Chapter 14 The Mathematics Framework

§ 1. The Definitional Problem of Mathematics

What is mathematics? This question echoes the question, "What is language?" from chapter 13. Like language, the definition of mathematics is taken for granted by most people, including most mathematicians. And, as is usually the case for ideas we take for granted, the question goes deeper than one might think. Unlike 'language,' the term 'mathematics' has a *technical* dictionary definition accepted by the community of mathematicians. It is:

> **mathematics** The study of numbers, shapes, and other entities by logical means. It is divided into *pure mathematics* and *applied mathematics*, although the division is not a sharp one and the two branches are interdependent. Applied mathematics is the use of mathematics in studying natural phenomena. It includes such topics as statistics, probability, mechanics, relativity, and quantum mechanics. Pure mathematics is the study of relationships between abstract entities according to certain rules. It has various branches, including arithmetic, algebra, geometry, trigonometry, calculus, and topology. [Nelson (2003)]

Insofar as the named topics are concerned, these are passively accepted in the American institution of education and in the institutions of other nations as well. However, it is important to recognize that merely naming a list of topics does not define what 'mathematics' is any more than 'knowledge' can be defined by naming items of knowledge such as 'knowledge of shoemaking' or 'knowledge of furniture making' [Plato (date uncertain), 146a-148c]. Only the first sentence above *might* be a definition of 'mathematics.' But is it?

First, it says mathematics is the *study* of something. This means that balancing your check-book is not mathematics. When a physicist studies some natural phenomenon, that is not mathematics. When a civil engineer designs a bridge, that is not mathematics. In none of these cases is the doer "studying numbers, shapes, and other entities by logical means" *unless we declare that whatever activity he is doing* ***is*** *'mathematics'* – a definition by fiat. One can argue that in cases like these and countless others the doer is "using" mathematics and that "using mathematics" means he is making use of *knowledge* from studies implied by the dictionary definition. This is not an unreasonable way to look at it and most people accept the "using" distinction. But "using knowledge" is not the same thing as studying something. Logically, it would have to be concluded that *mathematics* is done by almost no one except professional mathematicians, schoolchildren, and some college students – mathematicians because it is their job, children because our Society makes them do it, and collegians either because someone makes them do it or because they want to do it. But what about 'mathematics' makes it worthwhile for *everyone* to learn it? What is it we want schoolchildren to learn? What is "mathematical knowledge"?

In my "mind's ear" I can hear complaints that I am quibbling about this, so let's move on to my second point. What makes 'numbers, shapes, and other entities' objects of mathematics? If you logically plan out how you are going to roll your bowling ball or select your golf club for your next shot, why aren't these "mathematics"? Or shall we say they are? If we did, my friends in the math department would be horrified, and I don't want to horrify them. They're good folks and really are "seekers of truth" provided the truth being sought is "mathematical truth." What, then, is an "object of mathematics" and, equally important, what is not? Is the study of what sorts of knots can be tied with a rope an object of mathematics? You might not think so, but that topic is called "knot theory" and is officially regarded as a branch of geometry. If he were a real person, presumably Popeye the Sailor would have been a mathematician specializing in knot theory. If your mother or grandmother knits, presumably she, too, would have been a mathematician at one time. Popeye, your mom, or your grandma would each have been an "applied" mathematician.

The plain fact is that any study of any "entity" that the mathematics community *agrees to call* an object of mathematics is *ipso facto* made to be "mathematics." The dictionary definition above, consequently, is one that is so open-ended that it cannot properly be called a definition at all.

And that is my point regarding this 'official definition' of mathematics. It doesn't actually define anything. This won't do if we are to understand what to teach learners and how to instruct them within the mathematics framework of public education. The thing causing the problem is the way people "look at the world." The prevailing ways of looking, focused as they are on objects, are nothing else than divers species of *ontology-centered* metaphysics. The definitional problem of mathematics will not be solved until we take an *epistemology*-centered metaphysic as our "way of looking at the world." I will be presenting this way of looking at it shortly.

To understand all these things, it helps to begin with a look back in history to find out where the notion of "mathematics" came from in the first place. Mathematics historians generally agree that mathematics properly so-called began with the Greeks around 600 BC and developed out of empirical ideas and practices known to the Phoenicians and the Egyptians [Ball (1908), pp. 1-10]. Math historian D.E. Smith tells us,

> When we attempt to define "mathematics" we find ourselves encircled by unexpected limitations, and these limitations are still more in evidence when we change the term to "elementary mathematics." If mathematics means that "abstract science which investigates deductively the conclusions implicit in the elementary conceptions of spatial and numerical relations," as the Oxford Dictionary defines it, then the history of mathematics cannot, strictly speaking, go back much earlier than the time of Thales (*c*. 600 BC), a relatively modern writer if we consider the antiquity of the [human] race. Such a limitation, however, would not be a satisfactory one, for it would withdraw from our consideration those early steps in the development of the science which have great interest to the student and which are of value in considering the education of the individual.
>
> It is well, therefore, to discard such niceties of definition and to take a broader view of the case, seeking to tell the story of the genesis of mathematics even before the period in which the science, as it was defined above, began to exist. [Smith (1923), pp. 1-2]

Historians, as Smith illustrates here, are not immune to the definitional problem. What is usually done, as a result, is to try to draw a fine distinction between a "science" of mathematics and what is likely best called a "craft" of "the art of calculating" and a "craft" of land surveying. The former came to the Greeks mainly from the Phoenicians, the latter mainly from the Egyptians.

Transformation of these ancient crafts into what would become "mathematics" coincided with the birth of philosophy and is historically credited to Thales of Miletus (*c*. 640-550 BC) and Pythagoras of Samos (*c*. 569-500 BC). What distinguishes Thales is that he was the first Greek thinker to reject mythology in explanations of the natural world and to introduce the use of formal deductive reasoning. Pythagoras is credited with founding the first "school" of philosophy and of what would become "mathematics." Our word "mathematics" derives from the Greek word *mathema* (μάθημα), which itself derives from *mathein* (μαθεîν) or "that which is learned." In this Greek root we can draw the first contrast between the dictionary definitions above and what was the *essential* character of Greek mathematics. The modern definitions specialize the word to refer to objects such as numbers or shapes. It is true that the ancient Greek mathematicians did quickly come to use such objects as objects of mathematics, but note that coming to this usage *followed* their practice rather than *defined* it. To put this another way, the Greeks did not start out with numbers, shapes, etc. and invent a science (mathematics) for them; rather they started out with a practice for deducing truths about the physical world and found numbers, shapes, etc. to be the concepts that seemed to them most useful and fecund for doing so. Perhaps this is a gossamer distinction, but it is one that is fundamental to answering the question, "What is mathematics?"

There are two things I think are important in understanding the original idea of *mathematikos* (μαθηματικος) held by the early Greek philosophers. The first is that the Greeks were metaphysical *realists* in the way they looked at the world. The second is that they were pragmatic theorists in the "business" context of the word pragmatic. They were not motivated to explore geometry and numbers by idle curiosity but rather by desire and intent to put knowledge of these things to work. We are told by ancient authors that Thales made himself a rather wealthy man by putting his *mathema* ("lessons") to work for himself. Archytas the Pythagorean (*c*. 428-347 BC), a contemporary and friend of Plato, was reported by Porphyry to have written,

> The mathematicians seem to me to have arrived at true knowledge, and it is not surprising that they rightly conceive the nature of each individual thing; for, having reached true knowledge about the nature of the universe as a whole, they were bound to see in its true light the nature of the parts as well. Thus they have handed down to us clear knowledge about the speed of the stars, and their risings and settings, and about geometry, arithmetic and sphaeric, and, not least, about music; for these studies appear to be sisters. [Thomas (1939), pg. 5]

Archytas is famous today as an ancient mathematician but, like other ancient Greeks, he in no way fits the not-uncommon modern picture of a Greek philosopher as a couch potato spending his days lying about lost in dreamy contemplation. Diogenes Laertius tells us,

> Aristoxenus says that [Archytas] was never defeated during his whole generalship, though he once resigned it owing to bad feeling against him, whereupon the army at once fell into the hands of the enemy.
>
> He was the first to bring mechanics to a system by applying mathematical principles; he also first employed mechanical motion in a geometrical construction, namely, when he tried, by means of a section of a half-cylinder, to find two mean proportionals in order to duplicate the cube. In geometry, too, he was the first to discover the cube, as Plato says in *Politeía*. [Diogenes Laertius (*c*. 3rd century AD), vol. II, pp. 395-397]

No contemporary mathematicians who are also generals leading armies come easily to mind. In like manner, if an ancient Roman general was still around we might ask him if he regarded Archimedes of Syracuse to be an idle or impractical speculator. He would answer, "No."

Realism and scientific pragmatism characterized mathematics from its beginning and were held as tenets by mathematicians until the late 19th century. After Plato the view of "what 'reality' really is" took mathematicians down a trail leading to what would have to be called 'idea realism' or 'immaterialism.' For example, in *c*. 100 AD Nichomacus of Gerasa wrote,

> The ancients, who under the leadership of Pythagoras first made science systematic, defined philosophy as the love of wisdom. Indeed, the name itself means this, and before Pythagoras all who had knowledge were called "wise" indiscriminately – a carpenter, for example, a cobbler, a helmsman, and in a word anyone who was versed in any art or handicraft. Pythagoras, however, restricting the title to apply to the knowledge and comprehension of reality, and calling the knowledge of the truth in this the only wisdom, naturally designated the desire and pursuit of this knowledge 'philosophy,' as being the desire for wisdom.
>
> He is more worthy of credence than those who have given other definitions, since he makes clear the sense of the term and the thing defined. This "wisdom" he defined as the knowledge, or science, of the truth in real things, conceiving "science" to be a steadfast and firm apprehension of the underlying substance, and "real things" to be those which continue uniformly and the same in the universe and never depart even briefly from their existence; these real things would be things immaterial, by sharing in the substance of

> which everything else that exists under the same name, and is so called, is said to be "this particular thing," and exists. [Nichomacus (*c*. 100 AD), pg. 811]

By the time of the Enlightenment the community of philosophy and science had come to regard mathematics as the last bastion of rationalism and the sole bulwark resisting the rising tide of British empiricism. The practical successes of empiricism following Newton's *Principia* seemed to be sweeping over all of science, but the *Principia* itself relied upon mathematics for its deductive power. This put the philosophy of rationalism and that of empiricism at odds with one another in a pitched battle (and this despite the mortal wounds to both philosophies that the skepticism of Hume had inflicted; both camps answered Hume's criticisms by ignoring them). To the Enlightenment thinkers it did seem that mathematical rationalism was unassailable and that what Davis & Hersh call "the Euclid Myth" was not at all mythical. All this changed radically in the 19th century when mathematics was overtaken by what came to be called 'the crisis in the foundations' of mathematics. Davis & Hersh tell us,

> There are two strands of history that should be followed. One is in the philosophy of mathematics; the other is in mathematics itself. For the crisis was a manifestation of a long-standing discrepancy between the traditional ideal of mathematics, which we can call the Euclid Myth, and the reality of mathematics, the actual practice of mathematical activity at any particular time. . . .
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> What is the Euclid myth? It is the belief that the books of Euclid contain truths about the universe which are clear and indubitable. Starting from self-evident truths, and proceeding by rigorous proof, Euclid arrives at knowledge which is certain, objective, and eternal. Even now, it seems that most educated people believe in the Euclid myth. Up until the middle or late nineteenth century, the myth was unchallenged. Everyone believed it. It has been the major support for metaphysical philosophy, that is, for philosophy which sought to establish some a priori certainty about the nature of the universe[1]. . . .
>
> In [the battle between rationalism and empiricism] both sides took it for granted that geometrical knowledge is not problematical, even if all other knowledge is. . . . For the rationalists mathematics was the best example to confirm their view of the world. For empiricists it was an embarrassing counterexample which had to be ignored or somehow explained away. . . . This embarrassment is still with us; it is a reason for our difficulties with the philosophy of mathematics. . . .
>
> In the nineteenth century, several disasters took place. One disaster was the discovery of non-Euclidean geometries, which showed that there was more than one thinkable geometry. A greater disaster was the development of analysis so that it overtook geometrical intuition, as in the discovery of space-filling curves and continuous nowhere-differentiable curves. These stunning surprises exposed the vulnerability of the one solid foundation – geometrical intuition – on which mathematics had been thought to rest. The loss of certainty in geometry was philosophically intolerable, because it implied the loss of all certainty in human knowledge. [Davis & Hersh (1981), pp. 323-331]

Davis & Hersh go on to describe the desperate but all-in-vain efforts of mathematicians up through the first third of the twentieth century to fix 'the crisis in the foundations' and restore apodictic certainty to mathematics. Their strenuous efforts met their Waterloo with the discovery of Gödel's theorems, which put a stake through the heart of mathematicians' efforts to resolve 'the crisis in the foundations' and moved them to give up on it. This did not mean that empiricism won out. Empiricism had been dealt a deathblow by Hume, the Great Skeptic whose criticisms of empiricism's foundations are unanswerable by any ontology-centered metaphysic.

This brings us to the Critical Philosophy and epistemology-centered metaphysics. Analyzed

[1] This 'metaphysical philosophy' by another name is called 'rationalism.'

from epistemology-centered grounds, the eventual failure of the ancient metaphysical position taken up by mathematics practitioners was inevitable and unsurprising because that metaphysic seeks to draw a real division between 'mathematics' and the mathematician who carries out the mathematical work. But there ***is*** no mathematics without the mathematician. Echoing Protagoras, *Man* ***is*** *the measure of all things*, including those things we call 'mathematical.' To find the *real explanation* (*Realerklärung*) of 'mathematics,' we must put the mathematician back in the picture because there is no knowledge without a knower and the mathematician is the knower who knows mathematics.

§ 2. What Mathematics Is

§ 2.1 Mathematics in the Theoretical Standpoint of Critical Metaphysics

Kant discussed mathematics in *Critique of Pure Reason* and in numerous other places within the Kantian *corpus*. Although he lived before the 'crisis in the foundations' of mathematics, Hume had raised some very serious fundamental questions about mathematical knowledge which, as a philosopher, Kant regarded himself as duty-bound to answer and thereby save mathematics from sinking into the pit of skepticism. This could not and cannot be done by any ontology-centered metaphysic, but it can be and is quite successfully done by his epistemology-centered metaphysic. To begin with, Kant tells us,

> *Philosophical* knowledge is rational cognition from *concepts*, mathematical [knowledge] that from the *construction* of concepts. But to *construct* a concept means to *present a priori* the intuition corresponding to it. For the construction of a concept, therefore, a *non-empirical* intuition is required, which consequently, as intuition, is an *individual* Object but that must, as the construction of a concept (of a general representation), nevertheless express in the representation general validity for all possible intuitions that belong under the same concept. [Kant (1787), B741]

From this explanation of mathematical *knowledge*, the *Realerklärung* of mathematics follows in a straightforward way:

> Mathematics is the science of the construction of concepts [Kant (1776-95), 18: 141].

This is the real explanation from the theoretical Standpoint of Critical metaphysics. There is in this very brief statement a great deal for us to understand. First, every human being is engaged in the construction of concepts his entire life. It requires no training or instruction for a person to construct concepts. The capacity to do so is innate in the phenomenon of mind and is present in a newborn infant. However, the capacity to construct concepts and a *science* of the construction of concepts are two different things. A science is a doctrine constituting a system in accordance with a principle of a disciplined whole of knowledge that is understood by means of said science. Thus mathematics is, first of all, a science but not an *empirical* science. All mathematical objects are what Kant called "made objects" – objects that are what they are because we *say* they are this way and no other by *definition*. Mathematicians ought to be happy about this[2]; the fact mathematical objects are defined objects means that mathematical knowledge of them can be apodictically certain. What a mathematical object is *not* is an object of any possible empirical experience. Our knowledge of an object of this latter sort is contingent knowledge and so is never *certain* know-ledge. The challenge for any natural science is to *connect* these two different types of objects.

[2] They *ought* to be, but some of them will not be because mathematical objects are *never* objects of sensible *physical* Nature, and the Euclid Myth wishes them to be precisely that. Neither you nor I nor anyone else has ever had or will ever have a direct and immediate experience of, say, an irrational number *per se*.

If it was impossible to connect mathematical objects and physical objects then mathematics would have to be called an "unnatural" science. Fortunately, it *is* possible to forge connections between them [Wells (2006), chap. 23; (2009), chap. 1] and so it is correct to call mathematics a *non-natural* science. It is probably more important for other scientists to understand this than it is for mathematicians to understand this because: (*i*) if, e.g., a physicist or a psychologist wants to use mathematics in his science the burden is on him to make the corresponding connections between the phenomena he studies and the mathematical objects he uses to understand these phenomena; and (*ii*) there are mathematicians who could not care less whether or not anyone other than a few other mathematicians has any use for or interest in the objects he defines and studies; mathematicians who feel this way couldn't care less if mathematics is called an *un*natural or a *non*-natural science.

