second is said to have a frequency of 1 kHz (1000 cycles per second). In 1807 the French mathematician and physicist Joseph Fourier showed that more complicated regular events can be broken down into a series of frequencies, e.g., 1 kHz, 2 kHz, 3 kHz, etc., and he claimed that *any* signal can be decomposed into this form[3]. Bandwidth is simply the range of frequencies present in the Fourier representation of a signal. Fourier gave no proof for his generalized claim but later mathematicians were able to prove it was true subject to certain mathematical conditions. Today students of electrical engineering devote multiple semesters of study to Fourier analysis.

A great deal of the work performed by electrical engineers consists of the generation, processing, and

---

[1] The Institute of Electrical & Electronics Engineers.

[2] The Shannon Lecture is delivered by the IEEE Information Theory Society's winner of the prestigious Claude E. Shannon Award for consistent and profound contributions to the field of information theory. This award is the Society's equivalent of the Nobel Prize. Slepian was the first person to win the Claude E Shannon Award (other than Claude Shannon himself). His Shannon Lecture was regarded as so significant that it was reprinted in the IEEE's flagship journal, *Proceedings of the IEEE*, in 1976.

[3] Primitive cases of what eventually came to be called Fourier analysis were known as far back as ancient Babylon, where it was used to compute tables of astronomical positions. French astronomer Alexis Clairaut used a version of it in 1754 to compute a planetary orbit. Ptolemaic astronomers used a related method to explain retrograde motions of the five planets known to their science.

transmission of signals. Television and radio are two examples of many kinds of commercial electronic products where the transmission of information from one place to another is the central purpose of the device. Most of the IEEE audience members at Slepian's 1974 lecture were people professionally involved in the engineering of communication devices and systems but the bandwidth paradox also arises in other quantitative sciences such as physics. Oddly enough, this paradox is seldom mentioned in college courses taken by science and engineering students. If mentioned at all, it is usually presented to graduate students; but a few especially bright undergraduate students occasionally discover it for themselves – an event that has been known to embarrass their teachers from time to time when the student asks one of them about it. Perhaps one factor in the stir Slepian's lecture produced had to do with many members of his audience being surprised to learn of it and none of them ever having thought of how to resolve it.

Slepian's solution of the bandwidth paradox is obtained by a shift in "how one looks at the world"; in particular, it adopts an epistemology-centered view. Out of this shift comes an important principle I am calling Slepian's Principle. Slepian said,

> My starting point is to recall to you that each of the quantitative physical sciences – such as physics, chemistry, and most branches of engineering – is comprised of an amalgam of two *distinctly different* components. That these two facets of each science are indeed distinct from one another, that they are made of totally different stuff, is rarely mentioned and certainly not emphasized in the traditional college training of the engineer or scientist. Separate concepts from the two components are continually confused. In fact, we even lack a convenient language for keeping them straight in our thinking. I shall call the two parts Facet A and Facet B.
>
> Facet A consists of observations on, and manipulations of, the "real world." . . . For the electrical engineer this real world contains oscilloscopes and wires and voltmeters and coils and transistors and thousands of other tangible devices. These are fabricated, interconnected, energized, and studied with other real instruments. Numbers describing the state of this real world are derived from reading meters, thermometers, counters, and dial settings. They are recorded in notebooks as *rational real numbers*. (No other kind of number seems to be *directly* obtained in this real world.)
>
> Facet B is something else again. It is a mathematical model and the means for operating with the model. It consists of pencils and papers and symbols and rules for manipulating the symbols. It also consists of the minds of the men and women who invent and interpret the rules and manipulate the symbols, for without the seeming consistency of their thinking processes there would be no single model to consider. When numerical values are given to some of the symbols, the rules prescribe numerical values for other symbols of the model. [*ibid*.]

Slepian's words contain and carry an echo of Kant across the centuries:

> There is no doubt whatever that all our [empirical] knowledge begins with experience; for how else should the faculty of knowledge be awakened into exercise if not through objects that move our senses and in part themselves bring about representations, in part bring the activity of our understanding into movement to compare these, to connect or separate them, and thus to work up the raw stuff of sensuous impressions into a cognition of objects that is called experience? [Kant (1787) B: 1]

Slepian's Facet A is the world of tangible objects of sensible experience. On the other hand, Facet B is a mental world of abstract concepts and ideas of supersensible experience, and this world is where all mathematics resides. Facet A can be called the physical world; Facet B can be called the mathematical world. Slepian said,

> Our mathematical models are full of concepts, operations, and symbols that have no counterpart in Facet A. Take the very fundamental notion of a real number, for instance. In Facet B, certain symbols take numerical values that are supposed to correspond to the readings of instruments in Facet A. Almost always in Facet B these numerical values are elements of the real-number

> continuum, the rationals *and irrationals*. This latter sort of number seems to have no counterpart in Facet A. In Facet B, irrational numbers are defined by limiting operations or Dedekind cuts – mental exercises that with some effort and practice we can be trained to "understand" and agree upon. After years of experience with them, we theoreticians find them very "real," but they do not seem to belong to the real world of Facet A. *The direct result of every instrument reading in the laboratory is a finite string of decimal digits – usually fewer than six – and a small integer indicating the exponent of some power of 10 to be used as a factor*. Irrationals just cannot result directly from real measurements [Slepian (1976)].

Neither, for that matter, can rational numbers with an unlimited number of non-zero digits. This brings Slepian to another important observation:

> Now there are several ways in which we can handle this fundamental lack of correspondence between symbol values in Facet B and measurements in Facet A. . . . [The simplest is] the scheme adopted and known to you all. We admit the real-line continuum into Facet B and we impose yet another abstraction – continuity. In the end, if the model says the voltage is π, we are pleased if the meter in Facet A reads 3.1417. We work with the abstract continuum in Facet B, and we round off to make the correspondence with Facet A.
>
> Mathematical continuity deserves a few words. It is another concept with no counterpart in the real world. It makes no sense at all to ask whether *in Facet A* the position of the voltmeter needle is a continuous function of time. Observing the position of the needle at millisecond or micro-second or even picosecond intervals comes no closer to answering the question than does measurement daily or annually. Yet continuity is a vital concept for Facet B. By invoking it, by demanding continuous solutions of the equations of our models, we make the parts of our model that correspond to measurements in Facet A insensitive to small changes in the parts of the model that do not correspond to anything in Facet A. . . . There are certain constructs in our models (such as the first few significant digits of some numerical variable) to which we attach physical significance. That is to say, we wish them to agree quantitatively with certain measureable quantities in a real-world experiment. Let us call these the *principal quantities* of Facet B. Other parts of our models have no direct meaningful counterparts in Facet A but are mathematical abstractions introduced into Facet B to make a tractable model. We call these *secondary constructs* or *secondary quantities*. . . . It is my contention, however, that a necessary and important condition for a model to be *useful* in science is that the *principal quantities of the model be insensitive to small changes in the secondary quantities*. Most of us would treat with great suspicion a model that predicts stable flight for an airplane if some parameter is irrational but predicts disaster if that parameter is some nearby rational number. Few of us would board a plane designed from such a model. [*ibid*.]

Slepian's contention – *the principal quantities of a model must be made insensitive to small changes in secondary quantities* – is Slepian's Principle. He used this principle in the remainder of his lecture to explain the Bandwidth Paradox. This explanation is somewhat lengthy and involves mathematical arguments that require some significant training in order to be understood, but his final conclusion can be explained fairly simply. If two Facet B signals, $s_1(t)$ and $s_2(t)$, have energies that cannot be distinguished by measuring instruments in Facet A, *then the signals must be considered "the same."* By this he means that if s_1 is time-unlimited in Facet B, s_2 in Facet B is a time-limited *truncation* of s_1, and the energies of these two cannot be distinguished through measurements in Facet A, then they are from a *practical* point of view *indistinguishable* and to be regarded *in Facet A* as *both* time *and* bandwidth limited. As he put it,

> A consequence of these definitions is that *all* signals of finite energy are both bandlimited to some finite bandwidth W and time limited to some finite duration T. [*ibid*.]

