Math 175, Fall 2017
Exam 3 (100 pts) Name: _____

Section (circle): 9:30 (Hardy) 11:30 (Parker) 12:30 (Jacobs) 2:30 (Meek)

Instructions: Work neatly and show all necessary work to receive credit. Answers without appropriate supporting calculations will not receive credit. Clearly indicate your final answer. Closed-book and closed-notes, except for notes provided. No calculators needed or allowed.

1. (10 pts) Give explanations for the following:

(a) If $\lim_{k\longrightarrow\infty} a_k = 0$, is it possible for the series $\sum_{k=1}^{\infty} a_k$ to diverge?

Yes! For example, the harmonic series $\sum \frac{1}{n}$ is an example of a series that diverges even though the terms have limit 0.

(b) What is the sequence of partial sums for the series $\sum_{k=1}^{\infty}(-1)^k = -1+1-1+1-\cdots$?

Does this series converge?

The sequence of partial sums is $-1, 0, -1, 0, \ldots$

This sequence does not converge, so the series also does not converge.

2. (5 pts) Write a formula for the nth term of the sequence $a_1 = -1/2, a_2 = 3/4, a_3 = -5/8, a_4 = 7/16, a_5 = -9/32, a_6 = 11/64, \ldots$

$$\boxed{a_n = (-1)^n \frac{2n-1}{2^n}}$$

3. (5 pts) Write the first three terms of the sequence of partial sums for the series $\sum_{k=1}^{\infty} \frac{1}{2k}$.

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{3}{4} + \frac{1}{6} = \frac{11}{12}$$

4. (10 pts) For each sequence below, either find the limit or show that the limit does not exist. Justify your answer.

$\left\{\frac{1-n}{2+n}\right\}$

$$\lim_{n\to\infty} \frac{1-n}{2+n} = \lim_{n\to\infty} \frac{(1-n)\cdot\frac{1}{n}}{(2+n)\cdot\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n}-1}{\frac{2}{n}+1}$$

$$= \frac{0-1}{0+1} = \boxed{-1}$$

$\left\{\frac{n\cos(n)}{n^2+1}\right\}$

$$\frac{-n}{n^2+1} \le \frac{n\cos n}{n^2+1} \le \frac{n}{n^2+1}$$

Both $\lim_{n\to\infty} \frac{-n}{n^2+1}$ and $\lim_{n\to\infty} \frac{n}{n^2+1}$ are zero.

So by the Squeeze Theorem, $\lim_{n\to\infty} \frac{n\cos n}{n^2+1} = \boxed{0}$

5. (10 pts) Find the sum of the geometric series $\sum_{k=2}^{\infty} \frac{3\cdot 2^k}{5^k}$ or show that the series diverges.

$$3\cdot\frac{4}{25} + 3\cdot\frac{4}{25}\cdot\frac{2}{5} + 3\cdot\frac{4}{25}\cdot\frac{2^2}{5^2}$$

$$= \frac{12}{25}\left(1 + \frac{2}{5} + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \cdots\right)$$

$$= \frac{12}{25}\cdot\frac{1}{1-\frac{2}{5}} = \frac{12}{25}\cdot\frac{5}{3} = \boxed{\frac{4}{5}}$$

6. (12 pts) Use the Integral Test to determine if $\sum_{k=1}^{\infty} \frac{1}{(2k+5)^{3/2}}$ the series converges or diverges.

$$\int_1^{\infty} \frac{1}{(2x+5)^{3/2}}\,dx = \lim_{c\to\infty} \int_1^{c} (2x+5)^{-3/2}\,dx$$

$$= \lim_{c\to\infty} \left[-(2x+5)^{-1/2}\right]_1^{c} = \lim_{c\to\infty} \left(-(2c+5)^{-1/2} + 7^{-1/2}\right)$$

$$= \lim_{c\to\infty} \left(\frac{1}{\sqrt{7}} - \frac{1}{\sqrt{2c+5}}\right) = \frac{1}{\sqrt{7}}$$

$\Rightarrow$ Series converges by the Integral Test

7. (12 pts) Use the Ratio Test or Root Test to determine if $\sum_{k=1}^{\infty} \frac{5k^2}{3^k}$ converges or diverges.

$$\lim_{k\to\infty} \frac{5(k+1)^2/3^{k+1}}{5k^2/3^k} = \lim_{k\to\infty} \frac{5(k+1)^2 \cdot 3^k}{5k^2 \cdot 3^{k+1}} = \lim_{k\to\infty} \frac{k^2+2k+1}{3k^2}$$

$$= \frac{1}{3} < 1$$

$\Rightarrow$ Series converges by the Ratio Test

8. (12 pts) Use the Comparison Test or Limit Comparison Test to determine if $\sum_{k=1}^{\infty} \frac{k^2}{k^3+4}$ converges or diverges.

$\frac{k^2}{k^3+4} < \frac{k^2}{k^3} = \frac{1}{k}$ but $\sum \frac{1}{k}$ diverges and "less than divergent" doesnt tell us anything. So we need Limit Comparison Test!

$$\lim_{k\to\infty} \frac{k^2/k^3+4}{1/k} = \lim_{k\to\infty} \frac{k^3}{k^3+4} = 1,$$ so since $\sum \frac{1}{k}$ diverges, $\sum \frac{k^2}{k^3+4}$ also diverges by the Limit Comparison Test

9. (12 + 12 pts) Use any of the series convergence tests to determine whether the series below converge or diverge:

$$\sum_{k=1}^{\infty} \frac{1}{e^k + k}$$

$\frac{1}{e^k+k} < \frac{1}{e^k}$ and $\sum \frac{1}{e^k}$ is a convergent geometric series $\left(\frac{1}{e} < 1\right)$

So: <u>Converges by the Comparison Test</u>.

$$\sum_{k=1}^{\infty} \left(\frac{5k+1}{3k+5}\right)^k$$

Root Test!

$$\lim_{k\to\infty} \left(\left(\frac{5k+1}{3k+5}\right)^k\right)^{1/k} = \lim_{k\to\infty} \frac{5k+1}{3k+5} = \frac{5}{3} > 1$$

$\Rightarrow$ <u>Series diverges by the Root Test</u>

Extra Credit. (up to 10 pts) What is the smallest value of n for which the nth partial sum of $\sum_{k=1}^{\infty} \left(\frac{1}{2k+1} - \frac{1}{2k+3}\right)$ is within 10^{-3} of the actual sum of the series?

$$= \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{9}\right) + \cdots$$

So the nth partail sum is $S_n = \frac{1}{3} - \frac{1}{2n+3}$

So the sum is $\lim_{n\to\infty} S_n = \frac{1}{3}$, and S_n is within 10^{-3} of this sum when $\frac{1}{2n+3} < \frac{1}{1000} \Rightarrow 2n+3 > 1000$

$\Rightarrow 2n > 997 \Rightarrow n > \frac{997}{2}$, so $\boxed{n=499}$ will do.