Exam Rules:

- Do not open this exam until instructed to do so.
- No questions will be taken during the exam.
- Calculators or other electronics are not allowed to be used on this exam.
- Calculators, phones, laptops, or other devices must be put away out of sight during the exam.
- Phones must be off or set to "silent".

**Note:** You must use correct mathematical notation, show all of your work, and present your work in a neatly organized manner in order to get full credit for your answer.

If you need more space during the exam, you may use the back of this sheet.
Problem 1. (12 points) Evaluate \[ \int \sin^3 x \cos^2 x \, dx. \]

\[ \int \sin^2 x \sin x \cos^2 x \, dx = \int (1 - \cos^2 x) \cos^2 x \sin x \, dx = \int (u^2 - u^4) \, du \]

\[ u = \cos x \]
\[ du = -\sin x \, dx \]

\[ = \frac{u^5}{5} - \frac{u^3}{3} + C \]

\[ = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C \]

Problem 2. (12 points) Evaluate \[ \int \tan^2 x \sec^4 x \, dx. \]

\[ \int \tan^2 x \sec^2 x \tan x \, dx = \int \tan^2 x (1 + \tan^2 x) \sec^2 x \, dx \]

\[ u = \tan x \]
\[ du = \sec^2 x \, dx \]

\[ = \int u^2 + u^4 \, du \]

\[ = \frac{u^3}{3} + \frac{u^5}{5} + C \]

\[ = \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C \]
Problem 3. (15 points) Evaluate \( \int x^3 \sqrt{1 - x^2} \, dx \), by using a trig substitution.

\[
\int x^3 \sqrt{1 - x^2} \, dx = \int \sin^3 \theta \cos \theta \cos \theta \, d\theta \\
\left[ \text{by 1} \right] = \frac{\cos^5 \theta}{5} - \frac{\cos^3 \theta}{3} + c \\
= \frac{(1 - x^2)^{3/2}}{5} - \frac{(1 - x^2)^{3/2}}{3} + c
\]

Problem 4. (15 points) Evaluate \( \int x^3 \sqrt{x^2 - 1} \, dx \), by using a trig substitution.

\[
\int x^3 \sqrt{x^2 - 1} \, dx = \int \sec^3 \theta \tan \theta \sec \theta \tan \theta \, d\theta \\
= \int \tan^2 \theta \sec \theta \tan \theta \, d\theta \\
\left[ \text{by 2} \right] = \frac{\tan^3 \theta}{3} + \frac{\tan^5 \theta}{5} + c \\
= \frac{(x^2 - 1)^{3/2}}{3} + \frac{(x^2 - 1)^{5/2}}{5} + c
\]
Problem 5. (13 points) Estimate the value of $\int_{0}^{6} (x + 1)^2 \, dx$ by numerical integration using the midpoint rule with $n = 3$ (three rectangles). Also, find the error of approximation.

\[
\int_{0}^{6} (x + 1)^2 \, dx = 114
\]

$\Delta x = \frac{6 - 0}{3} = 2 \Rightarrow x_0 = 0, \ x_1 = 2, \ x_2 = 4, \ x_3 = 6$

\[
M_3 = 2 \left[ f \left( \frac{0 + 2}{2} \right) + f \left( \frac{2 + 4}{2} \right) + f \left( \frac{4 + 6}{2} \right) \right]
\]

\[
= 2 \left[ 4 + 2b + 3b \right]
\]

\[
= 114
\]

(error: $|114 - 112| = 2$

Problem 6. (13 points) Evaluate \[
\int \frac{e^{2x}}{1 + e^x} \, dx
\]

by using any method(s) you want.

\[
\begin{align*}
    u &= e^x \\
    du &= e^x \, dx
\end{align*}
\]

\[
\int \frac{u}{1 + u} \, du
\]

\[
= \int \frac{u + 1 - 1}{1 + u} \, du
\]

\[
= \int \frac{u + 1 - 1}{1 + u} \, du
\]

\[
= u - \ln |1 + e^x| + C
\]

\[
= e^x - \ln |1 + e^x| + C
\]

OR

\[
\begin{align*}
    u &= e^x + 1 \\
    du &= e^x \, dx
\end{align*}
\]

\[
\int \frac{u - 1}{u} \, du
\]

\[
= u - \ln u + C
\]

\[
= e^x - \ln |1 + e^x| + C
\]

(error constants)
Problem 7. Evaluate each of the following improper integrals. Remember to use correct notation, writing each as a limit of proper integrals.

(a) (10 points) \( \int_0^1 \frac{dx}{\sqrt{1 - x^2}} \)

\[ \text{Integrate DO NOT use } x = 1 \]

\[ \int \frac{dx}{\sqrt{1 - x^2}} = \arcsin(x) + C \]

\[ \lim_{t \to 1^-} \left[ \arcsin(x) \right]_0^t = \lim_{t \to 1^-} \left[ \arcsin(t) \right] = \frac{\pi}{2} \quad (\text{for } t < \frac{\pi}{2}) \]

(b) (10 points) \( \int_0^\infty \frac{dx}{(3x + 1)^2} \)

\[ \text{Infinite bound} \]

\[ \int \frac{dx}{(3x + 1)^2} = -\frac{1}{3(3x + 1)} + C \]

\[ \lim_{t \to \infty} \left[ \frac{1}{3(3x + 1)} \right]_0^t = \lim_{t \to \infty} \left( \frac{1}{3} - \frac{1}{3(3t + 1)} \right) = \frac{1}{3} \quad (\text{for } t \to \infty) \]

Extra Credit. (10 points) Solve

\[ \int_0^1 \frac{dx}{\sqrt{x + 1 + \sqrt{x}}} \quad (\text{both bounds}) \]

\[ \int_0^1 \frac{dx}{\sqrt{x + 1 + \sqrt{x}}} = \int_0^1 \frac{dx}{\sqrt{x + 1} - \sqrt{x}} \]

\[ = \left. \frac{2}{3} (x + 1) \right|_0^1 - \frac{2}{3} \int_0^1 x^{3/2} \, dx \]

\[ = \frac{4\sqrt{2}}{3} - \frac{2}{3} - \frac{2}{3} \]

\[ = \frac{4\sqrt{2}}{3} - \frac{4}{3} \]