Slepian also pointed out that when measuring instruments become available that permit measurements detailed enough to be able to distinguish two signals that were previously indistinguishable *then the Facet B constructs* of bandwidth and time duration could be altered. Put another way, measurement instruments *can be used to extend the range of our senses* and convert what were *super*sensible objects of Facet B into *sensible* objects of Facet A.

2. Slepian Dimensioning

Slepian's principle turns out to be of fundamental pertinence to other developments in mathematics and engineering that were first introduced before his 1974 lecture and have been undergoing continued development since then. One of these in pure mathematics is called "non-standard analysis" [Robinson (1996)]; another is called "set membership theory" or SMT [Combettes (1993)], [Deller *et al*. (1993)].

Robinson introduced the idea of non-standard analysis in a 1960 Princeton seminar. SMT was first introduced in 1968 by control system engineers as a method for dealing with particularly nasty real world engineering problems [Schweppe (1968)], [Witsenhausen (1968)]. Since then it has also been applied to technical problems in communication system theory [Wells (1996)] and digital magnetic data storage [Wells (1995)], [McCarthy & Wells (1997)]. Slepian's idea of "indistinguishability" is central to all of these.

There are also other scholarly fields where Slepian's Principle lies implicitly in their theories. One example is the theory of quantum electrodynamics, the theory for which Feynman, Schwinger, and Tomonaga received their Nobel Prizes. Feynman unintentionally gave us a hint of this [Feynman (1985), pp. 124-129]. The hint came while he was discussing the measured rest mass of an electron (*m*), the measured charge on an electron (*e*), the "rest mass of an ideal electron" (*n*), and the "charge on an ideal electron" (*j*). The first two (*m* and *e*) can be measured experimentally; the latter two (*n* and *j*) have to be calculated theoretically and this calculation depends on the assumed distance between the electron and the point in space where the calculation is made. The numerical values of *n* and *j* depend on this distance, and this distance is many orders of magnitude smaller than anything we can measure. Feynman said,

> Schwinger, Tomonaga, and I independently invented ways to make definite calculations to confirm that [the theory] is true (we got prizes for that). People could finally calculate with the theory of quantum electrodynamics!
>
> So it appears that the *only* things that depend on the small distances between coupling points are the values for *n* and *j* – *theoretical numbers that are not directly observed anyway*; everything else, which can be observed, seems not to be affected.
>
> The shell game that we play to find *n* and *j* is technically called "renormalization." [Feynman (1985), pg. 128]

In other words, *n* and *j* are what Slepian called secondary quantities of Facet B.

To facilitate an understanding of Slepian's Principle it is useful to introduce an epistemological idea we shall call "Slepian dimensioning." This idea derives directly from Kant's Critical theory of concepts[4]. A concept is one type of mental representation, and representation is a logically primitive idea in Critical epistemology because, as Kant put it,

> cognition always presupposes representation. And this latter cannot be explained at all. For we would always have to explain *what is representation?* by means of yet another representation. [Kant (1800) 9: 34]

What he means by this is that we cannot provide any ontological *real explanation* (*Realerklärung*) for representation in *objective* terms. Like other primitives in Kant's epistemology, the only *real meaning* for the term "representation" is a *practical* one:

[4] The Critical theory is presented in Wells (2006) and Wells (2009). The reader is referred to these sources for its explanation.

> Representation is mental (internal) determination where a thing is being referred to as if it were separable from myself. . . . But I call it related if its quality is conformable to the quality of outer things, or if it gives shape to external things-in-the-world. [Kant (1753-59) 16: 76-77]

On the practical level, representation is a mental act human beings carry out; thus this explanation is a practical explanation, not an ontological one, and obtains its practical validity from the fact that we *know* human beings are capable of *doing* what Kant describes here. Representation is necessary for the possibility of human knowledge as we understand the nature of what it is to be a human being. Kant also tells us,

> All our knowledge has a *twofold* relationship, *first* to a relationship to the *Object*, *second* a relationship to the subject[5]. In the first consideration, it refers to *representation*, in the latter to *consciousness*, the universal condition of all cognition in general. (In reality, consciousness is a representation that another representation is in me.) [Kant (1800) 9: 33]

Thinking is *cognition* [6] *through concepts*, and this brings us around to the idea of a "concept." A concept is the representation of a *rule* [7] constructed (mentally) *by* a human being through which he is able to consciously *re*-present to himself an appearance of an object. We can call the act of *making* a concept an *act of judgment* out of which *experience* is produced. Broadly speaking, an act of judgment is either a *judgment of perception* (which is a conscious representation with merely subjective validity, i.e., validity only for the person making the judgment; its representation is called an "intuition") or a *judgment of experience* (in which case the object referred to in the representation is objectively valid, i.e., valid for every person's experience; its representation is called a "concept").

Through acts of thinking and judgment, human beings gradually *structure* their knowledge of objects through a complex and hierarchical organization of concepts that Kant called *the manifold of concepts* [Wells (2006; 2009; 2011; 2012)]. The act of constructing this structure is what Critical epistemology calls *understanding*. Kant tells us,

> The logical acts of understanding, through which concepts are begat as to their forms, are:
>
> 1. *Comparation* of representations among one another in relationship to unity of consciousness;
>
> 2. *Reflexion*, i.e. reconsideration as to how various representations can be comprehended in one consciousness; and finally
>
> 3. *Abstraction* or separation of everything else in which the given representations differ. [Kant (1800) 9: 94] [8]

During the process of thinking concepts can be synthetically combined to produce additional concepts without the constituent concepts losing their original matters (sensations) and forms of representation. The act of abstraction then produces a concept Kant called a "mark" of the two or more concepts going into its synthesis. The "mark" concept contains all that is common to them and nothing that differs between them. The mark is said to *coordinate* the concepts standing under it and so is also called a *coordinate concept*. Figure 1A illustrates this idea. Concept *M* is said to be "contained in" concepts *A* and *B*, while concepts *A* and *B* are said to be "contained under" concept *M*. Because of the act of abstraction, concept M contains less *in* itself than either concept standing under it.

---

[5] By "subject" Kant means the human being said to have knowledge.
[6] I use the word "cognition" to mean the act or process of knowing including both awareness and judgment.
[7] A rule is an assertion made under a general condition. A concept asserts reproduction of a particular representation of the appearance of an object.
[8] In the language of modern mathematics, the act of *Comparation* is the synthesis of what is called an equivalence structure. The act of *Reflexion* is the construction of what is called a congruence structure. For the mathematically formal definitions of these terms see Preparata and Yeh (1974), pp. 45 and 136.

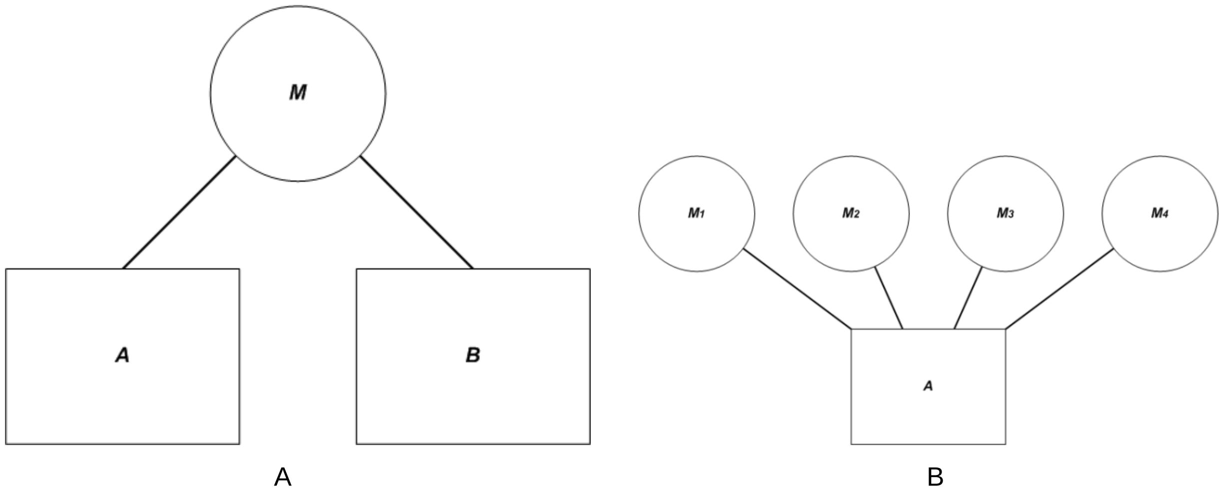

Figure 1: Coordination of concepts in the manifold of concepts. (A) Concepts *A* and *B* combined to produce a higher concept *M* called a "mark" of *A* and *B*; (B) Multiple marks M_1 through M_4 of a concept *A*.