Second – and I think it likely there will be quite a few mathematicians who won't like this very much – the science of mathematics is inseparably tied to the mental processes of intuition, the free play of imagination and understanding, and the subjective processes of reflective judgment. There can be no mathematics without the mathematician, as I said before, and getting knowledge through the *construction* of concepts inevitably involves all these processes. All mathematics is the produce of human creativity. I will say more about this later in the discussion of mathematics from the judicial Standpoint of Critical metaphysics.

But although all *doing* of mathematics is *subjectively* grounded, insofar as the real possibility of mathematics is concerned, mathematics as a science is *not* a subjective science. It is *objective*, and this because, as a science, mathematics is devoted to understanding mathematical *objects*. All one need do to see this is look at the way mathematicians go about their work. Kant noted,

> Mathematics, as synthetic *a priori* knowledge, grounds its possibility on the fact that its concepts can be built up; for they have to do only with space and time[3], in which Objects of intuition can be given *a priori*. These, however, are *quanta*, thus mathematics is a science of *quantis*. But it also considers quantity by means of number, by means of amount which can be built up in time by counting. [Kant (1776-95), 18: 240]

To properly understand the point Kant is making here, one needs to know something about the character of mathematical structures. Probably the most important contribution the Bourbaki mathematicians of the 1940s and 50s made was the discovery that all of mathematics can be reduced to combinations of three so-called "mother structures": topological structure, order structure, and algebraic structure. Now, the process of the pure intuition of space in sensibility is a process of topological structuring; that of the pure intuition of time is a process of order structuring; and in the free play of imagination and understanding is the capacity for algebraic structuring. This means that the capability for human beings to *create* mathematical knowledge is grounded in mental processes inherent in human sensibility and the free play of imagination and understanding. What Kant tells us here boils down to, "Pure mathematics is possible because of the way sensibility works." It does not depend on Plato's "remembering of ideas" [Plato (*c*. 399-387 BC)], divine revelation, or the "self-evident truth" of axioms. Mathematics is the way it is because, in a nutshell, it is a manifestation of *homo noumenal* human Nature.

But Kant's deceptively brief little note has an implication that goes further. The possibility of pure mathematics is *grounded* in human intuition and imagination, but it doesn't stop at this. To *construct* mathematical knowledge requires reasoning and judgmentation and, for its concepts to exhibit objective validity, this reasoning process requires *discipline* – i.e. the reasoning must be *scientific* reasoning.

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[3] More precisely, the pure forms of outer and inner sense, i.e. the pure intuitions of subjective space and subjective time in the synthesis in sensibility [Wells (2009), chap. 3].

In the past, many mathematicians rejected what Kant tells us because of connotations the non-technical use of the word "intuition" carries in everyday language. They presume that any practice that employs "intuition" must be the antithesis of the rigorous process of definition and proof that constitutes the core work of today's professional mathematician. The Bourbaki mathematicians were inclined to think this way. Mathematical knowledge is indeed knowledge *a priori* but it is not *innate* knowledge. "Knowledge *a priori*" only means "knowledge prior to actual experience" and, because no mathematical object is an object of any possible human experience, mathematical knowledge is indeed knowledge *a priori* but it is *constructed* knowledge *a priori*. That it requires discipline and rigor to establish mathematical knowledge *that is objectively valid* is why mathematics is a *science* and not just "an intuition." Kant notes elsewhere,

> Mathematical *analysis* is always a philosophical *synthesis*, only there I think the whole prior to thinking the parts; if, however, I think the parts prior to [thinking] the whole, then it is a mathematical *synthesis*. All *synthesis* rests on coordination and takes place through understanding. But *analysis philosophica* rests on subordination and takes place through reason. The ground is not part of the consequence, nor conversely. [Kant (1764-68), 17: 261]

Every science has developed principles of its practice and mathematics is no different. It is not incorrect to say that the developed principles of mathematical practice comprise what the ancient Greeks called *ortho dóxa* ("right opinion") in regard to the objects of mathematics because the objects of mathematics are not *merely* the way a mathematician defines them to be but ***are*** the way they are defined to be because they *must* be made this way to prevent contradictions from being built into the mathematical structures that mathematicians synthesize.

This, then, is mathematics from the theoretical Standpoint. This Standpoint is the determining Standpoint for understanding what mathematics is, but to fully understand what mathematics is – and how to teach it – we must also consider mathematics from the other two Critical Standpoints.

§ 2.2 Mathematics in the Judicial Standpoint of Critical Metaphysics

The theoretical Standpoint is the Standpoint for objectivity in Critical metaphysics. In contrast the judicial Standpoint is the Standpoint for subjectivity. As an object, mathematics is a science, as just discussed. What, however, is mathematics regarded *subjectively*? To ask this is to ask if mathematics is in some way *part of human Nature* and, if so, what part? If the answer is yes – and I am about to explain that it is – then mathematics regarded subjectively vindicates a *part* of the philosophy of rationalism that nineteenth and twentieth century mathematicians worked so hard, albeit unsuccessfully, to try to establish.

It is not surprising that Kant did not devote any extensive coverage to mathematics in *Critique of the Power of Judgment*. The surprise is that he devoted to it any coverage at all. In it he tells us that

> mathematics certainly has not the least share in the charm and emotion that music produces; rather, it is only the indispensable condition (*conditio sine qua non*) of that proportion of the impressions [of senses], in their combination as well as in their alteration, by means of which it becomes possible to grasp them [the data of sensations] together and to prevent them from destroying one another, so that they instead harmonize in a continuous movement and animation of mind by means of consonant affects and hereby in a comfortable self-enjoyment. [Kant (1790), 5: 329]

Conditio sine qua non literally means "flavoring without which [there is] nothing." Our more usual interpretation of this Latin phrase, "condition without which [there is] nothing," i.e., "indispensable," is a transference of contextual connotation. Because Kant brought up mathematics in the context of judgments of taste, the notion of mathematics as some sort of "flavoring" seems to me a more insightful way to analyze this peculiar remark about mathematics. In the discussion wherein the paragraph quoted above occurs, Kant was tying the notion of mathematics to the notion of aesthetic Ideas (which I previously discussed in chapter 13).

Kant's paragraph expresses a strange and vague notion of mathematics foreign to our modern ears. What he is describing in it is not mathematics as a science but, rather, mathematics as an *a priori* condition (i.e., "flavoring" or "giving flavor to") bringing about a *harmony* in aesthetic Ideas. Scientists and mathematicians often try to describe an aesthetic "sense" or "feeling" they have when they have come up with some important new understanding of something. Feynman said of the discovery of new physical laws of nature,

> One of the most important things in the 'guess-compute consequences-compare with experiment' business is to know when you are right. It is possible to know when you are right way ahead of checking all the consequences. You can recognize truth by its beauty and simplicity. It is always easy when you have made a guess, and done two or three little calculations to make sure it is not obviously wrong, to know when it is right. [Feynman (1965), pg. 171]

The renowned mathematician and polymath Henri Poincaré said of mathematical discovery,

> Such are the facts of the case, and they suggest the following reflections. The result of all that precedes is to show that the unconscious ego or, as it is called, the subliminal ego, plays a most important part in mathematical discovery. . . . Now we have seen that mathematical work is not a simple mechanical work, and that it could not be entrusted to any machine . . . It is not merely a question of applying certain rules, of manufacturing as many combinations [of ideas] as possible according to certain fixed laws. The combinations so obtained would be extremely numerous, useless, and encumbering. The real work of the discoverer consists in choosing between these combinations with a view to eliminating those that are useless, or rather not giving himself the trouble of making them at all. The rules which must guide this choice are extremely subtle and delicate, and it is practically impossible to state them in precise language; they must be felt rather than formulated. . . .
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> It is certain that the combinations which present themselves to the mind in a kind of sudden illumination after a somewhat prolonged period of unconscious work are generally useful and fruitful combinations which appear to be the result of a preliminary sifting. . . . More commonly the privileged unconscious phenomena, those that are capable of becoming conscious, are those which, directly or indirectly, most deeply affect our sensibility.
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> It may appear surprising that sensibility should be introduced in connection with mathematical demonstrations, which, it would seem, can only interest the intellect. But not if we bear in mind the feeling of mathematical beauty, of the harmony of numbers and forms, and of geometric elegance. It is a real aesthetic feeling that all true mathematicians recognize, and this is truly sensibility. [Poincaré (1914), pp. 56-59]

I can attest to you that I have personally experienced the sort of 'real aesthetic feeling' that Feynman and Poincaré try to describe here. Somewhere in these descriptions do we perhaps find the quintessence of mathematics *as a human faculty*? In Critical terminology, a ***faculty*** is the form of an ability insofar as the ability is represented in an idea of organization. 'Faculty' is a term that denotes a representing of how that ability is exhibited in experience. *As a science*, mathematics is a systematic doctrine for constructing concepts, in many ways akin to formal logic, and this is an idea of objective organization. But *subjectively mathematics* is *the form of the ability **to make** objective organizations out of manifold presentations in sensibility*. This, then, is what mathematics is when viewed from the judicial Standpoint.

We would have to mark this down as merely an interesting psychological speculation if it were not for the fact of an unexpected discovery of an immediate relationship between the mental development of children and the discovery by the Bourbaki mathematicians that all mathematics can be expressed as combinations of the three Bourbaki "mother structures." Piaget informs us,

> A number of years ago I attended a conference outside Paris entitled "Mental Structures and Mathematical Structures." This conference brought together psychologists and mathematicians for discussion of these problems. For my part, my ignorance of mathematics then was even greater than what I admit to today. On the other hand, the mathematician Dieudonne[4], who was representing the Bourbaki mathematicians, totally mistrusted anything that had to do with psychology. Dieudonne gave a talk in which he described the three mother structures. Then I gave a talk in which I described the structures I had found in children's thinking, and to the great astonishment of us both we saw that there was a very direct relationship between these three mathematical structures and the three structures of children's thinking. We were, of course, impressed with each other, and Dieudonne went so far as to say to me: "This is the first time that I have taken psychology seriously. It may also be the last, but at any rate it's the first." [Piaget (1970), pg. 26]

I previously mentioned that the capacity for constructing two of the three Bourbaki mother structures are found in the pure intuition of space (topological structuring) and the pure intuition of subjective time (order structuring) in the synthesis of sensibility. The capacity for constructing the third mother structure (algebraic structure) is found in the free play of imagination and understanding in the synthesis of judgmentation. That the three mother structures are known to be sufficient for the possibility of mathematics-as-a-science has been rigorously established by the work of the Bourbaki. However, because *all* of the objects of mathematics-science are objects that lie beyond the horizon of possible human experience, the marvel and wonder of mathematics *per se* has always been *that it is possible at all*. The number mysticism of the Pythagoreans and the speculations of philosophical rationalism were both attempts to explain this marvelous human capacity. Mathematics as the form of a capacity for the construction of concepts (by means of the synthesis of aesthetic Ideas) *is the real explanation for the possibility of mathematics-science*.

An immediate implication of this is that *all human beings have the potential ability to learn and practice mathematics as a science*. Mathematical ability is not a special gift possessed only by a few people. In its objective character, it is also not 'preformed' innate knowledge *a priori* – and this is why mathematics as a science, and even as a craft, must be *learned*. In principle there is no impediment to any person learning and becoming skilled in mathematics-as-science. Perhaps you might feel, as I do, that this is one of the many sublime aspects of being-a-human-being. Shakespeare, I think, said it best:

> What a piece of work is a man! how noble in reason! how infinite in faculty! in form and moving how express and admirable! in action how like an angel! in apprehension how like a god! the beauty of the world! the paragon of animals! [Shakespeare (1600-01), Act II, Scene II]

But while there is theoretically no principle hindering any person from becoming a skilled mathematician, there are practical considerations of circumstances that, I think quite obviously, deserve and even command our attention as teachers. This brings me to mathematics regarded from the practical Standpoint of Critical epistemology.

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[4] Jean Dieudonne (1906-1992) was one of the founders of the Bourbaki movement and perhaps its most well known member and spokesman.

§ 2.3 Mathematics in the Practical Standpoint of Critical Metaphysics

The Kantian *corpus* is almost silent in regard to mathematics from the practical Standpoint and most of what it does contain is not pertinent to purposes of this treatise. Kant tells us in many places that mathematics is an *instrument* of philosophy and natural science but says little about what makes it instrumental. What Kant's idea of mathematics from the practical Standpoint was can be gleaned from examining how he uses the idea *en passant* as he speaks of other things. One remark he recorded that is useful for this purpose is,

> A philosophy exists (and this is metaphysics) which employs mathematics merely as an instrument in order to organize *empirical* representations of sense according to an *a priori* principle . . . and which classifies *a priori* pure intuitions according to their form in order to present the schematism of concepts of reflection in a system. [Kant (1804), 22: 490]

Kant uses the word "instrument" in its connotation as a tool that can be applied to understand metaphysics systematically. He elsewhere states a similar relationship between mathematics and natural science and he expressly states that mathematics is not part of philosophy. Indeed, Kant strongly criticized Newton's *Principia* for holding that mathematics was part of philosophy. His position on this is similar to saying a hammer is not part of the house it is used to build.

What sort of "tool" is mathematics? Because mathematics is not any sort of corporeal object, the only answer that suffices for this question is that mathematics as a "tool" is a *skill*, specifically a skill that is used to construct systematic schematisms of concepts. These schematisms are constructed *a priori* (prior to objective experience), and this assigns mathematics-as-a-tool to one place only: the manifold of rules in practical Reason. That means mathematics from the practical Standpoint is viewed as *the developed maxims* by means of which Reason carries out (by ratio-expression) its *regulation* of constructions of topological and order structures in intuition as well as its regulation of constructions of algebraic structures in the free play of imagination and understanding. However, practical maxims are themselves empirical constructs – and for that reason contingent constructs – and so an understanding of mathematics from the practical Standpoint is an understanding that is concerned with how ratio-expressions of pure Reason effect systematic construction of concepts. To understand this requires a synthesis of the theoretical and judicial Standpoints of mathematics.

Let me denote mathematics from the theoretical Standpoint as "theoretical math." Let me likewise denote mathematics from the judicial Standpoint as "judicial math." The synthesis of the two Standpoints (which produces the practical Standpoint according to the Critical Logic of Kantian metaphysics) then yields mathematics from the practical Standpoint, and I will denote this by calling it "practical math." Symbolically,

$$\text{theoretical math} \;+\; \text{judicial math} \quad\rightarrow\quad \text{practical math.}$$

Properly, this synthesis merits a full treatment of its thorough deduction in its own right. However, I think this treatise is not the appropriate place to present this and it should be relegated to a paper devoted exclusively to this purpose. For what is pertinent to present purposes, a summary of the results of this synthesis suffices.

First, consider what the role of the objects of mathematics is in the theoretical Standpoint. These objects are all, without exception, *noumena* of ideas in the manifold of concepts that stand at or beyond the horizon of possible human experience. This is to say that mathematical objects are placed at and beyond the frontiers of knowledge in the structure of Critical ontology. Figure 1 illustrates the Critical structure of ontology. I have previously provided a short discussion of this structure in Wells (2011) but a brief explanation of what the figure represents is appropriate here.

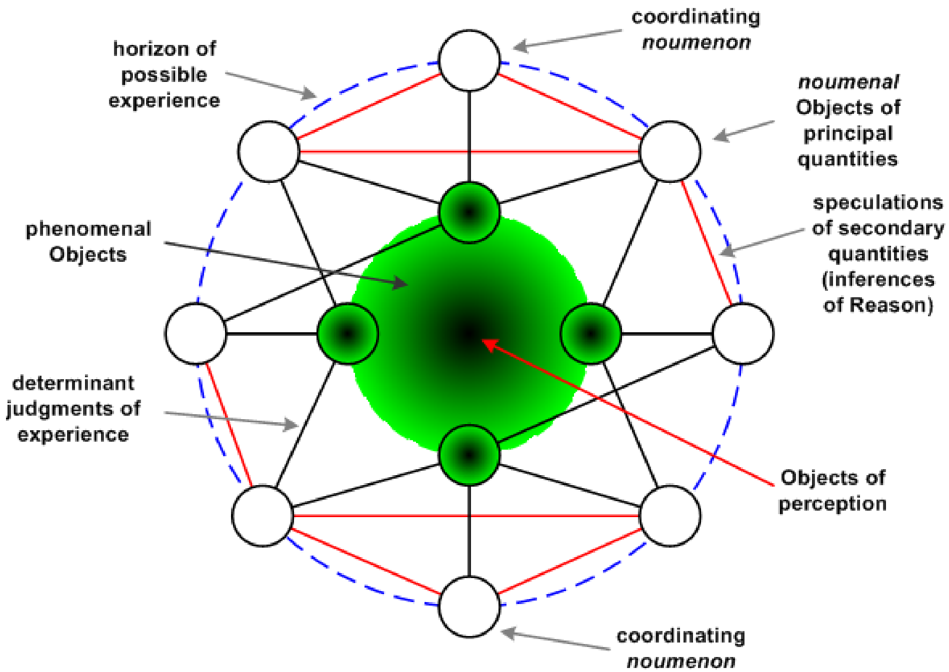

Figure 1: Illustration of the structure of Critical ontology in science.