The marks of a concept generally depend on what other concepts it is synthetically combined with because differences represented in these other concepts determine what the act of abstraction separates out in representing the mark. A concept *A* therefore can have multiple marks (as illustrated in figure 1B) as a result of its being combined with various other concepts. Each coordinate mark in this figure has at least one other connection (not shown) to another concept because coordinate concepts always involve at least two lower concepts. It is said to "under stand" the concepts it coordinates, hence we call this synthetic processing of concepts "understanding."

A mark, in its turn, can also be synthetically combined with other concepts to produce a "higher" mark-of-a-mark that under stands it. At each successively higher level, marks contain less information than the concepts standing under it. Because of differences in the sensational content (matter) in a mark, the process of understanding, in producing successively higher concepts, eventually reaches a point where abstraction has removed all sensational matter and retains only rules of pure form. This is illustrated in figure 2.

However, the object of a concept containing no sensation information cannot be an object of sense. For this reason, that object is said to be "supersensible" and is not an object of any possible human experience. The concept of such an object is called an **idea**. The existence of the object of an idea can be inferred with objective validity from the still-sensible objects of the concepts standing immediately under it, but the object itself cannot be experienced because its idea contains no sensations. Kant called such an object a *noumenon*. In contrast, an object of a concept in which matters of sensation representation are still contained is called a *phenomenon*. Only phenomena can be objects of possible experience.

When understanding has progressed so far to first produce an idea (a concept devoid of sensational content), the *noumenon* it represents is said to "stand at ***the horizon of possible experience***" (figure 2). The object can properly be called "a thing-as-we-know-it." For example, in Newtonian physics the phenomenon of gravity is understood through gravity's various *effects* but Newton was unable to find a *cause* of gravity. Its effects are phenomena but the unknown cause was a *noumenon*. Gravity as a thing-as-we-know-it is known only by its effects; where none of these effects can be found we say "no gravity is presented." An unknown cause of a known effect is always a *noumenon*. In Kant's Critical metaphysic, *noumena* at the horizon of possible experience mark the end of objectively valid Critical ontology.

However, the mere fact that the idea of a *noumenon* has reached this horizon does not put a stop to the process of *thinking*. While all ideas lack sensational content, they still contain representations of form, and forms can differ between ideas. Therefore, *thinking* can continue to produce *higher* ideas of *noumena*.

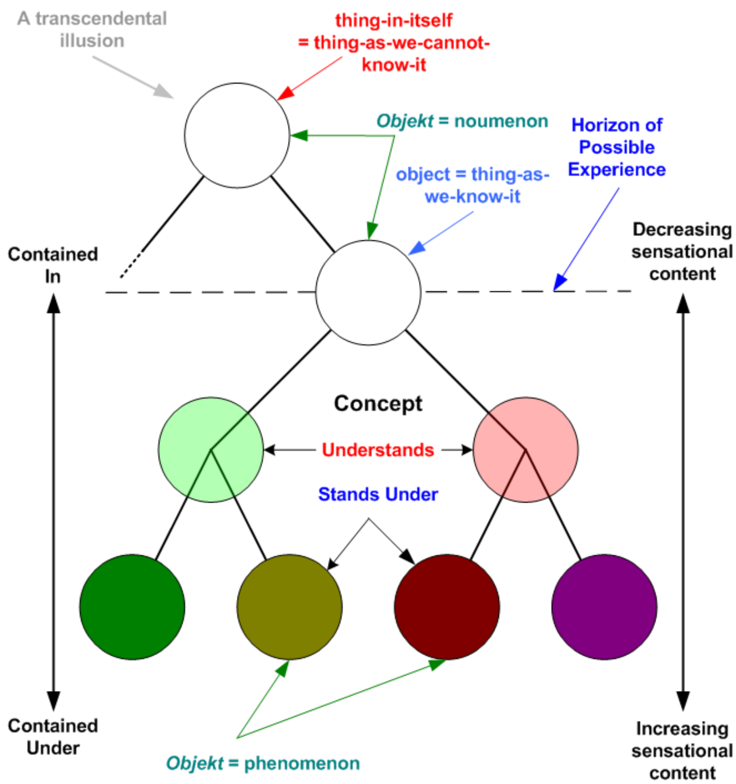

Figure 2: Illustration of the horizon of possible experience. I translate the German word *Objekt* as Object.

Figure 2 illustrates this continued progression of thinking. An idea produced by making abstraction of other ideas has for its object of representation a different kind of *noumenon* we can call "a thing-as-we-cannot-know it" or "a thing-in-itself" (in German, a *Ding an sich selbst*) because this object can never be an object of experience. Such an object has *epistemological significance* for human understanding, but it can have *no ontological significance whatsoever*. If we mistake such an object for being ontologically significant we subject ourselves to what Kant called a "transcendental illusion."

But this is not to say that all *noumena* beyond the horizon of possible experience are insignificant. *All objects of pure mathematics are* noumena *beyond the horizon of possible experience* but these objects are nonetheless *epistemologically significant for the practice of* science and mathematics. There is no illusional "pi in the sky" (as Davis & Hersh humorously put it in chapter 1), but where would math and science be without the idea of the number π ? Where, indeed, would they be without the idea of "real numbers"? Slepian noted,

> We could build a mathematical model in which only a finite number of numbers can occur, say those with 10 significant digits and one of a few hundred exponents. Differential equations would be replaced by difference equations, and complicated boundary conditions and rules would have to be added to treat the roundoff problem at every stage. This model would be exceedingly complex. [Slepian (1976)]

But because the idea of a "number" is itself an idea (and its object a *noumenon*), a philosopher might say even this idea of Slepian's is problematic.

In point of fact, every scientific theory (including the science of mathematics) is a doctrine of ideas. The challenge for scientists is not so much the challenge of coming up with ideas but, rather, the challenge of coming up with *ideas that work*, i.e., that accurately and reliably describe natural behaviors. Feynman said of scientific theorizing,

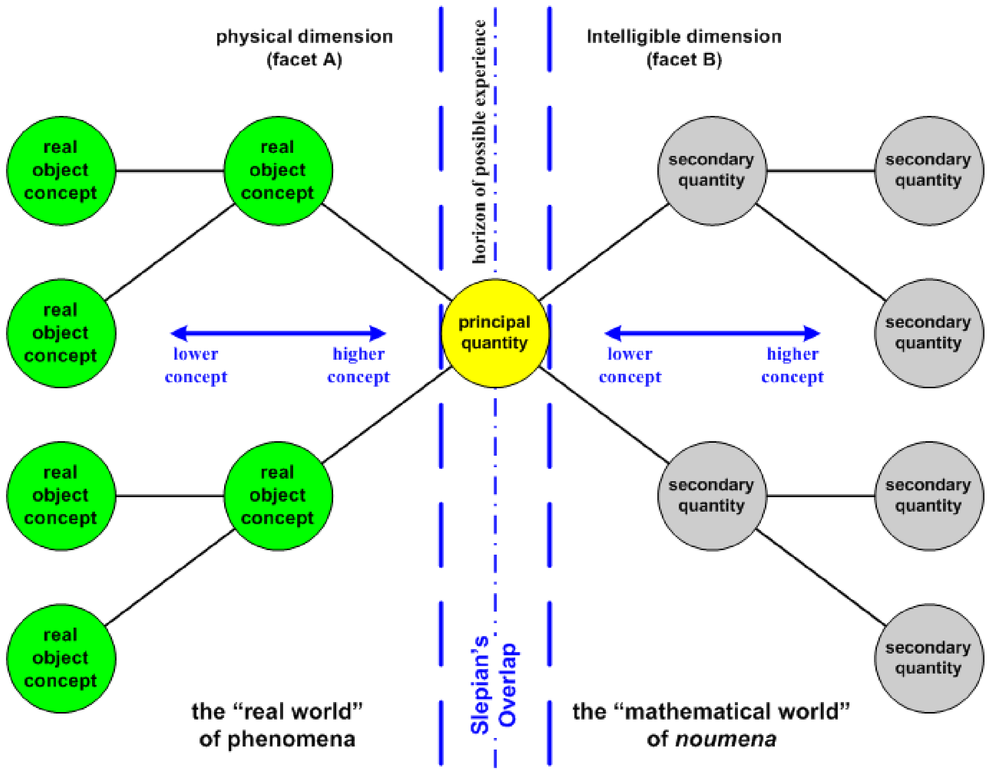

Figure 3: Slepian dimensioning of concepts into a "physical dimension" (concepts of phenomena) and an "intelligible dimension" (ideas of *noumena*). From ideas at the horizon of possible experience we get our principal quantities of Facet B. Other mathematical concepts of secondary quantities of Facet B represent the abstract objects of pure mathematics.

> What we need is imagination, but imagination in a terrible strait-jacket. We have to find a new view of that world that has to agree with everything that is known, but disagrees in its predictions somewhere, otherwise it is not interesting. And in that disagreement it must agree with nature. If you can find any other view of the world which agrees over the entire range where things have already been observed, but disagrees somewhere else, you have made a great discovery. [Feynman (1965), pg. 171]

This lengthy prologue brings us around at last to the idea of Slepian dimensioning (figure 3). Slepian makes a logical division of concepts (and their objects) into those whose contexts which lie in a physical dimension (the "real world" or "world of phenomena") and those whose contexts lie in an *intelligible* dimension (the "mathematical world" or "world of *noumena*"). In Critical epistemology the context of a concept is extremely important. Context is what delimits the applicable scope of a concept insofar as this concept can be applied *with objective validity*.

This is as much as saying context determines the nature of the reality of a concept's object. In Critical epistemology, *every* object is *real* (objectively valid) *in some contexts*, it is *unreal* (lacks objective validity) in others, and *non-real* (inapplicable) in still others. For example, "the ghost of Hamlet's father" is *real* in the context of being a character in Shakespeare's play *Hamlet*; it is *unreal* in the context of being a spirit that actually haunts some castle in Denmark; and it is *non-real* (does not apply) in the context of the price of a barrel of oil. The epistemological significance of a concept is determined by one's understanding: those contexts where the object is real; those where it is unreal; and those where it is non-real. The number pi, *as an object*, is *real* in the context of geometry; it is *unreal* in the context of Pythagorean number mysticism, i.e., "the nature of things is number" [Zeller (1980), pg. 35]; it is *non-real* in the context of a game of soccer. For concepts of objects of Facet A, their contexts are generally established by sensuous relationships with other phenomenal concepts. Every empirical perception contains sensations; for this reason Kant called phenomenal objects "the real of sensation" [Kant (1787) B 207] because sensation is to be regarded as an *effect* on the perceiving person *caused* by an object [*ibid*., B: 34].

But when pure *noumena* are represented by concepts of secondary quantities, which contain no matter of sensation, how can anyone say these objects are "objectively valid" or "real in a context"? Obviously, since these objects lie beyond the horizon of possible experience, no one can have any *direct* sensuous experience of them, and direct sensuous experience is what most people mean by "being real" or "having objective validity." Here it is important to distinguish between two kinds of objective validity recognized in Critical epistemology. These are "theoretical objective validity" and "practical objective validity."

To understand what these mean, we must further examine combinations of concepts in the manifold of concepts. In particular, we must look at some *structures within the manifold of concepts*. Indeed, the *idea* of a "context" cannot be adequately understood without this further examination[9]. For this we need to look at three kinds of concept structures. Kant called these: the polysyllogism; the Classification; and the disjunctive inference of Reason.

§3. Polysyllogisms

One reason present day logicians so badly misunderstand Kantian Logic is, in part, because Kant used technical terminology he took from the ontology-centered logics of his day[10]. He used the same *words* but he gave them different (epistemology-centered) *meanings*. He also seemed to regard the logics of his day as so trivially deducible from his Critical Logic that his logic lectures barely differentiate between them. A modern reader who has no or only a passing familiarity with *Critique of Pure Reason* cannot be blamed if he, too, fails to notice that Kant's Critical Logic is something quite different from "traditional" logic.

Let us begin with those concept structures Kant called *polysyllogisms* [Wells (2011)]. The concept diagrams we have seen thus far *explicitly* illustrate nothing beyond simple connections of coordination between higher and lower concepts in the manifold of concepts. While connections of coordination are vital for thinking basic propositions, a little reflection on your part might soon convince you that this kind of concept connection is completely inadequate to explain the wondrous variety and complexity of your everyday thoughts. Even so simple a classroom example as

> All lions are big cats;
> All big cats are predators;
> All predators are carnivores;
> Therefore all lions are carnivores,

cannot be captured by a simple chain of coordinated concepts (figure 4). To go from the first proposition in this example to the final conclusion requires multiple cycles of thinking in which the different propositions are *seriated*, e.g., lions $\rightarrow$ big cats $\rightarrow$ predators $\rightarrow$ carnivores. The conclusion cannot be reached without *thinking* a *series connection* running from carnivores all the way back to lions. But this kind of connection *is not indicated* by the diagram of a simple chain of coordinated concepts.

[9] That the concept of a "context" is an idea (contains no sensuous matter) becomes clear as soon as you understand that you have never had a *direct sensuous* experience with a *thing* called a "context." You do, of course, *recognize* your contexts in your process of thinking. Put rather simply, your context is the scope of what you are thinking *about*. But if thinking is cognition through concepts, what sort of *concept* represents a "context"? This is the sort of question we need the structures of polysyllogisms, Classifications, and disjunctive inferences in order to answer.

[10] The education most students receive today on the subject of logic is so impoverished that most students are unaware that there have been and still are many different "kinds" or "schools" of logic. These different logics are discussed in Kneale & Kneale (1962). The most prevalent logics used in Kant's day were the Port Royal Logic [*ibid*., pp. 315-320] and the logic of Leibniz [*ibid*., pp. 320-336].

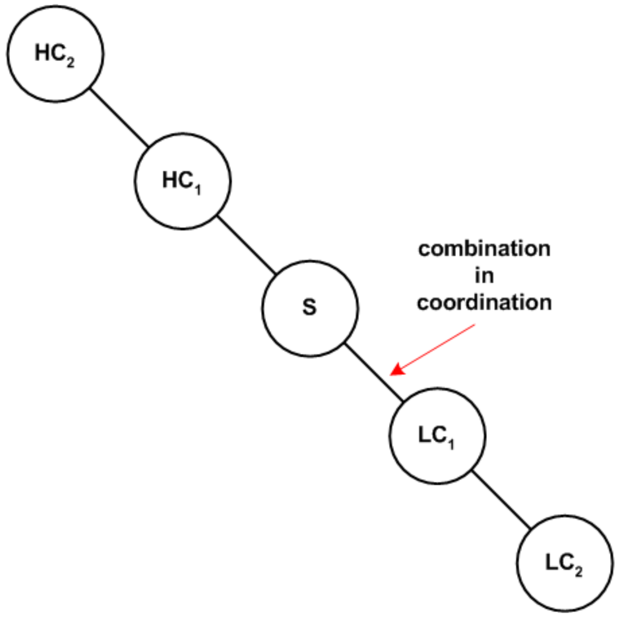

Figure 4: Illustration of a simple chain of coordinated concepts. HC denotes concepts that are higher than the subject concept S. LC denotes concepts that are lower than concept S.