Each successively higher concept in the manifold of concepts is abstracted from the lower concepts it coordinates, and this means that at each successively higher level the concept contains less sensuous matter in its representation. When the point is first reached where all sensuous matter has been removed by abstraction, the concept is called an idea and the object of that idea is called a *noumenon*. All lower concepts standing under this idea are concepts of phenomenal objects that Slepian described as "belonging to the real world" [Wells (2009), chap. 1]. The ideas standing at the horizon of possible experience have their objects denoted *mathematical* objects. Those objects corresponding to coordinations of phenomena are called ***principal quantities*** of mathematics. Those which coordinate mathematical *noumena* with one another and are also placed in a relationship with a phenomenon are called *coordinating* noumena.

It remains possible for higher concepts, beyond the horizon of possible experience, to be synthesized as abstractions from principal quantities, and these are called ***secondary quantities*** of pure mathematics. Because their objects are beyond this horizon, objects of secondary quantities *have no ontological significance whatsoever*. (This is why figure 1 does not display them). They do, however, have *epistemological significance* in the "mathematical world" beyond the horizon of possible human experience. Their *function* is to provide relationships among principal quantities through inferences of Reason. Most theoretical mathematics is concerned with the construction of concepts of secondary quantities and these quantities constitute the bulk of ordinary mathematical theory. The secondary quantities of pure mathematics are indicated by the red lines in figure 1 that supply connections among the principal quantities. They have *no connection whatsoever* with phenomena. Secondary quantities must be *mathematically* valid, but they never have *objective* validity for sensuous phenomena of real experience.

The process of pure Reason is the master regulator of all non-autonomic human activity, both mental and physical. The synthesis of understanding in the process of determining judgment falls immediately under regulation by speculative Reason (because determining judgment does not determine its own employment), while the synthesis in sensibility and in reflective judgment are *indirectly* regulated by Reason via the free play of imagination and understanding. Corresponding

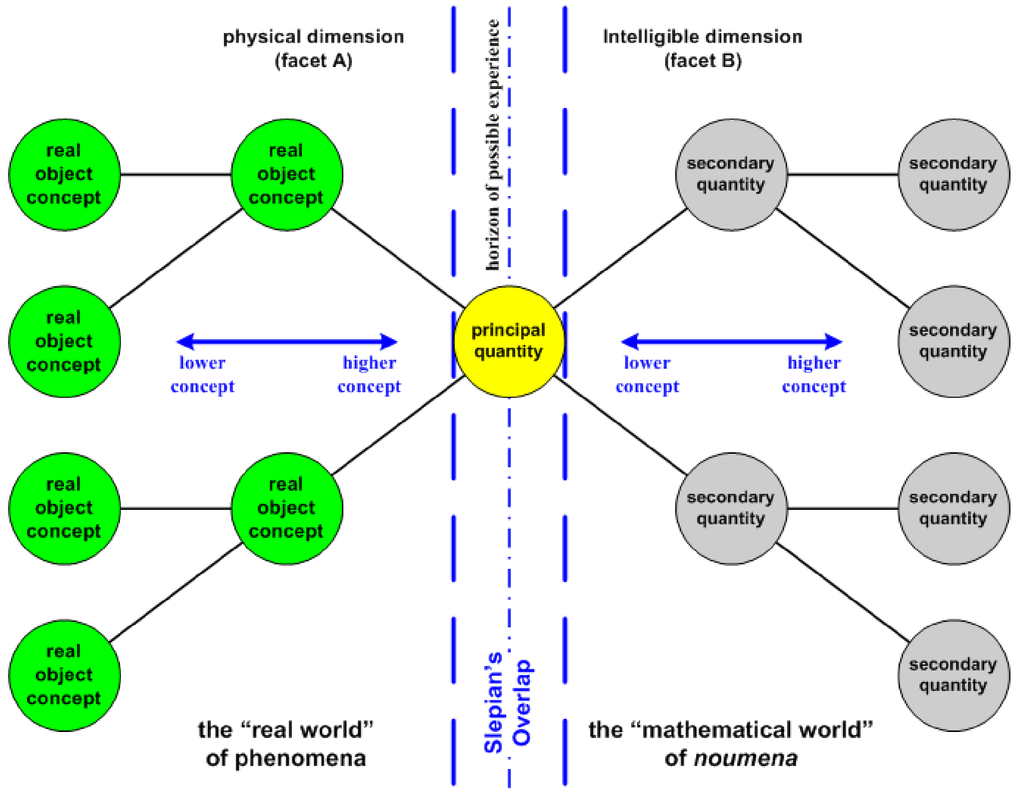

Figure 2: Partial manifold of concepts illustrating Slepian's facet division of knowledge into dimensions of the world of real phenomena (facet A) and the world of mathematical *noumena* (facet B).

to the structure of Critical ontology in figure 1 there is a logical structure in the manifold of concepts in which are found both concepts of sensible phenomena and concepts of pure form that have no immediate connection to phenomena and are ideas of pure *noumena*. The latter concepts are the secondary quantities and they are concepts of pure mathematics. Figure 2 is an illustration of a partial manifold of concepts detailing the distinction between concepts within the horizon of possible human experience and those that go beyond this horizon. The concepts of secondary quantities are those beyond this horizon. Slepian described the logic of this structure by defining two orthogonal "dimensions" of understanding, *viz*., the "real world" of physical phenomena and the "mathematical world" of supersensible *noumena* [Slepian (1976)].

Slepian's dimensioning of understanding somewhat parallels ancient Greek divisions dating back to Parmenides ('the way of what is, what is-not, and what is-and-is-not') and Plato (the 'world of the senses' and the 'world of the mind') [Marías (1956), pp. 19-25, 43-51] but it does so without the omnipresent realism that characterized all of ancient Greek philosophy. The Slepian model reflects the effect of what Kant called the transcendental dialectic of pure Reason. The process of Reason knows no objects and in its regulating drive for the perfection of equilibrium it makes transcendent uses of the process of determining judgment that enable human beings to pass beyond the *dabile* of the senses *to unify the manifold of concepts* with mathematical ideals.

Figure 2 illustrates the transcendental place of mathematical Objects in understanding. This place tells us where the regulations of practical mathematics lead to as *teleological ends*. Next we must explore how maxims in the manifold of practical rules combine this endpoint with the reflecting Subject who conceptualizes knowledge by means of mathematics. To do this, it is useful to classify theoretical, judicial, and practical math structure in terms of the general ideas of representation [Wells (2009), chap. 2].

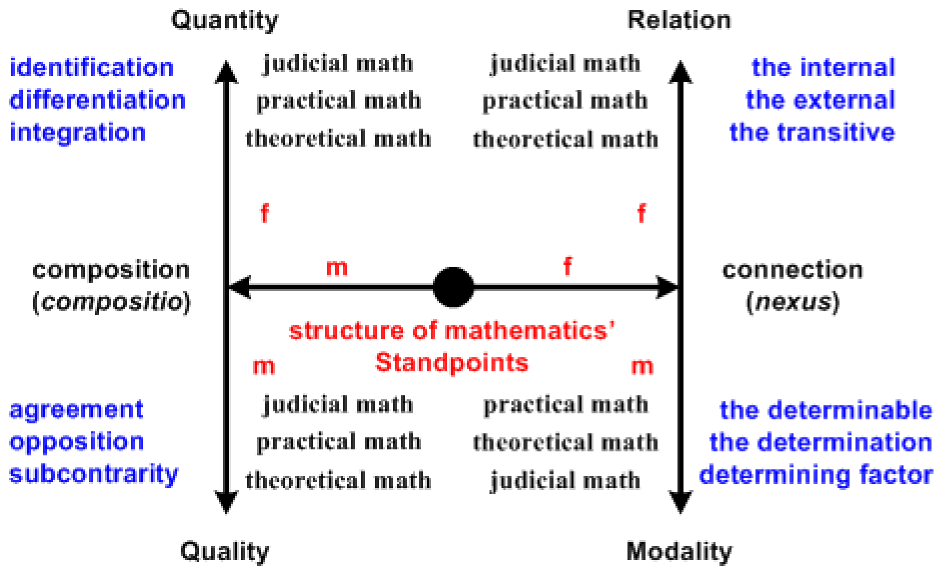

Figure 3: The 2LAR structure of mathematics' Standpoints in relationship to the general ideas of representation in Critical metaphysics.

Figure 3 illustrates this structure in 2LAR form and shows the correspondences between the three Standpoints of mathematics and the ideas of representation in general of Critical metaphysics. The object-oriented headings (Quantity, Quality, and Relation) pertain to the object being represented ('mathematics' in this case). The Subject-oriented heading (Modality) pertains to the relationship between object representation and the person (the Subject) who does the representing.

Judicial math stands with the general idea of identification in Quantity because the representations of judicial math are *consequences* of aesthetic Ideas (representations in sensibility from the synthesis in continuity of the aesthetic Idea [Wells (2009), chap. 7]). Kant described an aesthetic Idea as "that representation of imagination that occasions much thinking without it being possible for any determinant thought, i.e., *concept*, to be adequate to it" [Kant (1790), 5: 314]. Judicial math is not itself an aesthetic Idea, but is rather its consequence for reflective judgments of the manifold in sensibility. Aesthetic Ideas have a Quality of something sublime, which is an energizing Quality, but the uncontainable character of an aesthetic Idea means aesthetical expedience in sensibility cannot be judged without "singling out" some part of the manifold in sensibility. To single out a *part* of the manifold in sensibility is to make a *judgment of taste*. Kant tells us that judgments of taste are logically singular [Kant (1790), 5: 215], and the logically singular corresponds to the general idea of identification in Quantity.

Judicial math stands with the general idea of agreement in Quality because of its character of *harmonizing* the data of sensation (as Kant explicitly stated in the earlier quote given in § 2.2). A representation of judicial math can in this context be said to be a coalescing function.

Judicial math stands with the general idea of internal representation in Relation because its placement in affectivity makes representations of judicial math entirely subjective, and thus the making of these representations is entirely bound up with the Subject himself. This is Relation internal to the Subject unconnected with any external object. Finally, judicial math stands with the general idea of the determining factor in Modality because its representations are subjectively sufficient and necessitated for the possibility of object-oriented mathematical representations.

Theoretical math stands with the general idea of integration in Quantity because mathematics from the theoretical Standpoint is a science of constructing concepts out of the manifold in sensibility. A construct is a whole comprised of parts, and synthesizing parts to make a whole of representation is an act of integration by the technical definition (*Realerklärung*) of that term. Theoretical math stands with the general idea of subcontrarity in Quality because it always follows a fundamental principle of non-contradiction, i.e., within the system of a mathematical structure no part of that system can contradict another part of it. Indeed, non-contradiction is the basis of the idea of mathematical truth. *Removal* of contradiction is an act of subcontrarity in synthesis.

Theoretical math stands with the general idea of transitive representation in Relation because *as the doctrine of a science* theoretical math must effect a Relation of community between knowledge of mathematical objects and what the mathematician holds-to-be-true. It is, in other words, a relationship between objects external to the mathematician and judgments internal to the mathematician himself. This, however, is the technical condition for the transitive form of connection in Relation. Finally, theoretical math is the outcome of mathematical reasoning, i.e., it is a determination and so stands with the general idea of the determination in Modality.

We get the idea of practical math from synthesis of the ideas of judicial and theoretical math. In Quantity, practical math stands with the general idea of differentiation. Maxims of practical math (in the manifold of rules) must *pick out* how concepts are to go into the construction of knowledge but must do so without *knowing* what these concepts are. This is because Reason is objectively blind but a concept is an objective representation. It is, in other words, a practical functional skill for *directing* the employment of determining judgment in the synthesis of understanding rather than being the employment itself. Practical maxims exercised through ratio-expression can do this only by differentiations among the regulative principles of speculative Reason, i.e., by differentiations among the transcendental Ideas. These principles are:

(a) the principle of unity in the synthesis of appearances;

(b) the principle of absolute unity in the thinking Subject;

(c) the principle of absolute completion in the series of conditions; and

(d) the principle of absolute unity of the condition of all objects of thinking in general [Wells (2009), chap. 2].

In Quality, practical math stands with the general idea of opposition (*Widerstreit*). New practical maxims are never constructed in the manifold of rules unless the person experiences a disturbance to equilibrium that he cannot bring to reequilibration by means of type-α compensation behaviors. Furthermore, practical judgments in pure practical Reason are fundamentally negative judgments because what Reason is *able* to judge is *in*compatibility of the existing manifold of rules with the formula of the categorical imperative of practical Reason. Indeed, this is the character of pure Reason that underlies saying that human beings exhibit "free won't" rather than "free will." The constructs of practical rules are, consequently, the outcomes of opposition between the condition of the Subject and the dictate of the formula of the Reason's categorical imperative. Gainsaying (failure) of anticipated expectations provokes new rule construction.

Practical math stands with the general idea of the external Relation because practical rules in the manifold of rules are constructed to connect the Subject with the 'not-me' of external Nature. The notion of such a connection is not contained in the idea of the Subject himself or in the idea of the environment in which he lives. Consequently, the form of connection comes under the logical idea of hypothetical Relation and the notion of causality-and-dependency in understanding. This understanding of Relation in practical math is consistent with Piaget's empirical finding that cognizance begins "at the periphery" (goals and results) [Piaget (1974), pp. 333-337].

Finally, practical math stands with the general idea of the determinable in Modality. The manifold of rules is a constructed manifold, hence all of its maxims are undetermined prior to the Subject undergoing disturbances to equilibrium gainsaying impetuous judgments of expedience made by the process of reflective judgment. New maxims that are formed in the manifold of rules are formed if and only if the accommodations they make to the manifold restores agreement of the manifold with the dictate of the categorical imperative. But because this restoration can again be upset by future experiences, congruence of the accommodated manifold with the dictate of the categorical imperative is only a *possible* congruence, and this places it with the general idea of the determinable in Modality.

The constructions of practical math lead to cognition of "how to" practices (algorithms and heuristics) in the manifold of concepts. Practical math is, as Kant indicated, mathematics regarded as a tool for constructing empirical knowledge, and this is its *Realerklärung* from the practical Standpoint of Critical metaphysics. Mathematics *instruction* has for its aim the cultivating of the development of this tool. This brings us to the next issue, namely, how can instruction do this?

§ 3. Real Mathematical Skill is Cultivated By Cultivation of Taste

The intended outcome for all instruction in mathematics is to cultivate in the learner skills in *doing* mathematics and *understanding* the objects of mathematics. Thus these outcomes target the learner's practical manifold of rules and his theoretical manifold of concepts. Teachers, however, are confronted by a formidable challenge: *Nothing* that a teacher can do can *immediately* effect structures in either the manifold of rules or the manifold of concepts. Only the learner can do that and, in a manner of speaking, the teacher finds the doors to these manifolds closed to him and locked from the inside. The human Nature of "the gatehouse to the citadel of mind" is grasped by examining the thinking and judgmentation structure of the phenomenon of mind. Figure 4 re-depicts this structure here for your convenience. The only access a teacher has to "the citadel of mind" is through the learner's synthesis in sensibility via receptivity. A teacher's access to the learner is effected through his actions that provide environmental stimulation to the learner and evoke *reflective* judgments. A teacher's actions can *immediately* reach sensibility and affect the manifold of Desires but can *never* immediately touch determining judgment or appetition.

Because a learner's practical and theoretical math skills are his alone to *develop*, a teacher's ability to *cultivate* them is restricted to his ability to influence and cultivate the learner's faculty of *judicial* math. This is as much as to say that mathematics instruction must take *direct* aim at learner ***affectivity***. Instruction *téchne* peculiar to mathematics education must be given its particular specialization from the practical and theoretical goals being sought. This is, of course, true of every kind of instruction; but it begs for especial emphasis in the case of mathematics instruction because of a modern peculiarity in mathematics pedagogy for which it is not unfair to blame the Bourbaki mathematicians. Mathematics is often viewed as the most objective and quantitative of topics – and I do not fundamentally disagree with this viewpoint – but that does not change the fact that all effective mathematics *instruction* begins by appeal to learner affectivity. There is more truth than one might suspect contained in saying of some particular learner "he has a *feel* for math." But it is important to understand that this is a *cultivated feeling* and not an *innate talent*.

Instruction traditions down through the centuries have always, at least until relatively recently, recognized at some level that effective instruction depends on subjective appeals. How it depends on this was not particularly well understood by teachers from antiquity to the 19th century. Prior to Pestalozzi, this understanding was simplemindedly brutish. Marrou tells us,

> There were schools for the training of the scribe . . . the ruins of which Mesopotamian archaeologists claim to have discovered here and there – as, for example, in recent times at Mari on the Euphrates. There, in the ruins of the palace destroyed by fire at the end of the second millennium, A. Parrot unearthed two classrooms . . .

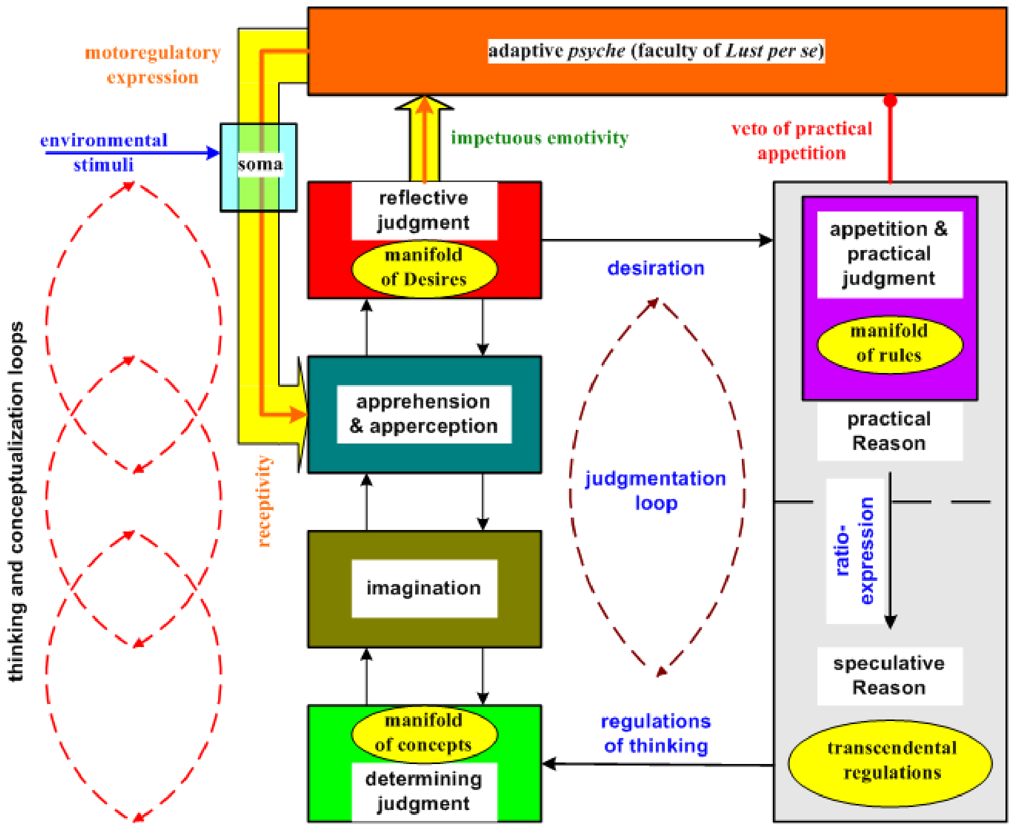

Figure 4: Thinking and judgmentation structure of the phenomenon of mind.

> The method of instruction was very elementary, and called for no initiative in the pupil; it depended for its effectiveness on his docility and therefore, as we might expect, made use of the most drastic corporal punishment, as did the classical education of a later date. The Hebrew word *musar* means both instruction and correction or chastisement. Here again the most vivid descriptions come from Egypt. "The ears of the stripling are on his back. He hears when he is being beaten." "You brought me up when I was a child," declares a grateful pupil to his master; "you beat me on the back and your teaching penetrated my ears." [Marrou (1948), pg. xvi]

I have a hunch you and I would agree that a desire to avoid taking a beating is an affective portal. But what sort of learning does it produce? Etch A Sketch® learning? That is not unlikely. Our modern Taylorite institution is more genteel: we use grades to beat the pupil psychologically; it leaves fewer visible bruises and scars. We have even simplified our simplemindedness in 21st century America by using pupils' performances on standardized tests to beat, or threaten to beat, their teachers economically. Such is the boundless wisdom of institutionalized Taylorism. It would be called child abuse if Taylorism did not regularly abuse adults psychologically too.

A special peculiarity of mathematics instruction since the mid-20th century has been a gradual but persistent effort to ignore affectivity and focus instead on features of formalism. Attempts to make mathematics interesting or otherwise *positively* appealing to learner affectivity are kept to a minimum if they are kept at all. Formalism began as a mathematician's method motivated by the "disasters" of 19th century mathematics and the attending "crisis in the foundations." From there it hardened into a pseudo-philosophical attitude and gradually seeped out from institutionalized professional *method* in mathematics and into mathematics *instruction*. The historical route it took started with the logicians (e.g. Bertrand Russell), traveled to David Hilbert, died briefly from a severe case of Gödel's theorems, and was resurrected by the Bourbaki. Davis & Hersh tell us,

> In the mid-twentieth century, formalism became the predominant philosophical attitude in textbooks and other "official" writing on mathematics. . . . Contemporary formalism is descended from Hilbert's formalism, but it is not the same thing. Hilbert believed in the reality of finite mathematics. He invented metamathematics in order to justify the mathematics of the infinite. This realism-of-the-finite with formalism for the infinite is still advocated by some writers. But more often the formalist doesn't bother with this distinction. For him, mathematics, from arithmetic on up, is just a game of logical deduction.
>
> The formalist defines mathematics as the science of rigorous proof. In other fields some theory may be advocated on the basis of experience or plausibility, but in mathematics, he says, either we have a proof or we have nothing. . . . In brief, to the formalist mathematics is the science of formal deductions, from axioms to theorems. Its primitive terms are undefined. Its statements have no content until they are supplied with an interpretation. . . . Indeed, brief reflection shows that the formalist view is not plausible according to ordinary mathematical experience. . . .
>
> The most influential example of formalism as a style in mathematical exposition was the writing of the group known collectively as Nicolas Bourbaki. Under this pseudonym, a series of basic graduate texts in set theory, algebra and analysis was produced which had a tremendous influence all over the world in the 1950s and 1960s. [Davis & Hersch (1981), pp. 339-344]

Hilbert's formalism was, in part, a response to the collapse of the efforts of Russell and others to restore "mathematical certainty" (rationalism) by means of the "logicians' program" of the late 19th and early 20th centuries. This effort collapsed, ironically enough, because of the discovery of the Russell Paradox. As a mathematics philosophy, the logicians' program and formalism are both bankrupt pseudo-metaphysics. As teaching philosophies they are unmitigated disasters. Poincaré was extremely critical of both the logicians' movement and of Hilbert's formalism:

> [Many mathematicians] have become so familiar with transfinite numbers that they have reached the point of making the theory of finite numbers depend on that of Cantor's cardinal numbers[5]. In their opinion, if we wish to teach arithmetic in a truly logical way, we ought to begin by establishing the general properties of the transfinite cardinal numbers and then distinguish from among them quite a small class, that of the ordinary whole numbers. Thanks to this roundabout proceeding, we might succeed in proving all the properties relating to this small class (that is to say, our whole arithmetic and algebra) without making use of a single principle foreign to logic.
>
> This method is evidently contrary to all healthy psychology. It is certainly not in this manner that the human mind proceeded to construct mathematics, and I imagine, too, the authors do not dream of introducing it into secondary education. But is it at least logical, or, more properly speaking, is it accurate? We may well doubt it. [Poincaré (1914), pp. 144-145]

Poincaré (1914) is a book[6] that launches an all out attack on the entire direction mathematics

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[5] Cantor's cardinal number is "a cardinal number greater than all ordinary cardinal numbers." The most popular symbol for it seen in textbooks is ∞ but technically the approved symbol is $\aleph_0$ (pronounced "aleph naught"). When you see a series of cardinal numbers written in a form like 1, 2, 3, . . ., Cantor's 'number' is supposed to somehow stand at the end of the continuation denoted by the " . . . " notation. To add juice to the cocktail, Cantor had another infinite ('transfinite') number as well, the "infinity" of the real numbers. Cantor's cardinal numbers utterly lack real objective validity.

[6] Poincaré died in 1912. The book I cite is the 1914 edition reprinted in 1996.

was taking in responding to the "crisis in the foundations." Whether or not the formalists dreamed of introducing it into secondary education, teachers did not. Mathematics education retained its old flavor – which I will call "Old Math" – until the 1950s and 1960s when, under the influence of the Bourbaki, formalism was introduced not only into college mathematics instruction but – under the name 'New Math' – into *primary* school mathematics instruction as well.

The Bourbaki movement began with a group of young, mostly French, mathematics graduate students who were, more or less, rebelling against the mathematics philosophy of their professors. That philosophy was basically the same philosophy as that of Poincaré. According to the folklore, what they found most objectionable about Old Math was its "intuitive" character. The young Bourbaki were very taken with Hilbert and formalism. They sought to have this view become the dominant view of mathematics. Bourbaki formalism was adhered to in the writing of their text-books and this formalism is the principal reason why almost all advanced mathematics books are opaque to those readers who have not been schooled in the ways of the Bourbaki. If a person set out to deliberately propagate ignorance of mathematics, he could likely find no way to do it more effectively than the way Bourbaki literature accomplishes it.

In the late 1950s, primarily as a result of the panic caused in the United States by the launch of Sputnik in 1957, an attempt was made to reform mathematics education. This was the New Math movement, which enjoyed a brief period of institutionalization from the 1960s to the mid-1970s. New books and teaching methods were developed primarily by mathematics professors without much influence by or advice from teachers. As Klein points out [Klein (2003)], the movement did not spread uniformly to all schools and, as early as 1962, it was even being criticized by some college mathematics professors. It was strongly opposed by teachers in the public school system. Dr. Ginger Warfield, Emeritus Principal Lecturer with the University of Washington Department of Mathematics, commented on her impressions of the relationship between the educator community and the mathematics community. She wrote that educologists tended to regard the mathematicians who produced the New Math program as

> arrogant clods trampling in where they had no business to be [Warfield (2007)],

and that the educator community at large

> regards mathematics as obscure and frightening and best left in the hands of the severely gifted. For the average teacher, the important thing was to protect children from it. [*ibid*.]

To the degree Warfield's personal impressions are accurate, this situation bodes ill for education. It is one of the profound ill effects of long-standing institutional isolation of colleges of education from the other isolated silos within American universities [Farkas & Johnson (1997); Mirel (2011)]. The notion that mathematics is suitable only for 'the severely gifted' is another of the educology establishment's many bigotries infecting public education in America.

By the mid-1970s almost everyone had abandoned the New Math program. After a brief 'back to basics' or Old Math interregnum in mathematics education, a new reform movement in mathematics education began. This one was prompted in large measure by publication of *A Nation at Risk* [Gardner *et al*. (1983)]. Educologists working through the National Council of Teachers of Mathematics (NCTM) put together this new reform movement, officially called the NCTM Standards and unofficially called Reform Math. Reform Math went into effect in 1989.

While it does keep some algorithms-and-heuristics features of Old Math, Reform Math is a reversion to the "projects method" idea of the Progressive Education Movement (PEM) in the 20th century [Klein (2003)]. This time it is presented under the label "constructivist math." The label vaguely refers to any of several mini-theories of education. Some of these mini-theories are based on reasonable psychology research. Jeremy Bruner's mini-theory is one example of these. Others of them are raised up on flimsy psychological pretexts with very dubious grounding. Papert's mini-theory of "constructionism" is an example belonging to this class. Most claim to be "based" on research by Piaget or Vygotsky but these claims seem to be of the same sort as we see when a Hollywood movie claims to be "based" on true events. More often than not the claim is like claiming the legends of Charlemagne are based on Charlemagne himself [Bulfinch (1863)].

The "projects" feature of Reform Math tends to arouse the passionate ire of back-to-basics partisans. Their opposition is not unjustified. Like the PEM projects movement, it suffers from a number of fundamental shortcomings, many of which were correctly criticized by one of the most prominent members of the Progressive Education Movement, Boyd Bode:

> It does not follow . . . that the projects method . . . can be made to cover the whole field. However cordially its merits may be recognized, as a universal method it suffers from certain obvious defects. By definition it takes no account of either logical organization or "social insight." Its spirit is the spirit of immediate practicality, which is the spirit of an exclusive vocationalism. This is no objection to the method unless we apply it too widely. If we do so, we find that our practicality overreaches itself. Learning that is limited to this method is too discontinuous, too random and haphazard, too immediate in its function, unless we supplement it with something else. Perhaps children may learn a great deal about numbers from running a play store or bank, but this alone does not give them the insight into mathematics they need to have. . . .
>
> This is not a criticism of the project method, but an attempt to show its limitations. Since the principle is limited to its application, it does not fully meet the demand for a kind of education that is not tied up so closely with immediate demands. So the idea naturally suggested itself that the method might be extended, that it might be possible to retain the virtues of the project method without this limitation. This has been attempted by various writers. It has proved a difficult undertaking, which is not surprising since it is not easy to discover anything that is distinctive about the method if we take away the feature of learning for the purpose of meeting an immediate practical need. In order to make the method cover other kinds of learning, which are not dominated by the idea of immediate practicality, it became necessary to change the meaning of the project method by identifying its essential nature with something other than this type of learning. The attempt to do this has introduced considerable confusion, so that the term project method had tended to become a name for a conglomeration of ideas and not for a definite guiding principle. [Bode (1927), pp. 150-151]

Changing the name to "constructivist math" does nothing to cure the ills Bode pointed out. Was Reform Math really a reform or is it just an repetition of an old PEM reform and therefore no reform at all? Or is it some of each? Whatever it is, it does not ask "what is math?" and sticks with traditional prejudices about the topic that date back to the ancient Greeks. The institution of Reform Math was the occasion for a raucous and largely propaganda-based national controversy that came to be called the Math Wars. Along with other education controversies, the Math Wars was a factor that motivated the still evolving reforms of the Common Core State Standards Initiative (CCSSI). I see no evidence that the CCSSI will turn out to be different. At the same time, though, "back to basics math" likewise does not escape Bode's vocationalism indictment.

The principal shortcoming exhibited by *all* the conflicting opinions on mathematics education is that they are not based on an epistemologically sound understanding of "what math is," namely, knowledge through construction of concepts. The essential task for mathematics instruction is to cultivate in the learner a capacity for knowledge-construction, and to do this the *téchne* must appeal to learner affectivity and not to the object-oriented *symptoms* of "doing math." Mathematics, as judicial math, is not the subjective capacity for aesthetical taste but it is grounded in that human capacity. Inasmuch as judicial math is manifested by approvals of taste, it can be called the faculty of knowledge that is exhibited by judgments of taste. By examining properties of the approvals of taste, the properties of how mathematical skill is cultivated are revealed.

§ 4. The Approvals of Taste and Properties of Judicial Math

Taste is *the aesthetical capacity for judgmentation of an object or mode of representation through a subjective satisfaction or dissatisfaction in which there is no objective interest.* Taste is a selection of that which is generally engaging according to the laws of sensibility. The ***approvals of taste*** are *the synthetic functions for judgments of taste:* patterning, coalescing, conceptualizing, and precisioning. These functions correspond, respectively, to Quantity, Quality, Relation, and Modality of approval. As Kant expressed it,

> In everything that is to be approved in accordance with taste there must be something that facilitates the differentiation of the manifold (patterning); something that promotes intelligibility (relationships, proportions); something that makes the pulling of it together possible (unity); and finally, something that promotes its distinction from all other possibilities (precisioning). [Kant (*c*. 1773-79), 15: 271]

The "something that promotes intelligibility" function is the conceptualizing function. The "something that makes the pulling together possible" is the coalescing function. The other two functions are expressly named in this quotation.

Taste plays the role of an optimizing function in reflective judgment and is an outgrowth of an on-going process by which a person makes his intellectual development more perfect:

> The perfection of a cognition in consideration of an object is logical, in consideration of the subject it is aesthetical. The latter magnifies the consciousness of life, since it magnifies the consciousness of one's state through the relationship in which one's senses are placed toward the object and through dedication, and is therefore called lively. Abstract representation practically cancels the consciousness of life. [*ibid*., 15: 299-300]
>
> All perfection seems to subsist in the harmonization of a thing with freedom, hence in expedience, general usefulness, etc. Since all things in empirical understanding are properly only that which they are taken to be in way of relationship to the law of sensibility, the perfection of objects of experience is a congruence with the law of the senses and this, as appearance, is called beauty; it is, so to speak, the outer side of perfection. [*ibid*., 15: 309]

While an aesthetic Idea is an energizing representation, a judgment of taste is a terminating one. This is to say that the activity sparked by an aesthetic Idea is concluded by a judgment of taste. In this context, taste is a mathematical grounding condition for developing "emotional intelligence" [Salovey, *et al*. (2000)]. Opinions and beliefs are grounded in judgments of taste but only on grounds that are subjectively sufficient conditions and are never objectively sufficient:

> Taste liberates from mere senses and makes a recommendation to understanding. Thus all that furthers the life of our knowledge pleases in taste. . . . Taste affords no doctrine but rather criticism. It requires practical understanding and, in order to preserve it, examples. [Kant (*c*. 1773-79), 15: 354]

How does the transcendental Logic of judgments of taste operate? By understanding this we establish a fundamental understanding of the properties of judicial math. In turn these properties lead to, first, the construction of practical maxims of practical math and, second, later objective constructions of theoretical math in the manifold of concepts. They are, to somewhat poetically put it another way, the "essence" of the human capacity for mathematics. Let us look at the Logic.