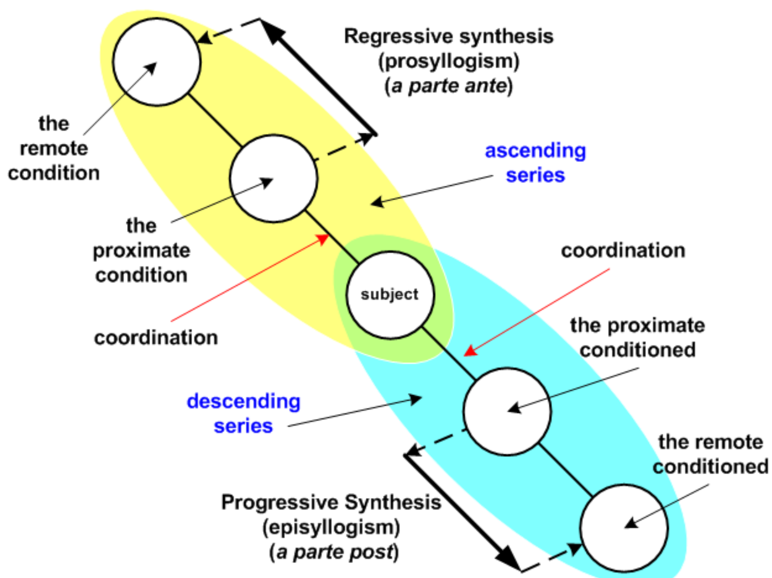

Figure 5: Illustration of the concept structures for prosyllogisms and episyllogisms. The colored ovals denote concepts that have been connected *as* a series.

The *construction* of a combination of concepts in series is what Kant means by "polysyllogism." The *structure* resulting from this is itself a concept, but one within which the series connection is represented. Kant called this structuring a "series of composite inferences" of Reason [Kant (1800) 9: 134]. One might call a polysyllogism structure a "composite concept." Within it the simpler concepts of which it is composed are conserved and remain distinct. The making of this series can proceed in two possible ways. If it runs from the lowest concept successively to the highest concept in the series, it is called a *prosyllogism*. If it instead runs from the highest concept to the lowest one in the series, it is called an *episyllogism*. Figure 5 illustrates these two cases. (You should note that these definitions of prosyllogism and episyllogism *are not the same* as the definitions used in the doctrines of ontology-centered logics).

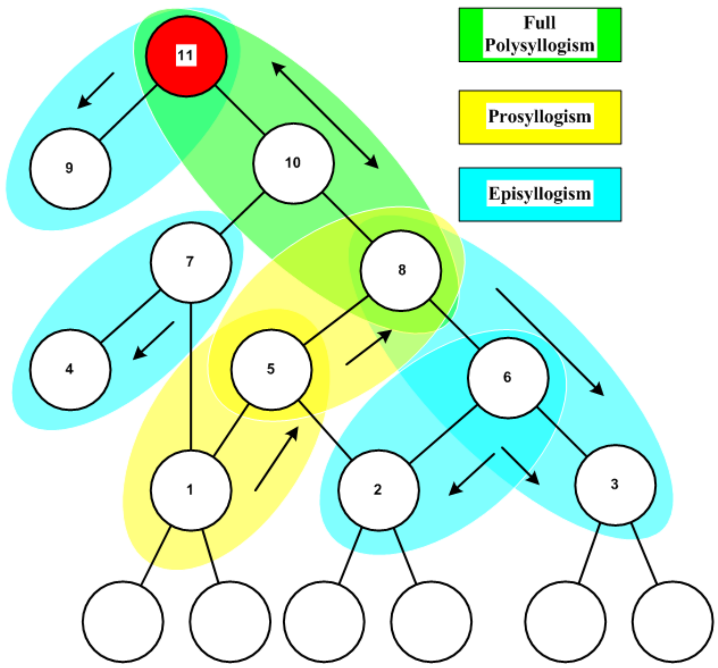

Figure 6: Multiple polysyllogism constructions within the manifold of concepts. A full polysyllogism is a structure containing both an episyllogism construct and a prosyllogism construct. Reasoning based upon a full polysyllogism can run in either direction, whereas reasoning through episyllogisms can run only from the higher to the lower concept, and that of a prosyllogism can run only from the lower to the higher concept.

Because concepts within a polysyllogism are conserved and remain distinct, a concept can be reused in synthesizing many polysyllogisms. This is illustrated in figure 6, where the ovals again indicate polysyllogisms. For example, the concept labeled "8" in figure 6 is part of the full polysyllogism 11 ↔ 10 ↔ 8, the episyllogism 8 → 6 → 3, and the prosyllogism 5 → 8. However, it does *not* participate in a series 1 → 5 → 8 nor in a series 8 → 6 → 2 because these series have not yet been synthesized (no single oval in the figure identifies either series). The thinking person has not yet made these inferences by means of his reasoning process; he simply "hasn't thought of them yet." There is a prosyllogism 1 → 5 and another one 5 → 8, but these two *composite* concepts have not yet been synthesized as *one* series.

The "1 → 5" and "5 → 8" example just given points to a subtle shortcoming in traditional logic doctrine. This doctrine deals solely with variables (e.g., "1", "5", "8") that are not related to *objects*. These variables have "truth values" *assigned* to them *by* the logician or mathematician but have in themselves no "real world" significance. Any significance assigned to them is assigned by the logician or mathematician or scientist, and this assignment necessarily requires some extra-logical concept be *added* to them from outside the domain of the formal logic discipline. The traditional logics are not "laws of thinking" as some people have proposed or assumed. They are what Aristotle originally said they were, i.e., methods

> by which we shall be able to reason from generally accepted opinions about any problem set before us and shall ourselves, when sustaining an argument, avoid saying anything self-contradictory. [Aristotle (*c.* 350 BC) pg. 273]

Aristotle tightly coupled his "science of deduction" to his system of metaphysics and by doing so excluded from it statements like "All griffins are fierce" that are fictitious or contrary to facts. The later Scholastics of the Middle Ages felt compelled to divorce "logic" from Aristotle's metaphysics because his metaphysics ran counter to Christian doctrine. By doing so, traditional logic became a methodology by which it was possible to make "true statements" about empirical absurdities. An example of this is provided by the following:

All horses are animals;
Animals with wings are birds;
The horses of Arabia have silver wings;
Therefore, the horses of Arabia are birds.

Lewis Carroll and other authors have had a great deal of fun at formal logic's expense over the years because of this divorce. For example, Mark Twain (Samuel Clemens) wrote,

> In the space of one hundred and seventy-six years the Lower Mississippi has shortened itself two hundred and forty-two miles. That is an average of a trifle over one mile and a third per year. Therefore, any calm person, who is not blind or idiotic, can see that in the Old Oölitic Silurian Period, just a million years ago next November, the Lower Mississippi River was upward of one million three hundred thousand miles long and stuck out over the Gulf of Mexico like a fishing rod. And by the same token any person can see that seven hundred and forty-two years from now the Lower Mississippi will be only a mile and three quarters long, and Cairo and New Orleans will have joined their streets together, and be plodding comfortably along under a single mayor and a mutual board of aldermen. There is something fascinating about science. One gets such wholesome returns of conjecture out of such a trifling investment of fact. [Twain (1883), pg. 88]

Lest you think things like this are harmless, bear in mind that in recent decades some physicists have come to regard their mathematics as being in some way ontologically significant. Out of this have come seriously proposed untestable transcendental illusions such as "the existence of parallel universes."

Formal logic does not describe how human beings think. Computers do *not* think even though their circuitry is designed by "logic designers" and despite claims sometimes made on behalf of so-called "artificial intelligence." Naturally, the medieval Scholastics recognized the vulnerabilities the divorce between logic and metaphysics produced, and they took mathematics-like measures to counter them. These include definitions specifying how such things as "truth" and "falsity" are to be regarded and by making distinctions between "empirical propositions" and "*a priori* propositions." Ambrose & Lazerowitz tell us,

> We shall distinguish in the first place between *a priori* and empirical truth or falsity. A proposition is said to be true *a priori* if its truth can be ascertained by examination of the propositions alone or if it is deducible from propositions whose truth is so ascertained, and by examination of nothing else. [Ambrose & Lazerowitz (1948), pg. 17]
>
> Empirical truths and falsities stand in sharp contrast to those which are *a priori*. A proposition is true, or false, empirically when its truth or its falsity is ascertainable not by inspection, but only by recourse to matters of fact, that is, by observation or by an experiment of some sort . . . Thus an empirical truth, sometimes called a "contingent" one, is determined by states of affairs in the world, so that if the states of affairs were otherwise, what is true would instead be false. By contrast, what is true or false *a priori* is in no sense contingent upon states of affairs. An *a priori* proposition is true, or false, regardless of what the world is like and of what happens in it. For this reason, *a priori* truths and falsities are characterized as "necessary" [*ibid*., pp. 17-18].