A \ B	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	$^{1}0$
2	2	3	4	5	6	7	8	9	$^{1}0$	$^{1}1$
3	3	4	5	6	7	8	9	$^{1}0$	$^{1}1$	$^{1}2$
4	4	5	6	7	8	9	$^{1}0$	$^{1}1$	$^{1}2$	$^{1}3$
5	5	6	7	8	9	$^{1}0$	$^{1}1$	$^{1}2$	$^{1}3$	$^{1}4$
6	6	7	8	9	$^{1}0$	$^{1}1$	$^{1}2$	$^{1}3$	$^{1}4$	$^{1}5$
7	7	8	9	$^{1}0$	$^{1}1$	$^{1}2$	$^{1}3$	$^{1}4$	$^{1}5$	$^{1}6$
8	8	9	$^{1}0$	$^{1}1$	$^{1}2$	$^{1}3$	$^{1}4$	$^{1}5$	$^{1}6$	$^{1}7$
9	9	$^{1}0$	$^{1}1$	$^{1}2$	$^{1}3$	$^{1}4$	$^{1}5$	$^{1}6$	$^{1}7$	$^{1}8$

^{1} denotes a carry

Figure 5: The addition table for decimal digits 0 through 9 with carries indicated.

§ 4.1 Patterning

Patterning is *the act of representing a pattern*. A ***pattern*** is *an arrangement of form as a grouping or distribution of elements*. A mathematical set is an example of a pattern. Judicial math in terms of Quantity in representation is, therefore, the aesthetical determination of patterns found in the data of the senses. This gives us one property of mathematics in general: ***mathematics is all about patterns***. Now, the general idea of a "pattern" is very broad. In formal theoretical math the constructs that are called ***theorems*** are, at their judicial roots, ***nothing more and nothing less than identified patterns***. For example, consider the addition table (A plus B) for the decimal digits zero through 9 (figure 5). If you look along the top ($A = 0$) row, you will see that this row is merely the regular counting sequence from 0 to 9. Now do the same for the first column ($B = 0$). You see the same pattern. These two *patterns* in the table are the conceptual *root* of the idea of "the additive identity element," zero, in mathematics.

Now look at the $B = 1$ column. What you see is the counting sequence beginning with B. Look at the $B = 2$ column. You see the counting sequence *beginning with* B *once again*. This *pattern* is repeated in every column in the table. But there is another as well. When the count reaches "10" the non-carry digit "wraps around" to remain within the 0 to 9 range with the "wrap around" going up by one box each time you move one column to the right. That's another pattern. This one is the root idea for what mathematicians call an "equivalence class," although if you try to read the typical definition of this idea in a post-Bourbaki-era math textbook you'll probably get dizzy and might start to feel nauseous. Formalism goes to great lengths to disguise and hide how simple every one of its constructs really is and how aesthetically pleasing they can be. Formalism is, as Poincaré said, contrary to all healthy psychology.

Warfield remarked that mathematicians "were not able to imagine that their beautiful field, if presented with a carefully thought out axiomatic structure, could fail to be clear and inspiring in the eyes of the learner" [Warfield (2007)]. She is not wrong about this. Where the mathematicians developing New Math made their mistake was in thinking it was the axiom structure that made math "clear and inspiring." No. What makes math "clear and inspiring" are its patterns. The error comes from focusing on object cognition as the thing that is important in learning mathematics and not recognizing the real importance lies with aesthetical functions of the approval of taste.

Look down the $B = 9$ column. From $A = 1$ through the rest of the column, the non-carry digit is "A take away 1." Look down the $B = 8$ column; the non-carry digit is the digit in the $B = 9$ column take away 1. Look at the $B = 7$ column; you get the digit in the $B = 8$ column take away 1. The same "take away" idea holds as you go leftward in the table. *This is the pattern of a pattern.* If you know about this pattern and *its* pattern, you can learn to do addition in your head. If you find this a little too challenging at first, ***use an abacus***. An abacus helps make the patterns very visible. An electronic calculator *hides the patterns*. If you want to make a child mathematically illiterate, teach him to first do arithmetic by punching buttons on a calculator.

If you make a game of looking for them, you might be surprised by how many patterns you can discover in the dull old addition table. For example, look at the right to down-and-left minor diagonals entries. Mathematicians call this "symmetry" and it underlies, among other things, the commutative property of addition. You can also come to the idea of the commutative property by observing that the same "game" I described above for the columns of the table also works for the rows. This is a pattern to the pattern of the patterns. There is a lot of mathematics, going well beyond basic arithmetic, "hiding" in the addition table. For example, start anywhere in the top row and go down the diagonal left-to-right (wrapping about to the left side of the table after you reach the right-hand edge). You'll see that this is "counting by twos."

All of mathematics consists of patterns and patterns of patterns. Patterns please aesthetically because, once recognized, they can be *anticipated* and so are *aesthetically expedient*. When I was a little boy first learning how to add, we were told to "memorize the addition table." Fortunately, as it turned out, this was something I found myself unable to do. There are a hundred entries in the addition table and I just could not keep that many entries in my head. But I did happen to notice the pattern in the $B = 9$ column of the table I have just described. I thought that pattern was "pretty neat" and I liked the fact it made "adding 9" a pretty trivial thing to do.

Then I noticed the pattern of the pattern in the $B = 8$ column. It was "almost the same" as the "9's pattern," and I thought *that* was pretty neat. And useful. Then I found the pattern of the patterns. By *using* them I was able to conceal from my teacher my inability obey her instruction to memorize the table *and* the fact that I was disobeying her – two things that were subjectively important to me because I didn't want to be called "dumb" or labeled as being "disobedient."

Step by step, as school went on to more advanced lessons in arithmetic, I kept finding new patterns of patterns and soon I had developed the habit of *looking* for them. It also gave me an aesthetic regard for math as "something that is neat and tidy" – and I liked things that were neat and tidy. Irrational numbers irritated me a little; to me they weren't tidy.

My teachers came to think of me as a math prodigy of some sort and some of them were puzzled when they discovered I really don't have all that keen a memory. But I was just a kid who had learned to make it a habit to find the patterns and use them. Instead of memorizing, I made myself become a little algorithm/heuristics developer. In Piaget's terminology, I developed mobile *schemes* for doing math. This, as it turned out, was my "secret" to understanding mathematics at all levels. (It also happens to be the "secret" to being a scientist). Mathematics is patterns and patterns of patterns. Mathematicians are people who have "developed a taste for patterns." Every human being has the capacity for patterning in making judgments of taste. Every person therefore has the latent capability to be a mathematician. Young children do not hate math until the instruction they receive makes mathematics so frustrating for them that they develop an aversion to it (which is a *learned* type-α compensation). Making the instruction non-frustrating means making it appeal to the learner's approval of taste functions.

§ 4.2 Coalescing

Coalescing is *the aesthetic function of syncretism in judgmentation*. Syncretism and juxtaposition are well known phenomena that have been studied in children [Piaget (1928), pp. 221-232]. Psychology theory has taken note of them going back at least as far as William James:

> Where the parts of an object have already been discerned, and each made the object of a special discriminative act, we can with difficulty feel the object again in its pristine unity; and so prominent may our consciousness of its composition be that we can hardly believe that it ever could have appeared undivided. But this is an erroneous view, the undeniable fact being that *any number of impressions, from any number of sensory sources, falling simultaneously on a mind* WHICH HAS NOT YET EXPERIENCED THEM SEPARATELY, *will fuse into a single undivided object for that mind*. The law is that all things fuse that *can* fuse, and nothing separates except what must. [James (1890), vol. I, pp. 487-488]

The capacities for *objective* syncretism ('fusing') and juxtaposition (inability to represent distinct parts of a whole *as* a whole) are built into the threefold synthesis of matter in sensibility, i.e., the *Verstandes-Actus* of *Comparation* → reflexion → abstraction in the synthesis of an intuition (figure 6) [Wells (2009), chap. 3]. The formation of an intuition, in terms of its matter, is

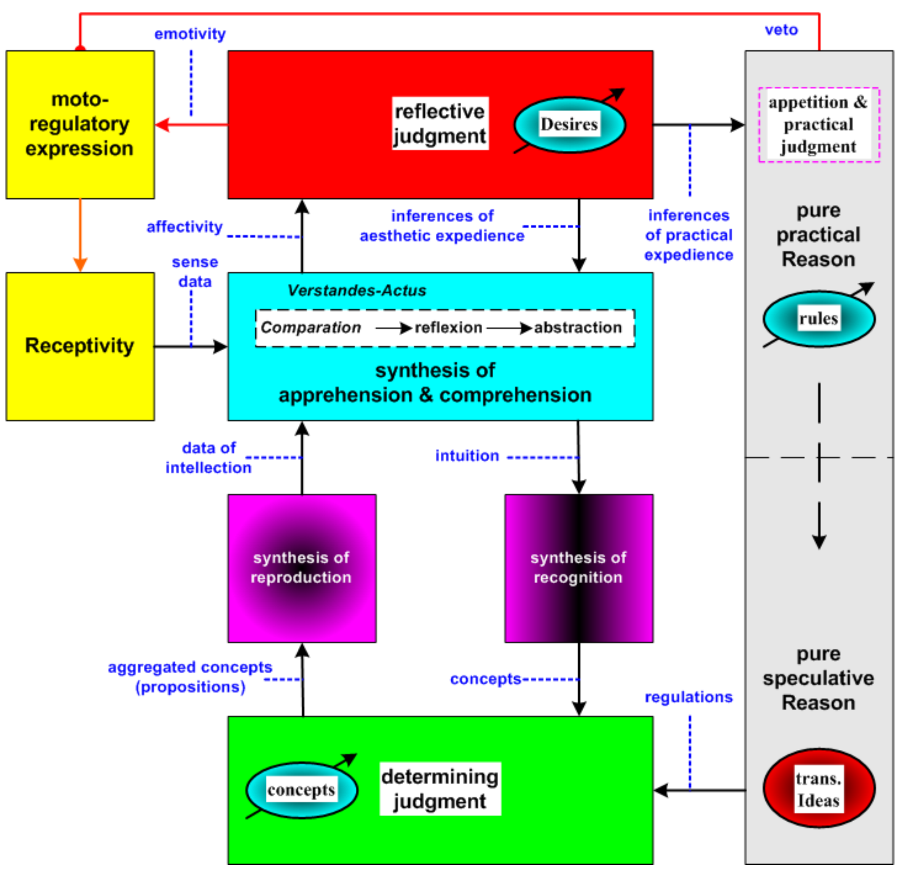

Figure 6: Functional organization of the processes of the motivational dynamic in judgmentation.

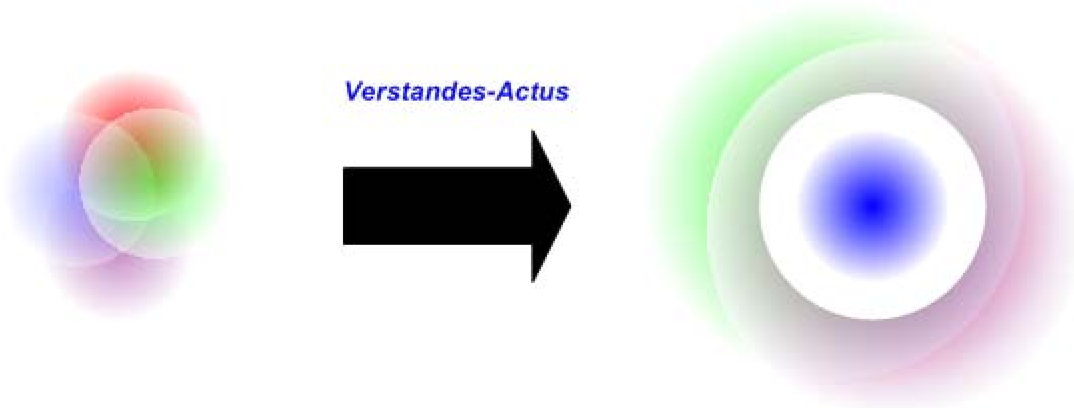

Figure 7: Accretion model of sensuous matter in an intuition during the synthesis in sensibility.

a process that can be pictured as a sort of coalescing of its *materia in qua* by means of a kind of accretion process (figure 7). During this process, other sensuous *materia* is excluded from the final intuition by one or the other of the three steps in the *Verstandes-Actus*. However, sensibility does not judge and formation (*Gestaltung*) of an intuition is adjudicated by reflective judgment. Of particular interest for the present topic-at-hand is the "flavor" of this judgment in regard to the process of reflexion. Here, Kant tells us,

> Reflexion (*reflexio*) does not have to do with objects themselves, in order to acquire concepts directly from them, but is rather the state of mind in which we first prepare ourselves to find out the subjective conditions under which we can arrive at concepts. It is the consciousness of the relationship of given representations to our various sources of knowledge through which alone their relationships among themselves can be directly determined. [Kant (1787), B316]

The coalition that occurs during the *Gestaltung* of an intuition is objective coalition, but the coalescing function of taste is a subjective function. It pertains to the entirety of the process of judgmentation (figure 6). Quality in taste services the motivational dynamic in judgmentation.

Coalescing is an agreement function and so immediately pertains to judicial math. Instruction in regard to this Quality function does not so much pertain to stimulating coalescing because the learner's laws of sensibility and judgmentation will perform coalition as a necessary part of the way the phenomenon of mind works. The aim of instruction, rather, is to try to get the learner's coalition to coalesce around an *intended* object of intuition at which the instruction is aimed. This is to say that the instruction is aimed at setting up a state of mind in the learner such that reflexion gathers together that which pertains to the intended lesson. To put it another way, Quality in judicial math instruction has the task of *combating the phenomenon of juxtaposition* rather than stimulating the phenomenon of syncretism.

Here James' remark quoted above is especially pertinent. Bearing in mind that theoretical math achieves knowledge through generalizing constructions of concepts, Quality in judicial math instruction must be concerned with gathering together previously learned mathematical concepts and seeking to integrate these in a concept structure of greater generality. This means *de facto* that divers objects are to be brought together and seen (by the learner) as parts of some greater mathematical object. There is a great difference between merely "getting an answer" and "getting a principle for getting answers." Too much of the traditional instruction in mathematics is aimed at "getting an answer" and too little at "getting a principle."

To illustrate what I mean, consider the progression of arithmetic → algebra → geometry →

trigonometry → analytic geometry → calculus. Each prior step in this progression is necessary for each succeeding one. This is shown by nothing more complicated than just looking at what goes into the increasingly more complex operations involved in each of these topics. I do not mean by this that *everything* pertaining to algebra is propaedeutic to everything in geometry – that is obviously not so – but, rather, that *some* concepts of algebra are needed by geometry, etc. For example, the proof of proposition 6 in Book III of Euclid's *Elements*, "If two circles touch one another, they will not have the same center," calls upon concepts of algebra. This is, of course, the reason that these courses are presented in the order they are presented in junior high school and high school mathematics curricula.

But the very fact that the divers parts of mathematics *are* taught in a sequence, and that each has objects peculiar to it, means that at the next step the learner is confronted by objects he has already experienced in prior classes as well as new ones being introduced to him at that moment. He therefore is psychologically confronted by a juxtaposition of objects he must somehow learn to 'fuse together' in a mathematical object of greater scope of generality. His challenge is not to "feel the object again in its pristine unity" but rather to come to know the object *for the first time*.

There is a lesson and warning for developing instructional *téchne* for mathematics to be found in the phenomenon of synthetic incapacity exhibited by young children in their drawings [Piaget (1928), pp. 221-227]. When a child draws an eye beside a head or an arm beside a torso, studies have found that the child thinks the one "goes with" the other rather than both being part of one whole. This is juxtaposition. As they grow older, children develop beyond synthetic incapacity of this sort, at least in regard to their drawings. However, just in the normal course of life they are bombarded with all sorts of experience that provides a sufficient number of disturbances to equilibrium that they develop maxims and concept structures for overcoming the incapacity. This, however, does not happen so readily with mathematics and for many people the only place where it *can* happen is in a mathematics class. Judicial math instructional Quality has to aim squarely at the issue of the synthetic incapacity of learners in regard to objects of mathematics.

One of the esoteric secrets mathematics departments do not share with outsiders is that the history of mathematics is a history of *designing mathematical structures*. For example, the everyday arithmetic we are all familiar with is a substructure within a greater one, namely the general structure of algebra. But this formal structure had to be originally *constructed* and mathematicians were the people who constructed it. For example, the first few formal structures one encounters on the road to algebraic structure in mathematics are:

1. the **groupoid**: a set of elements with a closed binary operation $\circ$ such that for any two elements x and y in the set the operation $x \circ y \rightarrow z$ yields an element z that also belongs to the set;
2. the **semigroup**: a groupoid in which the operation is associative;
3. the **monoid**: a semigroup whose set contains an identity element e such that $x \circ e \rightarrow x$;
4. the **group**: a monoid in which for every element of the set there is also an inverse element belonging to the set, i.e., there is an element x^{-1} for every x such that $x \circ x^{-1} \rightarrow e$.

If you are one of the overwhelming majority of human beings who have never heard of any of these, don't worry. The key point I want you to understand is that each successive structure in the list above is *built* by adding some property to its set or operation that the earlier structures do not have. Mathematics is the doctrine of knowledge through *construction* of concepts.

Personally, I think that designing mathematical structures is the most fun part of mathematics. In my engineering work, I often have to work with systems whose natures do not lend themselves well to solution by using any of the well known mathematical structures but are easily solved by inventing a special purpose mathematical structure. I call this "rolling my own math" and "making the math fit the problem instead of distorting the problem to fit the math." I do not quite understand why it is that mathematics departments choose to *not* tell people that if they become mathematicians then they'll get to do this fun thing – the *creative* part of mathematics – *and* get paid for doing it. In fact, I've yet to meet a mathematician who has admitted to me that he *does* design mathematical structures. Perhaps that reflects nothing more than the strength of the grip the Euclid myth has on the pseudo-metaphysics of mathematicians. After all, you can't *create* something if it already exists – as the Euclid myth claims. All you can do is "discover" it.

Nonetheless, *teaching* mathematics cannot be effective if it begins with this sort of step by step construction. This is because, for instance, the groupoid, the semigroup, etc. are abstract concepts. *They have no meaning for the learner unless he first has specific and concrete examples of 'doing math' to which he can refer in understanding the more abstract concept*. All human beings learn *from* the particular case *to* the general case. Once the learner has come to the general concept *then* he understands it through the divers meaning implications that stand under it and he can then, but *only* then, *use* it to construct additional specific concepts. Reversing this learning sequence will not be effective; that was the fundamental error and cause of the failure of the New Math experiment in 1960s mathematics education. It is still an error regularly committed at the college level and institutionalized in the formalism of math textbooks. I have seen it very often convert students who entered college with a liking for mathematics into students who are *trained* (a psychologist would say 'conditioned') to be highly averse to mathematics.

In the Logic of concept structuring in the manifold of concepts, a constructed higher concept of theoretical mathematics is a structure built in the form of a polytomy through determining judgments of Classifications and Co-Divisions [Wells (2012); Kant (1800), 9: 146-148]. These types of concept structures are constructed through inferences of Reason and multiple exercises of the cycle of judgmentation in reasoning. Juxtaposed concepts have to be brought together and 'fused' by means of judgmentations that gradually put these rich but very complex structures together. The key to enabling the learner to carry out these constructions, however, does not at all lie with coldly rational logic but, rather, with manipulation of learner affectivity and judgments of taste. Only then can learner juxtaposition be overcome by enabling his capacity for syncretism to be brought to bear on the juxtaposed concepts.

How can mathematics instruction combat the phenomenon of juxtaposition? To answer this it is important to understand the nature of how reflexion is adjudicated by reflective judgment. The process of reflexion in the synthesis in sensibility is a species of comparison. But in this process representations are not compared *to each other* but, rather, to the state of the thinking/reasoning Subject (the person) in regard to the compatibility of the representation with expedience for pure Reason. There are three ways representation in sensibility can be expedient for the Subject [Wells (2009), chap. 3, § 4.2]. All three involve harmonization, i.e., the act of making diverse representations compatible and homogeneous with one another so that they may be combined in composition. The first is harmonization of the free play of imagination and understanding, and this is equilibrium in *thinking*. The second is harmonization in the interaction between sensibility and reflective judgment, and this is called aesthetic harmony. The third is equilibrium in the cycle of judgmentation, and this is called the harmonization of reasoning. From the judicial Standpoint, ***reflexion*** is *the act of coalition in sensibility that produces any of these three forms of harmony*. Mathematically, reflexion is a synthesis that produces what mathematicians call "congruence classes"; this explanation is reflexion viewed from the theoretical Standpoint.

Representations in sensibility of juxtaposed objects that are able to be harmonized by reflexion can for this reason be brought into a unity of representation under a new concept that understands them all by means of polytomy structures. For example, you are familiar with the idea of 'distance' as 'how far apart two things are.' But 'how far apart' turns out to be a concept with a lot of different meanings *depending on context*. The one most people use most of the time is what mathematics calls "Euclidean distance." Its formula is that of the familiar Pythagorean theorem, "the square root of the sum of the squares of two sides of a right triangle equals the hypotenuse." But 'how far apart' might also mean "two days' walk from here." Or it might mean "how many letters are different in two words" (this is called a Hamming distance; the words 'cat' and 'cad' are a Hamming distance of 1 from each other because they differ in just one letter; they are "one letter apart in spelling"). Mathematics unifies *all* these different ideas of "distance" under one concept, which is called a "metric" and is used to define a "metric space" [Nelson (2003)].

Let us take a keen look at these examples to see what is going on and in what way mathematics can be said to "harmonize" these different versions of "distance." Each of these ideas of 'distance' has a specific *context* in which it "makes sense" and each does not "make sense" when we try to use it in one of the *other* contexts. For example, it is nonsense to say 'cat' is "one days' walk" from 'cad' (unless 'cat' and 'cad' are the names of two towns), and for most people it is even greater nonsense to try to work 'cat' or 'cad' into the notion of a triangle[7]. The general concept of a 'metric,' however, provides the idea of *a general meaning implication schema* under which the divers specifying concepts of these different "distances" *can all be understood under one set of conditions*. It provides, in other words, a general definition of the notion of "distance" under which divers *species* of distance are understood. This is the significance of *adjectives* such as "Euclidean" distance or "Hamming" distance or "walking" distance.

And here we can find a concrete practical explanation for what "harmonizing different representations" means. The divers concepts *with* their contexts (Euclidean, Hamming, and walking distance) are brought together and "made free of" the *specialization* given to each by their contexts so that what is left after "skinning" one of them of its context is in some way *the same as what is left after "skinning" the others*. The "way they are the same" might involve some fairly non-trivial practical scheme (the full definition of 'metric' and 'metric space' in Nelson's dictionary takes three fairly dense paragraphs to describe). But once the scheme has been worked out, then all the various meaning implications of the originally juxtaposed concepts are "the same despite being different."

This is not really as obscure a thing as my examples here might make it seem. You do these kinds of comparisons of reflexion every day. Barbara Bush is a human being. So are you. So was Albert Einstein. Obviously being Barbara Bush is not the same as being Albert Einstein but they are both, nonetheless, known to be human beings. They are "the same only different." And *that* is what the coalescing function of taste does: it makes juxtaposed things "the same only different." Mathematics *instruction*, in terms of the coalescing function, aims at guiding the learner to develop maxims of *looking for how things can be the same even though they are different*, and making this looking-for-sameness *habitual*. The grounding judgment for this "sameness" is a reflective judgment, which means that it is based on "feeling" two different things are somehow "the same." A teacher might, for instance, make a game of coming up with ways of seeing different things as being "the same" some*how*. She might, for example, show the pupils a circle and a square and ask them to find all they ways they are "the same" as they can find[8]. What the pupils come up with doesn't really matter (as long as they come up with something). What matters is the pupils develop schemes for looking-for-sameness. Once a learner has started to

[7] It can be done, though. Write down the letters of the alphabet in any planar arrangement you like on a piece of paper. Then draw a line from 'c' to 'a,' another from 'a' to 't,' and a third from 't' to 'c.' Now you have 'cat' represented as a triangle. Do the same for 'cad' and you can define at least one 'metric' (if you're very imaginative, you can probably define more than one) by which you can measure 'how far apart cat and cad are." Other than for the fun of mathematical frolicking, I don't know why you might want to do this, but the point is that if you wanted to, you could.

[8] In topology a circle and a square are "the same." They both have one "inside" and one "outside."

develop schemes of this sort, he can apply them to understanding the abstract *noumena* of theoretical math. Without them, the concepts of theoretical math are just "things to memorize" and the learner will be unable to *use* them outside the context of parroting back an answer on a test. That kind of "mathematics" deserves to be called "Polly want a cracker" math, which is the only kind of mathematics competency Taylorites' standardized math tests measure.

§ 4.3 Conceptualizing

Conceptualizing is the intelligibility function of approval of taste, but what is meant by saying this? What is 'intelligibility' to be taken to mean? 'Intelligibility' is an English rendering of the word Kant actually used, which was *Begreiflichkeit*. The verb *begreifen* means "to comprehend" and it denotes the highest of the seven levels of Kant's hierarchy classification of the degree of perfection of knowledge [Kant (1800), 9: 64-65]. Regarded in this Kantian context, judgments of taste are to be seen as aesthetical judgments that serve to promote the individual's perfection of his knowledge. Thus, Kant's *Begreiflichkeit* is best understood to imply ***comprehendability***.

The conceptualizing function pertains to the making of concepts, but this is not all there is to it. It pertains as well to *using* concepts to construct higher ones, specifically, *comprehensive* concepts. All concepts originally are formed because their corresponding intuitions satisfy some *motivation*. Critical motivation is *accommodation of perception* [Wells (2009), chap. 10]. Motivation is said to be *satisfied* when perception in sensibility is accommodated in such a way that the individual recovers a state of equilibrium. A judgment of taste makes no direct reference to the appetitive power of pure practical Reason, but *the symbolism built into a concept does*. Intuitions are *made symbolic* when they are formed. The symbolism is the meaning implication adjudicated by the process of teleological reflective judgment. The symbolism *marks the intuition as expedient in motivation* [Wells (2009), chap. 8, pp. 310-311].

From time to time philosophizers and philosophizing psychologists have questioned what it is that concepts and the ability to conceptualize *do* for human beings. Mental physics doctrine teaches that concepts, reintroduced into sensibility by the process of reproductive imagination, *serve to satisfy motivations* and thus they support reequilibration in judgmentation. They do so by means of the meaning implications that concepts symbolize. Because growth in concept structure multiplies the possible ways by which a human being can achieve reequilibration, the fact that human beings can conceptualize is what makes our species the extraordinary generalists that we are. In practical terms, *that* is what concepts and the ability to conceptualize do for us. As Piaget famously put it, "intellectual organization merely extends biological organization" [Piaget (1952), pg. 409]. The ability to conceptualize allows each of us to make himself become much more than he starts out as at birth and much more than mere biological growth and maturation could ever provide.

What sort of thing is 'conceptualizing'? William James put a finger on a very important aspect of this when he wrote,

> The word 'conception' . . . properly denotes neither the mental state nor what the mental state signifies, but the relation between the two, namely the function of the mental state in signifying just that particular thing. It is plain that one and the same mental state can be the vehicle of many conceptions, can mean a particular thing, and a great deal more besides. If it has such a multiple conceptual function, it may be called an act of compound conception. [James (1890), vol. I, pg. 461]

Here is a starting point for understanding the conceptualizing function of approval of taste in relationship to judicial math. That which is expedient for perfecting the manifold of concepts promotes comprehensive understanding by means of bringing more particular concepts to stand under a higher (and thereby more general) one. This expedience for the pure purpose of practical Reason is vested in making reequilibrations quicker and easier to achieve because the higher concept, when reintroduced into sensibility along with its enriched store of meaning implications represented by the concepts standing under it, promotes the ability to satisfy motivation (which, again, is accommodation of perceptions). Comprehending is perfecting of knowledge and this context points us to a purpose for mathematics instruction. In regard to Relation, the aim of mathematics instruction is to teach by: (1) presenting the learner with *examples* that supply the particular (lower) concepts to be subsumed under a higher concept of comprehension; and (2) to provoke the learner to *construct* that higher concept. To successfully accomplish the latter, the learner's ability to conceptualize *possibilities* must be cultivated. Within the present topic of discussion, those possibilities are in particular *the possibilities of mathematical objects*.

It is important to understand a few things about the psychology of possibility. First, Piaget has pointed out that

> Possibilities are in fact not observable, resulting as they do from subjects' active constructions. Even though the properties of objects play a role in these constructions, the properties always get interpreted in the light of a subject's acting on them. Such actions at the same time generate an ever-increasing number of new possibilities with increasingly rich interpretations. We are thus dealing with a creative process very different from the simple reading of reality invoked by empiricism. . . .
>
> In short, *possibility* in cognition means essentially invention and creation, which is why the study of possibility is of prime importance to constructive epistemology. [Piaget (1981), pp. 3-4]

Piaget and his coworkers were able to distinguish three levels of development in regard to children's ability to invent possibilities. These are:

> Level I (preoperational, age 4 to 6 years): possibilities generated by successive analogies characterized by the absence of reversibility, systematic inferences, or closure;
>
> Level II (concrete operations, age 7 to 10 years): formation of concrete co-possibilities;
>
> Level III (hypothetico-deductive operations, age 11-12 years and up): appearance of unlimited numbers of speculative co-possibilities. [Piaget (1981), pg. 145]

It *cannot be presumed* the learner will discover the possibility of subsuming lower concepts of mathematics under higher ones or that, even if a learner does, the higher concept he constructs will be one of fecund pertinence to mathematics. The cycle of judgmentation and all that happens during it is always under absolute regulation by the process of pure practical Reason and this is a process that regulates for one thing only – equilibrium. With objects of mathematics we are dealing with objects that lie beyond the horizon of possible experience. The most absurd fantasy a learner can imagine can serve the purpose of pure Reason just as ably as the most profound mathematics conception. Furthermore, the learner always has recourse to type-α compensation behavior – i.e., he might be able to satisfy the dictates of pure Reason by simply ignoring things. The task of judicial math instruction is to guide the learner to conceptualizing the higher concepts of mathematics that the lesson is intended he should come to construct.

Thus the instruction aims not merely at cultivating the learner's capability to conceive creative possibilities but at cultivating the learner's development of presentative schemes and procedural schemata that are fecund for mathematical reasoning. The seeds of this development are planted in only one way – by well designed *exercises*. As Piaget said above, possibilities depend on *active* constructions. Because learning comprehensive concepts only proceeds from lower to higher concepts – from particulars to generals – the learner must be provided with *examples* he can use in constructing his presentational schemes and his procedural schemata (chapter 10). Furthermore, the examples and exercises must gradually introduce equilibrium-disturbing cases that require the learner to *gradually* conceive broader possibilities and expand the scope of his concepts. If he is hit with too much too soon the result will be cycle rupture. Piaget found that

> To conceive of new possibilities, it is thus not enough to think of procedures oriented toward a particular goal . . .: one also needs to compensate for that actual or virtual perturbation that is the resistance of reality to explanation when it is conceived as pseudo-necessary. Such a compensation mechanism, once it has enabled subjects to conquer this obstacle (pseudo-necessity) in a particular situation, in addition leads them to realize almost immediately that if one variation is possible, others are also possible, beginning with the most similar or those that are opposite.
>
> It now becomes clear where these hypotheses lead us: if it is true that the notion of the possible derives from having overcome certain resistances of reality to explanation and from filling the gaps that are perceived as a result of having envisioned one variation . . . then it can be concluded that this dual process involves equilibration in its most general form. But although the system of presentative and structural schemes is characterized by intermittent or lasting states of equilibrium, the nature of the possible that is accessed via the procedural system is one of constant mobility, further strengthened by generalizations once a specific result is obtained. [Piaget (1981), pg. 6]

This is a point dramatically illustrated and passionately argued for in Lakatos (1974). Lakatos' book is composed as one very long dialogue that might be described as "Socratic inquiry without Socrates." If something like what takes place during this dialogue ever were to happen in a real mathematics class, I'm not confident a teacher would know how to handle it. But Lakatos was not trying to present a real-life situation. He was arguing that real mathematics is carried out as a series of propositions, counterexamples, and what Piaget called "resistances to explanations":

> The core of this case-study will challenge mathematical formalism . . . Its modest aim is to elaborate the point that informal, quasi-empirical mathematics does not grow through a monotonous increase of the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations. [Lakatos (1974), pg. 5]

Lakatos was not particularly known for being 'modest' about much of anything and his dialogue-format "classroom example" is not a very practical or realistic picture of real instruction in a real classroom. Nonetheless, what he called "the dialectic of the story" contains on-going illustrations of, as Piaget put it, "having to overcome resistances of reality to explanation and having to fill in the gaps" these "resistances" bring out. In this way, the dialogue has its uses as a vehicle for *teaching teachers how to teach* mathematics. What is different about Lakatos' imaginary classroom and a traditional mathematics class is that Lakatos' learners do not passively accept what they are told but, instead, actively seek to understand it by posing questions and raising problems.