This comes at the price of disconnecting formal logic from empirical science:

> Now in logic we have no concern whatever with whether any empirical proposition is actually true or actually false. This follows from the fact that we deal only with the formal elements in propositions and not with their subject matter, except in the indirect sense of noting how formal concepts function in *any* such context. This is one reason why logic is said to be a nonempirical study. Another reason is that it does not concern itself with empirical propositions except as these are constituents of *a priori* propositions, especially those that are *a priori* true. Out of these latter the deductive system is constructed. [*ibid*., pg. 18]

In contrast, Kantian Logic is not divorced from Critical metaphysics and it remains concerned with empirical propositions as well as formal statements.

There is a problem, though, that arises when we combine the idea of "*a priori* truths and falsity are characterized as 'necessary' " with the idea of divorcing logic from "empirical propositions." The issue centers on the question of what "necessity" *means*. Here, once again, we must turn to science for understanding what "necessity" is. The answer was revealed by Piaget's studies in 1983 of how children come to understand ideas as "possible" or as "necessary." He concluded,

> In short, necessity does not emanate from objective facts, which are by their nature merely real and of variable generality and therefore subject to necessary laws to a greater or lesser extent. They only become necessary when integrated within deductive models constructed by the subject. The necessity of *p* can thus not be characterized only as the impossibility of not-*p*, since new possibilities can always emerge, but must be described in Leibniz's manner as the contradiction of not-*p*, and this relative to a specific, limited model. [Piaget (1983), vol. 2, pg. 136]

If a person can only conclude something is "necessary" from the basis of a *model*, then "necessity" in logic cannot be divorced from empirical propositions because all deductive models have their bases in these.

The "1 $\rightarrow$ 5" and "5 $\rightarrow$ 8" example above also brings out a second point that bears emphasizing. A concept is not "automatically" combined within polysyllogisms merely because a coordinating judgment connects it to a higher or a lower concept. A human being needs a practical reason to make an inference, and his reasons are motivated by psychological needs that thinking and reasoning satisfy [Wells (2016)]. This property of the human nature of thinking and reasoning has been empirically borne out in psychological studies [Piaget (1974)], [Piaget (1983)]. It was also noted by mathematician Henri Poincaré:

> Great importance has been rightly attached to this process of [mathematical] "construction," and some claim to see in it the necessary and sufficient condition of the progress of the exact sciences. Necessary, no doubt, but not sufficient ! For a construction to be useful and not a mere waste of mental effort, for it to serve as a stepping stone to higher things, it must first of all possess a kind of unity enabling us to see something more than the juxtaposition of its elements. Or more accurately, there must be some advantage in considering the construction rather than the elements themselves. [Poincaré (1905), pg. 15]

I make this point in order to caution readers against interpreting my diagrams from a too automaton-like point of view such as that implicit in the 20th century "metamathematics" of formalism as championed by David Hilbert and by the "Nicolas Bourbaki" school of mathematical philosophy [Davis & Hersch (1981), pp. 335-344].

Polysyllogism structures provide one basic form of context. Their objective validity is at root *practical*, that is, rooted in motivated mental acts aimed at satisfying some purpose. However, there is another form of context equally important, and we look at this one next.

§4. Disjunctive Inferences and Classifications

Seriation is only one form of context formation. Another form is what is called "Classification," and Classifications are made possible by the synthesis of what Kant called "disjunctive inferences of Reason" [Wells (2012)]. There is evidence of the development of both capacities very early in life before the onset of language ability in a child:

> We therefore come to our final hypothesis, that the origins of classification and seriation are to be found in the sensorimotor schemes as a whole (which includes perceptual schemes as integral

> parts). Between the ages of 6-8 and 18-24 months, which is well before the acquisition of language, we find a number of behavior patterns which are suggestive of both classification and seriation. . . . The fact that we can observe various prototypes of classification and seriation at the sensorimotor and preverbal stage of development proves that the roots of these structures are independent of language. [Inhelder & Piaget (1964), pp. 13-14]

Bearing in mind that concepts are mediate representations of Objects, and that the idea of classifying something already presupposes distinguishable multiple Objects, Classifications are more complicated structuring activities than seriations. Polysyllogism constructs are contained in them, and disjunctive inferences distinguish them. Figure 7 illustrates the structure of a Classification. Kant explains,

> The several given judgments of which the disjunctive judgment is composed make up its *matter* and are called the *members of the disjunction* or *opposition*. In the *disjunction* itself subsists the *form* of these judgments, that is, in the determination of the relationship of the various judgments as reciprocally exclusive of each other and complementary members of the whole sphere of the classified cognition [C]. [Kant (1800) 9: 106]

The same manifold Object (the Object represented by C) can be subjected to multiple acts of Classification by which distinct multiple disjunctions (D) are placed under C. Multiple disjunctive inference structures under a concept C thereby establish multiple contexts in which C can be regarded. These divers disjunctive inferences are synthesized in separate acts of judgment and endow human reasoning with the enormous flexibility and creative capacity human beings are so clearly known to exhibit.

There is much more that can be said about inferences of Reason but doing so would draw us away from the topics of mathematics and science and into more psychological matters of the phenomenon of mind. This is best left to other discussions, e.g. Wells (2011) and Wells (2012). The discussion given here suits the purpose at hand, namely clarifying the idea of what "context" means. With this, we can return to the question that prompted this brief tangent – namely, "how can Facet B concepts be objectively valid?"

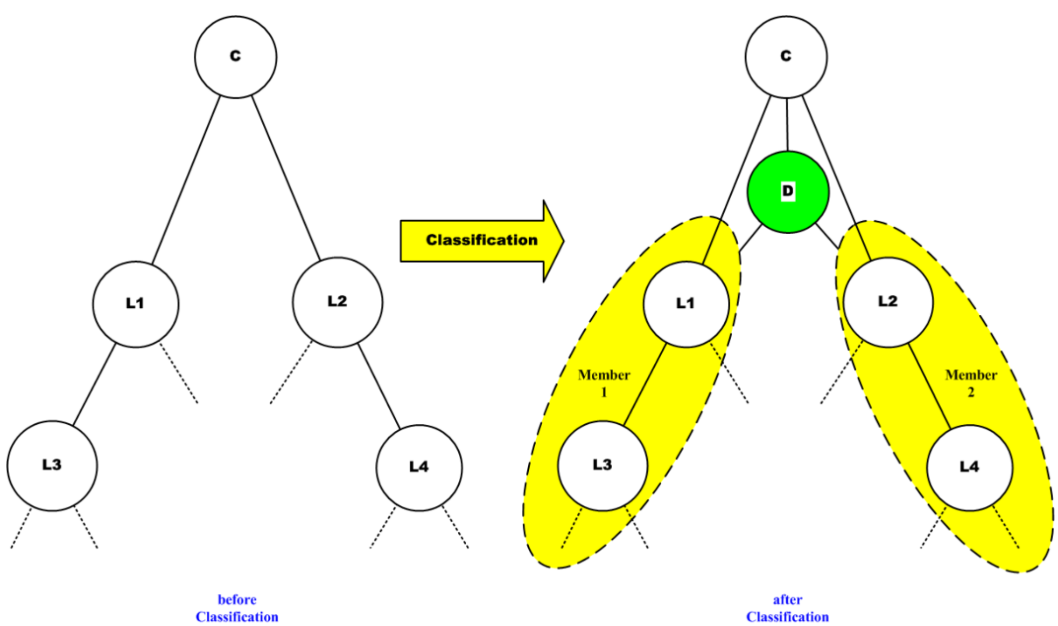

Figure 7: Manifold structure of a Classification. As a first step, concepts contained under concept C undergo polysyllogisms presenting new concepts (L_1-L_3 and L_2-L_4). Then the concept of a disjunctive inference (D) is formed, by which L_1-L_3 and L_2-L_4 are made "members of the disjunction." A transcendental disjunction has the property that Member 1 and Member 2 cannot both be presented in consciousness in the same intuition at the same moment in time.