Studies of children reinforce the requirement that mathematics instruction must not only build up from examples but that, furthermore, examples must be strongly tied to physical objects if the learner is to be able to formulate meanings for the supersensible objects of mathematics. Even such a basic concept as "a number" is not so basic as most of us presume. It is widely assumed that the ideas of numbers, counting, and addition grow out naturally and rationally from counting with one's fingers. Aside from the fact that this presupposition seems to be gainsaid by the fact that different cultures throughout history have used non-base-10 number systems, psychological facts likewise run counter to the presupposition. It is not enough for a learner merely to make a concept of a mathematical object. He must also conceive of that object as a *permanent* object in order for him to formulate the various conservations required to apply mathematics. Piaget wrote,

> Every notion, whether it be scientific or merely a matter of common sense, presupposes a set of principles of conservation, either explicit or implicit. . . . It is unnecessary to stress the importance in everyday life of the principle of identity; any attempt by thought to build up a system of notions requires a certain permanence in their definitions. In the field of perception, the scheme of the permanent object presupposes the elaboration of what is no doubt the most primitive of all these principles of conservation. . . .
>
> This being so, arithmetical thought is no exception to the rule. A set or collection is only conceivable if it remains unchanged irrespective of the changes occurring in the relationship between the elements. . . . A number is only intelligible if it remains identical with itself, whatever the distribution of the units of which it is composed. A continuous quantity such as a length or volume can only be used in reasoning if it is a permanent whole irrespective of the possible arrangement of its parts. In a word, whether it be a matter of continuous or discrete quantities, of quantitative relations perceived in the sensible universe, or of sets and numbers conceived by thought, whether it be a matter of the child's earliest contacts with number or of the most refined axiomatization of any intuitive system, in each and every case the conservation of something is postulated as a necessary condition for any mathematical understanding. [Piaget (1941), pp. 3-4]

Piaget documented three stages in children's conceptions of mathematical quantities. These stages are characterized by the following:

> Stage I: characterized by the absence of conservations; no exact correspondence and no equivalence;
>
> Stage II: characterized by the beginnings of construction of permanent sets; one-to-one correspondence but *without* lasting equivalence of corresponding sets; this stage serves as an intermediary stage between stage I and
>
> Stage III: conservations and quantifying coordinations; one-to-one correspondence *with* lasting equivalence of corresponding sets.

The essential importance of coordination between mathematical objects and objects to which they are applied cannot be overlooked and is not to be underestimated in mathematics instruction. The philosophy of Russell and the other logicians placed much stress on one-to-one correspondence in the definition of the whole and natural numbers, but psychologically this is not enough. Piaget tells us,

> Both counting on the fingers and the exchange of one object for another are indications of the considerable part played by correspondence in the synthesis of number. Yet although one-to-one correspondence is obviously the tool used by the mind in comparing two sets, it is not adequate, in its original form or forms, to give true equivalence to the corresponding sets, i.e. to give each set the same cardinal value, which is seen as constant as a result of the correspondence. As we saw . . . either the correspondence is held in check by perceptual factors or the correspondence itself develops from mere global correspondence, which issues in necessary equivalence and thus in cardinal invariance. . . .
>
> We must first make a distinction between two kinds of situations in which the child is led to discover or to make one-one correspondence. On the one hand, there are those cases in which he is required to assess the value of a given set of objects by comparing them with objects of the same kind. For instance, if two children are playing marbles and one puts four or six on the ground, his partner will want to put one opposite each of them and thus

> obtain an equivalent set without needing to be able to count. . . . On the other hand, there is an even simpler situation . . . namely, correspondence between objects that are heterogeneous but qualitatively complementary, a correspondence which is thus, as it were, provoked by external circumstances. . . . We must also include in this category the exchange of one object for another [Piaget (1941), pp. 41-42]

Lack of "provoked" correspondences has some significant effects. One of them is the *absence of associativity* in mathematical conceptions. The associative property of addition, i.e. A + B + C = (A + B) + C = A + (B + C), is taken more or less for granted by adults. However, *this is a learned correspondence* that very young children *do not make*:

> If we give 4- or 5-year-olds various cut-out shapes . . . they can put them into little collections on the basis of shape. The youngest children will make what I call figural collections; that is, they will make a little design with all the circles, and another little design with all the squares, and these designs will be an important part of the classification [into circles and squares]. They will think the classification has been changed if the design is changed.
>
> Slightly older children will forego this figural aspect and be able to make little piles of the similar shapes. But while the child can carry out classifications of this sort, he is not able to understand the relationship of class inclusion. It is in this sense that his classifying ability is still preoperational. He may be able to compare subclasses among themselves quantitatively, but he cannot deduce, for instance, that the total class must necessarily be as big as, or bigger than, one of its constituent subclasses. A child of this age will agree that all ducks are birds and that not all birds are ducks. But then, if he is asked whether out in the woods there are more birds or more ducks, he will say, "I don't know; I've never counted them." It is this relationship of class inclusion that gives rise to the operational structure of classification, which is in fact analogous to the algebraic structures of the mathematicians. The structure of class inclusion takes the following form: ducks plus other birds that are not ducks form the class of all birds; birds plus the other animals that are not birds together form the class of all animals; etc. . . . And it is easy to see that this relationship can be readily inverted. The birds are what is left when from all the animals we subtract all the animals but the birds. This is the reversibility by negation . . . A – A = 0. This is not exactly a group; there is inversion . . . but there is also the tautology, A + A = A. Birds plus more birds equal birds. This means that distributivity does not hold within this structure. If we write A + A – A, where we put the parentheses makes a difference in the result. (A + A) – A = A – A = 0, whereas A + (A – A) = A – 0 = A. [Piaget (1970), pp. 27-28]

It might seem as though these considerations become unimportant for teaching older learners, but this is not true. In conceptualizing mathematics there are always *two* types of concepts in play. One is the concept of the mathematical object or objects. *The other is the concept of the presentative scheme or procedural schemata operating on these objects*. Mathematicians I have known, and seemingly also mathematicians who write mathematics books, appear to be unaware of the *operational* aspects of mathematics as they *do* mathematics. At the least they appear to not be cognizant that the operational aspects are as key to *teaching* mathematics as the objects are. With young children, this synthetic necessity is more or less out in the open for a teacher to see. But it must be understood that older learners are learning about mathematical objects that are increasingly remote from the horizon of possible human experience and are therefore *fundamentally defined by operations and the transformations these produce*. The instructional import of this is that the development of maxims of mathematical thinking have to be cultivated so that they always maintain some connection back to principal quantities of mathematics – those mathematical objects that are *placed* by definition in correspondence with sensible objects of real human experience. Correspondences and set equivalences are not automatic consequences of thinking. They first require development of practical maxims leading to procedural schemata. Formalism in instruction is contrary to cultivation of such maxims.

If the habit of always connecting mathematical objects back to principal quantities is not cultivated in the learner, then those mathematical objects tend to become isolated in a sort of intellectual island universe, unable to be *applied* to serve useful purposes. Mathematics becomes a sort of game-of-symbols without any context outside of the manipulation of symbols. Equations do not come with an owner's manual directing, "Use me here and here but not there." One of the most noticeable characteristics of the majority of science and engineering students at the collegiate level is their habit of indiscriminately grabbing some equation and trying to use it to solve some problem without ever giving a thought to whether or not that equation has anything to do with that problem whatsoever. They do not know that they must connect it back to real objects before they can make use of it. Their mathematics skills are imprisoned in that symbolic island universe of context-free math. I have carried out my own experiments on college students over the course of thirty years, and here is what I have regularly seen: If I give the student a straight-up math problem (e.g., find the other two interior angles of a right triangle given the length of its sides), most of them solve it easily. What they **cannot** do is *develop a mathematical description of a problem from scratch and figure out what math to apply to it*. They are, as it were, lost in space with no way to navigate back to the universe where human beings live. This is nothing else than a failure to educate them when they were first learning mathematics. They have been well trained to manipulate symbols; they have not received an education in mathematics.

Today one hears much about "constructivist mathematics" in education. There is on the one hand no doubt that *some sort* of constructivist approach to mathematics instruction is a correct strategy. Mathematics is knowledge through the construction of concepts. On the other hand, any constructivist approach that neglects traditional presentation of explanations and methods of doing, say, arithmetic *ipso facto* neglects the necessity of the co-development of object concepts and operational scheme-development concepts (procedural schemata) linking mathematical objects to the world of experience via principal quantities. Such an approach will accomplish little no matter how much it *claims* to be "based" on Piaget's research – a rather remarkable claim in itself because Piaget did no education research, proposed no teaching methods, and the few text materials that exist and are used for teacher training in colleges of education are, in my opinion, caricatures of Piaget's research injecting unfounded and misleading speculations into it.

As an example of what I mean, there is much furor over NCTM algorithms for teaching pupils addition and multiplication. The furor, both from the educologists and from lay political groups who attack these methods, is based on the presupposition that there is a "one best method" of doing mathematics. That is hogwash. For example, my mother was a bookkeeper and had an old fashioned Addometer, a mechanical calculator you could use to add or subtract (figure 8). It had 8 wheels you turned with a stylus. You rotated the wheels clockwise to add, counterclockwise to subtract. When I was a little boy, I used to play with this thing for hours at a time.

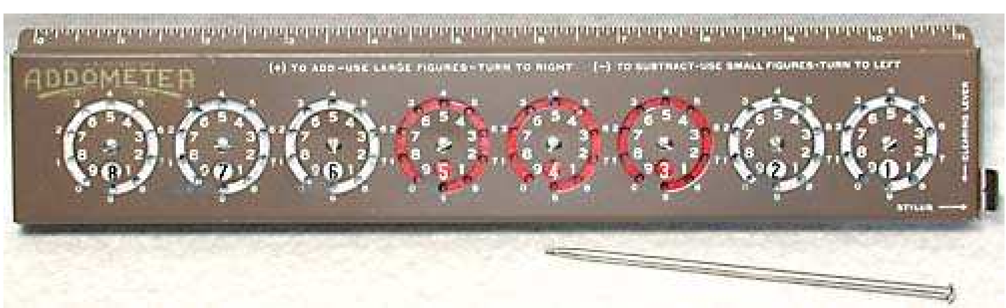

Figure 8: An Addometer, a mechanical calculator manufactured by the Reliable Typewriter and Adding Machine Company of Chicago from the 1900s until the 1960s. It was a very useful kind of abacus.

One of the things I discovered from playing with it was that it doesn't matter whether you add from right-to-left (which was the algorithm I was taught in school) or from left-to-right. The same is true for subtraction. The inner mechanics of the calculator took care of propagating carries or borrows from one digit to the next. In school, where a pupil had to add or subtract by hand, I always used the right-to-left method because I didn't want to get scolded for "doing it wrong." But from then to today I sometimes add or subtract large numbers left-to-right if all I want is a quick estimate of the answer rather than the exact answer. It turns out you can add digits in *any* order you want to so long as you remember to propagate the carries whenever they occur. From playing with the Addometer I formed concepts of principal quantities relating to the *process* of carrying out the "mechanics" of adding or subtracting that I still use to work problems today. Using a mechanical calculator-abacus (and, especially, playing with one) promotes the building of "mechanical" process concepts, the values of which go way beyond just "getting an answer."

"Doing" arithmetic and most other "mechanical" operations of mathematics always comes down to executing algorithms or using heuristics to manipulate quantities to the point where you can execute an algorithm. Some wonderful examples of this are found in Péter (1957). Different methods, from the "old way" favored by many of the lay groups to so-called "constructivist" methods introduced by educologists, have their uses in different circumstances. In my own professional practices, I pass back and forth between them at my convenience depending on what I am trying to do. Mathematics instruction *must not be confined to teaching just one method*; it should teach many different methods of getting to the same answer, including the "old" methods. To do otherwise is to starve the development of pupils' schemes for inventing possibilities and systematically comprehending them. The majority of students I have known are so rigidly locked into single algorithms that one might suspect they'd been toilet trained at gunpoint.

The Math Forum, a center for mathematics education on the Internet in the School of Education at Drexel University, describes "constructivism" in the following way:

> Students need to construct their own understanding of each mathematical concept, so the primary role of teaching is not to lecture, explain, or otherwise attempt to 'transfer' mathematical knowledge, but to create situations for students that will foster their making the necessary mathematical constructions.

Some of this statement is true, some of it is hogwash. As I have tried to explain here, it is true that learners have to construct their knowledge of mathematics. It is also true that teachers can only *immediately* reach them via judicial math. It is utter hogwash to say teaching's primary role is *not* "to explain or otherwise attempt to 'transfer' mathematical knowledge." That is *precisely* what instruction must have for its primary aim. Instruction is guidance and cultivation *through examples that develop procedural schemata*. You cannot do this if you do not present and explain procedural examples providing the learners with basal concepts necessary for the possibility of constructing higher ones. To cultivate the learner's knowledge and skill, a teacher must be a *pro-active* guide and role model, not a semi-passive facilitator. A teacher must also understand mathematics himself; a person who does not understand mathematics cannot teach mathematics.

There seems to be no shortage of so-called "constructivism" articles on the Internet today. I have no reasonable doubt that educologists who present these articles mean well and think what they write is true. But what I see there is not based on sound science; it is based on what I can only call "pseudo-Piagetianism" or "pseudo-Vygotskyism" or some hybrid of the two. These pseudo-theories illustrate the extent to which current educology is ignorant of or misunderstands Piaget's theory. In one 2013 Internet article from TERC's[9] Investigations in Numbers, Data, and

[9] TERC is the acronym-name of a non-profit education research and development organization. It has been around for about 40 years or so. The acronym originally stood for Technical Education Research Centers.

Space website, Clements & Battista wrote, "When a teacher demands that students use set mathematical methods the sense-making activity of children is seriously curtailed." This is a false statement. Piaget certainly never said any such thing. It is a prime example of the sort of pseudo-Piagetian hogwash that gets promoted and blamed on Piaget. One might as well base economics on astrology as base instruction *téchne* on such pseudo-theories. In the 20th century some of the most egregious and damaging mistakes in PEM reform were based on speculative and, as it turned out, unsound psychology interpretations. I see current educologists repeating this mistake through ignorance of the psychology theory on which they think they base their practices.

It is true that students studying to become teachers are exposed to the existence of Piaget's theory. But this exposure is limited to sound bites. The theory is enormous, extremely detailed, and its most important lessons are contained in its many *specific* observations. Encapsulated summaries of its findings can be extremely misleading because of overgeneralizations a student "reads into them" based on presuppositions of how he looks at the world via own private pseudo-metaphysic. Textbooks I have examined typically contain nothing more than a few fragments of Piaget's findings as parts of a few chapters. This is wholly inadequate. Even taking just one single course in Piaget's theory is inadequate. I estimate a sequence extended over about three or four years is likely necessary to adequately teach teaching students the theory. And this assumes that suitable textbooks for it are written. Piaget didn't write textbooks; he wrote research reports. Until an educologist has acquired such an education as this, he has no legitimate basis to claim his teaching theory is "based" on Piaget's theory. At most it is based on rumors of Piaget.

This is not to say positions advocated by lay groups – for example the group Utahns Against Common Core – are correct. "That's the way *I* was taught" is not a legitimate scientific argument. With the exception of conspiracy crackpots who infest organizations like these, laypeople are not-incorrect to oppose pseudo-scientific educology reforms. Taking a conservative view when it comes to doing experiments on children is well justified. But these groups are a long way from having the right answers for reforming public education. To the question, "Which side is right?" the only answer I can give is, "Neither of them. Their reforms would both be disastrous." It is near heresy in this country to say so, but democracy cannot dictate to nature. If you didn't already know this, I am sorry to have to tell you that nature doesn't care what your opinion is. That is why Mankind invented science.

§ 4.4 Precisioning

Precisioning is the act of making something precise. Its Latin root, *praecisio*, literally meant the act of amputating or "lopping off" an extremity. This is what a person does when he makes a concept precise. He lops off connections and contexts that are outside of the scope of the concept being made precise. It is this act Kant meant by "promoting the distinction of a manifold from other manifolds." Technical definitions used in science are products of precisioning. Lavoisier wrote,

> The impossibility of separating the nomenclature of a science from the science itself is owing to this, that every branch of physical science must consist of three things: the series of facts which are the objects of the science, the ideas which represent these facts, and the words by which these ideas are expressed. Like three impressions of the same seal, the word ought to produce the idea, and the idea to be a picture of the fact. And, as ideas are preserved and communicated by means of words, it necessarily follows that we cannot improve the language of any science without at the same time improving the science itself; neither can we, on the other hand, improve a science without improving the language or nomenclature which belongs to it. However certain the facts of any science may be, and however just the ideas we may have formed of these facts, we can only communicate false impressions to others while we want words by which these may be properly expressed.

[Lavoisier (1789), pp. xiv-xv]

The objective validity of Lavoisier's *dictum* traces back to the precisioning function of the approvals of taste. Indeed, this *subjective* function of Modality is what gives the expressions of mathematics the character of a *language* and their employment *as* language. To demonstrate what I mean by this, consider the learned writings that were characteristic of books and essays from the 17th century up through the first decades of the 20th century. One thing that is striking about these works is the authors' employments of Latin phrases embedded in their sentences. Here is an early 18th century example of this from a book written in English:

> Everyone that looks towards infinity does, as I have said, at first glance make some very large idea of that which he applies it to, let it be space or duration; and possibly he wearies his thoughts by multiplying in his mind that first large idea; but yet by that he comes no nearer to the having a positive clear idea of what remains to make up a positive infinite than the country fellow had of the water which was yet to come and pass the channel of the river where he stood:
>
> *Rusticus expectat dum defluat amnis, at ille*
> *Labitur, et labetur in omne volubilis ævum.* [Locke (1706), pg. 159]

These authors provide no footnote translations when they do this, nor are they trying to show the readers how educated they are. They clearly expect the readers to know their Latin well enough to understand the Latin passage. But why suddenly drop out of one's first language and into another, especially one that is a *dead* language no longer spoken in any nation on earth?

The answer comes out from the fact that Latin *was* a dead language. In a living language the meanings of words gradually undergo alterations. Eventually the usual fate of a word is to lose the crispness of meaning it originally had and to become burdened by so many metaphors that its meaning is no longer always clear to the reader. You can see this from even a casual inspection of any unabridged dictionary. But what if an author wishes to preserve some specific and exacting meaning in what he writes? His recourse is a simple one: *write it in a language that is no longer undergoing changes*. That is what Latin did for these authors: it preserved the specificity of what the author meant to say (assuming later readers could still read and understand Latin). Kant wrote that if you had something precise to say you should say it in Latin so that the meaning of your words would not change over time. His own lectures and writings are peppered with Latin phrases. Everywhere these occur a translator must take extra care in how he translates Kant's Latin phrases or he *will* alter the technical meaning Kant was expressing. He will lose the *precision* of Kant's thought. It is always a mistake to paraphrase Kant's Latin phrases.

This strategy, of course, fails when later readers no longer understand Latin. In modern day America very few people have any knowledge of Latin at all, due in large measure to the reforms of the Progressive Education Movement. PEM reformers' contempt of Latin and literary classics, and their willingness to cast away an entire heritage of Western civilization preserved in these works, can stagger one's imagination by the ignorant hubris it displayed. Speaking for myself, I see little difference between this aspect of PEM reform and the apocryphal legend of the burning of the Library of Alexandria, supposedly in 642 AD, attributed to an order supposedly issued by Umar ibn Al-Khattab (529-644 AD): "If these books are in agreement with the Quran, we have no need of them; and if these are opposed to the Quran, destroy them." There are many sound reasons to doubt the truth of the story blaming Umar for the burning of the Library of Alexandria, but there is no reasonable doubt about what the PEM reformers of the 20th century did to American public education and Americans' knowledge of America's Western heritage.

In science, engineering, economics, and, to a lesser extent, in psychology, mathematics has taken the place of Latin in books and papers *as a language for saying things precisely and in such* *a way that apodictic consequences of what is said can be deduced*. This is the *language aspect* of mathematics. It does not go too far to say mathematics has become the new Latin. This aspect of mathematics is one that I find entirely lacking in college students' understanding of mathematics – a lack of knowledge I attribute as much to the pseudo-philosophy of formalism in mathematics as to the way mathematics is taught. For example, here is an excerpt taken from a popular 1970s physics textbook for college freshmen and sophomores:

> A semiquantitative definition of [electric] flux is
>
> $$\Phi_E \cong \Sigma\, \mathbf{E} \bullet \Delta\mathbf{S}\,,$$
>
> which instructs us to add up the scalar quantity $\mathbf{E}\bullet\Delta\mathbf{S}$ for all elements of an area into which the surface has been divided. [Halliday & Resnick (1970), pg. 450]

Look carefully at the sentence structure here and how it is punctuated. The "symbolic hieroglyph" in the second line *is nothing else than a phrase within the overall sentence*. When we forego the use of "math language" (i.e., math symbols), this sentence reads,

> A semiquantitative definition of flux is that electric flux is approximately the total of the product of the normal component of electric field intensity multiplied by per unit surface area added up over an entire surface area, which instructs us to add up the scalar quantity given by the product of the normal component of electric field intensity times per unit surface area for all elements of an area into which the surface has been divided.[10]

To a person who has been adequately trained to understand the "hieroglyphs" used above, the first way of expressing the sentence appears to be much clearer than the second, although both are completely equivalent in terms of the physics objects being discussed. What I have found to be the case for the great majority of engineering students is that these students do not understand that the mathematical symbols are part of the sentence and it does not occur to them that the symbols can be "translated" into equivalent English words. Instead they view the symbols as if they were figures or graphs – i.e. as something that is not part of the sentence – and usually fail to connect the hieroglyphs with any physical "picture" of physical entities. Consequently, they do not understand what they are reading and tend to want to just "grab the equation" and use it without any notion of what it means. The symbols do not "speak" to the students because the students have never been taught to read them *nor have they even been taught that the symbols* ***can*** *be read*.

Indeed, even when I point out to students that the symbols are a kind of language and I translate it into English for them, most of the students greet this with deep skepticism and tend to ignore it. Their prior training in mathematics has left them with such a limited understanding of mathematical symbolism that they have become rigidly locked into a single maxim for how to use mathematics. And this locked-in maxim is astoundingly mistake prone.

Engineering and science students I have known over the past three decades also exhibit another mathematical language deafness. It is this: Symbols do not have to be algebraic in their appearances in order to be mathematical expressions. Figures and graphs also serve this function. This is perhaps better appreciated by freshman and sophomore college students in economics than it is in engineering or the physical sciences, perhaps because economics students receive much more graphical instruction than engineering and physics students receive. You may have noticed

[10] I will point out in passing that the approximate equation in this example is actually incorrect. It is missing a factor called the "permittivity of free space." Even physics professors are sometimes guilty of not proof-reading what they are saying when they write down equations. That is apparently what happened here. What the equation defines has to be called "the flux of an electric field intensity," which is not the same thing as "electric flux." Halliday & Resnick neglected to point out this difference.

and perhaps wondered about how few equations appear in this treatise. You may have even concluded from this that *The Institution of Public Education* is a non-mathematical and qualitative treatise. This is not so. Almost every figure that has appeared in this treatise makes a mathematical statement. These mathematical statements merely are not *algebraic* statements. Furthermore, they are mathematical statements geared towards helping the reader make fuller use of his visual sense to "make sense" of what he is being taught. I think not enough teachers reflect on the psychological implications inherent when a learner says, *"That makes sense."*

The engineering and science students I have known in my teaching career rarely ignore algebra-like mathematical statements because they recognize them as mathematical; on the other hand, they frequently ignore figures and graphs, deeming them of little to no importance "because they are not mathematics." Their maxims of thinking in this way have been so rigidly fixed by their prior mathematics training that getting them to break this habit rivals getting a heroin addict to give up his drug use. It is a habit that severely hinders their learning of science and technology. It is also a habit abetted by the pseudo-philosophy of formalism, which holds that mathematics ought to be taught with no references to objects or illustrations. This, indeed, is one of the most harmful legacies of the Bourbaki mathematicians. It is certainly one of the most profound errors that can be committed in mathematics instruction at any level of instruction.

Because I have no reason to assume you have received any training in mathematics that would have left you with an understanding different from that of my former students, I ask you to take a look at a copy of Euclid's *Elements*. Mathematical symbolism was not very advanced in 300 BC, and what you will find is that the greater part of Euclid's proofs are expressed in natural human language rather than in mathematical symbols. But if you look at geometry proofs in modern geometry textbooks, you will see the natural language largely displaced by the hieroglyphics of mathematics. Mathematics *is* used as a language but learners are not being taught how to speak it.

One thing this illustration brings out is that the precisioning function is, so to speak, a two-edged sword. To be useful concepts have to be made precise. But concepts can be made too precise and then a learner's capacity for creating possibilities or using his knowledge in new ways is severely crippled. When you hear someone exhorting another person to "think outside the box," the likelihood is that the situation is one in which over-precisioning is an important factor.

The precisioning function is a Modality function. This means that in judgments of taste the precisioning function pertains to *the relationship of the person to the concept* and not to how the concept represents its object. Modality functions pertain not at all to the object of the concept. The other headings (Quantity, Quality, and Relation) pertain to the representing of an object. The Modality heading pertains to the manner(s) in which the person views the concepts of the object, e.g., as possible or impossible, actual or unreal, necessary or contingent.

Because aesthetic Ideas are representations that give rise to so much to think about that they cannot be contained in a single concept, precisioning is that in judgmentation which, by 'lopping off' part of the representation, allows intuitions to be apprehended and keeps the Subject from being overwhelmed by feelings of sublimity. The feeling of sublimity is an act of aesthetical reflective judgment that occurs when a person is unable to concentrate all that he is trying to apprehend in one intuition. Kant wrote,

> Nature is thus sublime in those of its appearances whose intuition brings with it the Idea of its infinity. Now the latter cannot happen except through the inadequacy of even the greatest effort of our power of imagination in the estimation of the magnitude of an object. . . . Thus it must be the *aesthetic* estimation of magnitude in which is felt the effort at concentration which exceeds the capacity of imagination to comprehend progressive apprehension in one whole of intuition [Kant (1790), 5: 255].

The feeling of sublimity is a feeling disturbing to equilibrium. The accommodation of perceptions in judgmentation attempts to equilibrate this disturbance. Being unable to concentrate the data of sensibility all in a single intuition, and thereby produce an intuitive representation expedient for equilibrium, the act of precisioning is the sole recourse left in judgmentation for settling disturbances raised by aesthetic Ideas. The act of precisioning can in this sense properly be called a psychologically necessary approval of a judgment of taste in the phenomenon of mind.

However, because precisioning is a function of Modality, *how* an intuition and its concept is *made* precise by the thinking Subject carries no guarantee that the concept formed will have real objective validity. This is vividly illustrated by various childish conceptions of physical causality Piaget documented [Piaget (1929); Piaget (1930)]. Precisioning combined with the satisficing character of human judgmentation lies at the root of all superstitions.

The act of precisioning has profound implications for teaching. In the first place and especially for the age group from 5 to 15 years, a teacher ***must*** observe and evaluate whether precisioning is producing objectively valid concept structures in the learner or is instead giving rise to overly-rigid maxims of reasoning and objective misconceptions in understanding. The teacher must prod and probe to discover what the learner is learning from his lessons and how he is learning it. This aspect of teaching is profoundly important for this age range because this is the age range during which the learner is developing his maxims and habits of logical thinking that, for better or for worse, will dominate the rest of his life. Changing them later requires a psychological trauma.

One institutional consequence of this is that, in order for a teacher to be able to adequately carry out this key aspect of the teaching mission, pupil-teacher ratios in the earlier grade levels *must* be made significantly lower than at the higher grade levels. If this is not done, teachers will simply not have the capacity in terms of time and attention to do the prodding and evaluating of what habits of taste in reasoning and judgmentation the pupils are developing or for correcting these when they are shaping up in such a way as to hinder later learning abilities. *How one thinks and logically reasons are* ***developed*** *habits*. The importance of cultivating productive thinking and reasoning habits at the early ages cannot be overemphasized.

Two aspects of PEM reforms in the 20th century had severely debilitating effects in this regard. The first came about because of the PEM reform's zeal for differentiated curricula (and the accompanying institutionalized bigotry of "tracking" pupils into job tracks beginning at the age of twelve). In order to support curricular differentiation in junior high and high schools, in the teeth of budgets fixed by the amount of tax levy acceptable to the public, the PEM reformers single-mindedly emphasized raising teaching staff levels in the junior high and high schools at the expense of staffing levels in the grade schools. It was only the phenomenon of the "baby boom" from *circa* 1946 to *circa* 1964 that forced school districts to divert expenditures to increase the staffing of primary school teachers [Wells (2013), chap. 15, pp. 564-568].

The second aspect lies in the misguided and overzealous dogma of so-called "child-centered education" championed by some PEM reformers who seriously misunderstood what Dewey had meant by this term. So-called child-centered methods too often were interpreted to mean letting pupils more or less determine their own lesson objects in the mistaken belief that this was a more "natural" way for pupils to learn. Of course, teachers could not let educational anarchy reign in the classroom and therefore did prepare lesson plans and try to guide the learning activities. But, and this is my point, the dogma of child-centered education as the PEM envisioned it *paid no attention to observing and evaluating how learner judgments of taste were developing*. As a result no notice was paid to what intellectual effects precisioning was producing in the pupils. Without proper cultivation and guidance by the teacher, the effect on the majority of learners was, not to put too fine a point on it, calamitous. Lay groups opposed to "child-centered education" reforms were, I think, not able to well-articulate their objections to this movement, but I think that parents and others likely did have a vague intuition that the effects on schoolchildren were damaging. If so, this intuition was objectively valid. Real ***learner-centered education*** is education centered on cultivating the learners' powers of their persons. An important part of this is guiding and cultivating the direction the natural and necessary act of precisioning has on educational Self-development undertaken by the learner.

A teacher cannot control a learner's acts of precisioning. These acts are bound up in dynamics of judgmentation that serve the pure purpose of the process of practical Reason, namely equilibration. But while precisioning cannot be controlled by a teacher, it can be *guided* by the way that lesson objects are presented. Acts of precisioning happen because the learner *must* respond to disturbances to his equilibrium. Precisioning is guided by carefully introducing sequences of disturbances which are designed such that successively more robust equilibriums are achieved if the learner's construction of concepts coalesces around concept structures that accord with those of theoretical math.

Lakatos' fanciful dialogue in *Proofs and Refutations* recreates an historical series of arguments and counterexamples leading up to a proof famous among mathematicians (or at least some of them) and blissfully unheard of by all the rest of us. The specific conjecture around which his dialogue is built is not particularly important for our purposes here; the ideas contained in *Proofs and Refutations* that are important for us are: (1) the use of the method of Socratic inquiry in teaching mathematics; and (2) the use of heuristics in stating problems in mathematical form and then solving them. The refutations serve to introduce the factor of disturbance; the intermediate proofs serve to reestablish it by new constructions of mathematical concepts. However, this in no way minimizes the importance of establishing basic algorithmic procedures (e.g. those of basic addition and multiplication, fractions, long division, etc.) in elementary arithmetic. Before a learner can construct higher concepts he must construct basic practical maxims, without which he can do nothing at all. So it is, for instance, that multiple examples of different ways to add the same two numbers – all of which produce exactly the same answer – serve to introduce disturbance factors to counteract the development of rigidity in the learner's understanding of mathematics and to cultivate his capability to discover and invent new possibilities. Mathematics is not so much about correct answers as it is about *ways to get to* correct answers in divers contexts and for divers real-world problems. An answer merely validates the reasoning.

There are two natural tendencies in human judgmentation that a teacher's manipulation and guidance of precisioning must combat: (1) over-precisioning; and (2) overgeneralization. The first cripples a person's ability to use mathematics as a tool. The second is under-precisioning and leads to false beliefs. Bacon commented upon these two tendencies at the dawn of what historian Will Durant called the Age of Reason:

> In general, men take for the groundwork of their philosophy[11] either too much from a few topics or too little from many; in either case their philosophy is founded on too narrow a basis in experiment and natural history, and decides on too scanty grounds. For the theoretic philosopher seizes various common circumstances by experiment without reducing them to certainty or examining and frequently considering them, and relies for the rest upon meditation and the activity of his wit.
>
> There are other philosophers who have diligently and accurately attended to a few experiments and have thence presumed to deduce and invent systems of philosophy, forming everything to conformity with them. . . . There are, therefore, three sources of error and three species of false philosophy: the sophistic, empiric, and superstitious. [Bacon (1620), pg. 35]

[11] In Bacon's day "philosophy" and "science" were not distinct terms. Here he means "science."

The satisficing character of human judgmentation can combine with acts of precisioning to produce dogmatism and rigidity of thinking – both of which seriously retard Progress in any Society – if early instruction does not combat these factors in human reasoning. Bacon described the problem rather neatly:

> The human understanding, when any proposition has once been laid down (either from general admission and belief or from the pleasure it affords), forces everything else to add fresh support and confirmation; and although most cogent and abundant instances may exist to the contrary, yet either does not observe or despises them, or gets rid of and rejects them by some distinction, with violent and injurious prejudice, rather than sacrifice the authority of its first conclusions. [*ibid*., pg. 23]

I think it likely that one of the best sources for a teacher to learn how to construct "proofs and refutations" kinds of lessons is found in those Platonic dialogues providing relatively easy-to-follow examples of the Socratic method. The one thing the teacher must not copy from Plato, however, is Plato's persistent inability to get to any sort of answer to the questions he poses. Fortunately, in mathematics there *are* answers that *can* be reached, and the design of the lessons sequence can be made to steer the learner to these destinations.

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