§5. The Condition of Epistemological Significance for Secondary Quantities

Refer once again to figure 3 above, the diagram illustrating Slepian dimensioning. Secondary quantities are concepts that lie beyond the horizon of possible human experience. At no point are they connected by judgments of actual experience to concepts of objects belonging to Facet A. This lack of connection is what distinguishes a secondary quantity from Slepian's principal quantity.

What is meant by the qualifier "actual experience" here? A degree of caution is needed in explaining this because *all* objects are real in some contexts, unreal in others. We looked at "the ghost of Hamlet's father" in an earlier example. We might also look at the object known as "Mickey Mouse." Mickey Mouse is real in the context of being a cartoon character invented by Walt Disney or a costumed employee at Disneyland, but you would not expect a human-sized talking mouse to knock on your front door. We are imaginative beings and, indeed, science would not be possible without our imaginative abilities. Kant called this ability the human capacity for fiction, and some Kant scholars refer to this as our fictive faculty. Kant tells us,

> The ability to form a mental picture [fictive faculty] is the capacity for producing representations of things that we have never seen. This is either Imagination or fantasy. Imagination is when *we* play with the power of imagination and fabricate something for certain ends and purposes. Fantasy is when the power of imagination plays with *us*. The former is voluntary, for we can cancel and direct it as we please, but the latter is involuntary. Each fabrication must occur according to the analogy of experience, otherwise it is unbridled, unruly fantasy. We can therefore fabricate nothing materially, but rather only formally. If the fabrication is according to the analogy of experience then it is disciplined fantasy. If it is involuntary then it is specifically called unbridled fantasy. [Kant (1783) 29: 884-885]

We can feel pretty safe in supposing Walt Disney had some purpose in mind before he came up with his idea of Mickey Mouse as a means of satisfying this purpose. Mickey Mouse is the product of imaginative design and has nothing at all to do with the sciences of biology or zoology. The situation gets trickier when a product of imaginative design is made in an attempt to explain some quandary of science. An example of this is provided by Wolfgang Pauli's invention of the idea of neutrinos. In physics the idea of conservation of energy is one of its most important and fundamental theoretical principles. But in 1914 James Chadwick, who was studying radioactive decay, discovered what appeared to be a violation of this principle in experimental studies of a phenomenon called β-emission.

Some physicists, e.g. Niels Bohr, took the radical position that these experiments meant the law of conservation of energy did not hold for β-emission. Pauli was strongly opposed to Bohr's view. He speculated that the problem of β-emission was caused by the emission of some other yet undiscovered particle he called "the neutron" (it was later renamed "the neutrino" by Enrico Fermi after a different particle – now named the neutron – was experimentally discovered). It took until 1955 for Pauli's neutrino hypothesis to be experimentally confirmed by two Los Alamos physicists, F. Raines and C. Cowan. With this confirmation, the neutrino ceased to be merely a fantasy of imagination and became an established part of the ontology of physics.

The Los Alamos experiments did not observe neutrinos by the appearance of some *thing*; it wasn't a matter of "Oh, look! There it is!" Instead the experiments produced measurements of *effects* that matched what were expected to be effects *caused by* neutrinos if the neutrino "really existed in physical nature." It was this *correspondence* with phenomena within the horizon of possible experience that established the objective validity of the neutrino *as an Object of a principal quantity* of Facet B. Phenomena of Facet A thereby acquired an empirically *actual* connection to Facet B.

Above we saw Kant refer to "the analogy of experience." What this refers to is a principle of Critical metaphysics that goes by the name "the analogies of experience" [Kant (1787) B: 218-265]. Kant said of these principles,

> Our analogies [of experience] therefore precisely exhibit the unity of nature in the context of all appearances under certain exponents[11], which express nothing other than the relationship of time . . . to unity of apperception, which can only take place in synthesis in accordance with rules. Thus together they say: All appearances lie in one nature and must lie therein, since without this *a priori* unity no unity of experience, and thus also no determination of the objects in it, would be possible. [Kant (1787) B: 263]

The validation of the idea of Pauli's neutrino came about because it brought together a number of observable phenomena in accordance with Kant's analogies of experience and, by doing so, validated the idea of the neutrino as a proper object of physical ontology.

Although Kant provides a very thorough and very lengthy treatment of this, our purposes here can be summed up more succinctly by an explanation provided by physicist Henry Margenau. Margenau wrote,

> A formal connection is one which sets a construct in a purely logical relation with another construct; an epistemic connection is equivalent to and arises from a rule of correspondence which links the construct with data. Examples of formal connections are: all relations between geometric quantities which are provable on the basis of a suitable set of axioms, such as the relations between angles and sides of a triangle, the sine law, the cosine law, and so forth; the relation between a number and its square; a circle and its radius. In physics and chemistry also, every connection between entities that is derivable from postulates is a formal one: Examples are the relation between force and acceleration of a given mass (postulate: Newton's laws); the relation between a point charge and its electromagnetic field (postulate: Maxwell's equations); between the curvature of space and the quantity of matter in the universe (postulate: Einstein's law of general relativity); between the temperature and the mean kinetic energy of a gas (Gibb's statistical mechanics); between the structure of a molecule and its molecular weight. It is true that all formal connections are stable only so long as certain postulates are maintained, that they are in this sense hypothetical judgments. It is also true that their formal character becomes less obvious when they are empirically verified, as many of them are. . . .
>
> Epistemic connections will doubtlessly occur to the reader in considerable abundance: They exist between the objective tree and what is called the vision of it, between a force and an awareness of muscle exertion, between the weight of an object and a reading on a scale, between a wavelength and the discernment of a line on a photographic plate. All these experiences are linked by epistemic connections. One of the terms is a construct, the other is in Nature. [Margenau (1977), pp. 84-85]

Secondary quantities and their combinations in Facet B fall under what Margenau called formal connections. What he called epistemic connections are those relationships between phenomena and principal quantities. Objective validity for secondary quantities of Facet B is obtained when these ideas have a formal connection, by means of prosyllogisms, with one or more principal quantities of Facet B *provided these principal quantities have epistemic connections with concepts of objects in Facet A*.

What Margenau called "constructs" Kant calls ideas. Margenau tells us

> [Some constructs] stretch out two or more arms, either toward other constructs or toward Nature; they may be described as *multiply connected*. Every construct used in natural science will be seen to be of this type, permitting passage elsewhere in at least two ways. For example, from the idea of an electron one can pass on to its mass, its charge, its field. [*ibid*., pp. 85-86]

He also notes that one can have other constructs for which there is only one connection to some other construct. He called such a construct a "peninsular" construct and said of it,

> It hangs loosely within the system and obtains what meaning it has only from a coherent set of

[11] An "exponent" is something that expounds or interprets.

> others. An example of a peninsular construct is the color of an electron. No harm is done if color is assigned to it, but there is no way of substantiating this attribute, for it leads to no other significant knowledge by any formal route, no does it allow verification by any possible rule of correspondence. [*ibid*., pg. 86]

He also noted that it is possible to form sets of constructs that are mutually connected but have no epistemic connections. Here he remarked,

> They may be said to form an island universe, consistent in itself though unverifiable. Science sometimes generates such tantalizing curiosities, then looks for possible rules of correspondence which might give them significance. But they are dropped again unless such rules are found. [*ibid*.]

Mathematicians, it is to be noted, not-infrequently think up mathematical objects and structures that fall under Margenau's definition of "island universe constructs" so far as empirical science is concerned. This is not a criticism of pure mathematics, however, because with surprising frequency scientists find ways of forging epistemic connections to them. Non-Euclidean geometries provide one example of this. When they were first discovered physicists had no use for them. While they greatly unsettled the philosophy of mathematics, they produced no ripple of concern in the physics community. This situation changed radically with the publication of Einstein's theory of general relativity.

To summarize, secondary quantities of Facet B are "formal constructs" only. They only become epistemologically significant by means of connection to principal quantities of Facet B that make what Margenau calls epistemic connections to phenomena of experience in Facet A. Critical metaphysics, augmented by Margenau's explanation, brings us a schematic picture of scientific ontology depicted by figure 8. The Objects depicted by blank circles (*noumena*) standing at the horizon of possible experience are the Objects of Slepian principal quantities. They are blank to denote they lack sensuous matter. Secondary quantities do not appear in this diagram; they are *implicitly* depicted by red connection lines. This is because secondary quantities are epistemologically significant but not ontologically significant.

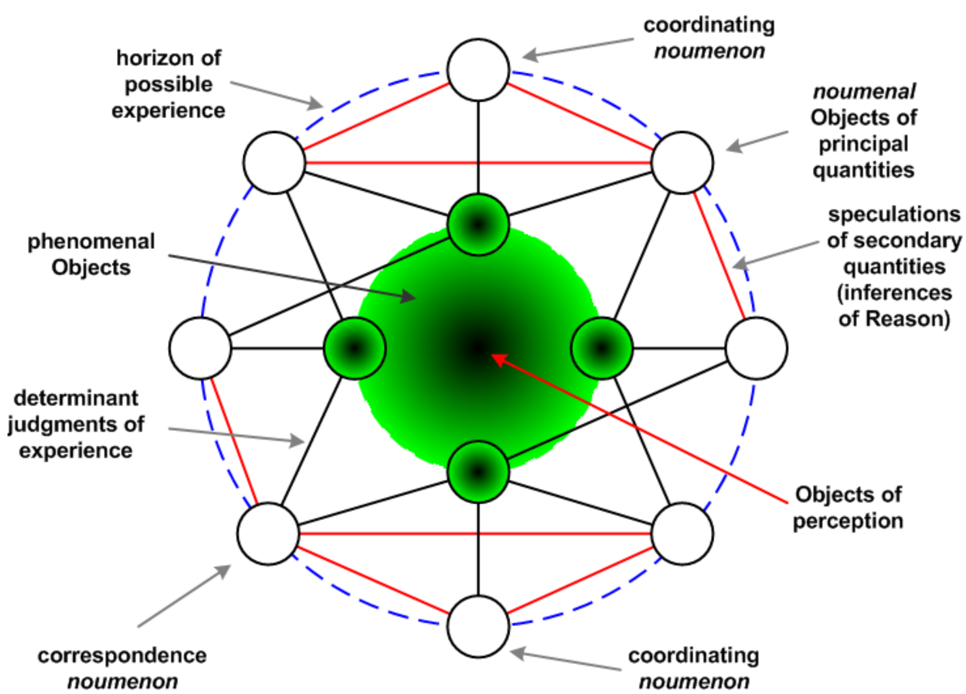

Figure 8: Schematic structure of scientific ontology.

There are two kinds of *noumena* depicted in figure 8. *Correspondence noumena* are Objects of ideas that have two or more direct epistemic connections to concepts of phenomenal objects. They generally stand in the role of hypothetical causes of actual phenomena in nature. *Coordinating noumena* have at least one epistemic connection to some phenomenon but do not stand in the role of the cause of an effect. Their primary function is, as the name implies, to provide theoretical coordination between correspondence *noumena* by means of secondary quantities. This coordination is made by *formal* connections we call physical laws of nature. As for the secondary quantities, the formal connections they make and the contexts they provide (through polysyllogisms and disjunctive inferences) are indicated by the red connecting lines.

As for the green colored interior of figure 8, this represents phenomenal Objects within the horizon of possible experience. The black connecting lines represent determinant judgments of experience [Wells (2009), chap. 5]. At the core of the schematic we have Objects of perception, which are raw sensuous intuitions of objects of human perception.

But this ontology raises another issue. What is needed for a principal quantity to have this property of epistemic connection? The answer to this question is a bit more complicated than one might expect. We will therefore devote chapter 3 to this question.

References

Ambrose, Alice & Morris Lazerowitz (1948), *Fundamentals of Symbolic Logic*, NY: Rinehart & Co.

Aristotle (*c*. 350 BC), *Topics*, in *Aristotle: Posterior Analytics; Topica*, Cambridge, MA: Harvard University Press, 1960.

Combettes, P. (1993), "The foundations of set theoretic estimation, "*Proceedings of the IEEE*, vol. 81, no. 2, pp. 182-208.

Davis, Philip J. & Reuben Hersch (1981), *The Mathematical Experience*, Boston, MA: Houghton Mifflin Co.

Deller, J., M. Nayeri, and S. Odeh (1993), "Least squares identification with error bounds for real-time signal processing and control, "*Proceedings of the IEEE*, vol. 81, no. 6, pp. 815-849.

Feynman, Richard P. (1965), *The Character of Physical Law*, Cambridge, MA: The MIT Press, 21st printing, 1994.

Feynman, Richard P. (1985), *QED: The Strange Theory of Light and Matter*, Princeton, NJ: The Princeton University Press.

Inhelder, Bärbel & Jean Piaget (1964), *The Early Growth of Logic in the Child*, London: Routledge & Keegan Paul.

Kant, Immanuel (1753-59), *Die Vernunftlehre*, in *Kant's gesammelte Schriften*, *Band XVI*, pp. 76-775, Berlin: Walter de Gruyter & Co., 1924.

Kant, Immanuel (1783), *Metaphysik Mrongovius*, in *Kant's gesammelte Schriften*, *Band XXIX*, pp. 743-940, Berlin: Walter de Gruyter & Co., 1983.

Kant, Immanuel (1787), *Kritik der reinen Vernunft*, 2nd ed., in *Kant's gesammelte Schriften*, *Band III*, Berlin: Druck und Verlag von Georg Reimer, 1911.

Kant, Immanuel (1800), *Logik*, in *Kant's gesammelte Schriften*, *Band IX*, pp. 1-150, Berlin: Walter de Gruyter & Co., 1923.

Kneale, William & Martha Kneale (1962), *The Development of Logic*, Oxford, UK: The Oxford University Press.

Margenau, Henry (1977), *The Nature of Physical Reality*, Woodbridge, CN: Ox Bow Press.

McCarthy, S.G. & R.B. Wells (1997), "Model order reduction for optimal bounding ellipsoid channel models," *IEEE Transactions on Magnetics*, vol. 33, no. 4, pp. 2552-2568.

Piaget, Jean (1974), *The Grasp of Consciousness*, Cambridge, MA: Harvard University Press, 1976.

Piaget, Jean (1983), *Possibility and Necessity*, in two volumes, Minneapolis, MN: University of Minnesota Press, 1987.

Poincaré, Henri (1905), *Science and Hypothesis*, NY: Dover Publications, 1952.

Preparata, Franco & Raymond Yeh (1974), *Introduction to Discrete Structures*, Reading, MA: Addison-Wesley Publishing Co.

Robinson, Abraham (1996), *Non-standard Analysis*, Princeton, NJ: Princeton University Press.

Schweppe, P. (1968), "Recursive state estimation: Unknown but bounded errors and system inputs," *IEEE Transactions on Automatic Control*, vol. AC-13, no. 1, pp. 22-28.

Slepian, David (1976), "On bandwidth," *Proceedings of the IEEE*, vol. 64, no. 3, Mar. 1976, pp. 292-300.

Twain, Mark (1883), *Life on the Mississippi*, NY: Dover Publications (2000).

Wells, Richard B. (1995), "Algorithms for threshold level selection and decoder logic design in decision-aided symbol-by-symbol data receivers for magnetic recording applications," *IEEE Transactions on Magnetics*, vol. 31, no. 5, pp. 2527-2535.

Wells, Richard B. (1996), "Application of set-membership techniques to symbol-by-symbol decoding for binary data transmission systems," *IEEE Transactions on Information Theory*, vol. 42, no. 4, pp. 1285-1290.

Wells, Richard B. (2006), *The Critical Philosophy and the Phenomenon of Mind*, available from the author's web site.

Wells, Richard B. (2009), *The Principles of Mental Physics*, available free of charge from the author's web site.

Wells, Richard B. (2011), "On the synthesis of polysyllogisms in Critical Logic," available free of charge from the author's web site.

Wells, Richard B. (2012), "On the synthesis of disjunctive inferences of Reason in Transcendental Logic," available free of charge from the author's web site.

Wells, Richard B. (2016), "Why people think," available free of charge from the author's web site.

Witsenhausen, H. (1968), "Sets of possible states of linear systems given perturbed observations," *IEEE Transactions on Automatic Control*, vol. AC-13, pp. 556-568.

Zeller, Eduard (1980), *Outlines of the History of Greek Philosophy*, 13th ed., NY: Dover Publications.