Instructions: Work neatly and show all necessary work to receive credit. Answers without appropriate supporting calculations will not receive credit. Clearly indicate your final answer. Closed-book and closed-notes, except for notes provided. No calculators needed or allowed.

1. (10 pts) Give explanations for the following:

   (a) If \( \lim_{k \to \infty} a_k = 0 \), is it possible for the series \( \sum_{k=1}^{\infty} a_k \) to diverge?

   Yes! For example, the harmonic series \( \sum \frac{1}{n} \) is an example of a series that diverges even though the terms have limit 0.

   (b) What is the sequence of partial sums for the series \( \sum_{k=1}^{\infty} (-1)^k = -1 + 1 - 1 + 1 - \cdots \)?

   Does this series converge?

   The sequence of partial sums is \(-1, 0, -1, 0, \ldots\) This sequence does not converge, so the series also does not converge.

2. (5 pts) Write a formula for the \( n \)th term of the sequence \( a_1 = -1/2, a_2 = 3/4, a_3 = -5/8, a_4 = 7/16, a_5 = -9/32, a_6 = 11/64, \ldots \)

\[
a_n = (-1)^n \frac{2n-1}{2^n}
\]
3. (5 pts) Write the first three terms of the sequence of partial sums for the series $\sum_{k=1}^{\infty} \frac{1}{2k}$.

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{3}{4} + \frac{1}{6} = \frac{11}{12}$$

4. (10 pts) For each sequence below, either find the limit or show that the limit does not exist. Justify your answer.

$$\left\{ \frac{1-n}{2+n} \right\} \quad \lim_{n \to \infty} \frac{1-n}{2+n} = \lim_{n \to \infty} \frac{\frac{1-n}{n}}{\frac{2+n}{n}} = \lim_{n \to \infty} \frac{\frac{1}{n} - 1}{\frac{2}{n} + 1} = \frac{0-1}{0+1} = -1$$

$$\left\{ \frac{n \cos(n)}{n^2 + 1} \right\} \quad \frac{-n}{n^2+1} \leq \frac{n \cos n}{n^2+1} \leq \frac{n}{n^2+1}$$

Both $\lim_{n \to \infty} \frac{-n}{n^2+1}$ and $\lim_{n \to \infty} \frac{n}{n^2+1}$ are zero.

So by the Squeeze Theorem, $\lim_{n \to \infty} \frac{n \cos n}{n^2+1} = 0$

5. (10 pts) Find the sum of the geometric series $\sum_{k=2}^{\infty} \frac{3 \cdot 2^k}{5^k}$ or show that the series diverges.

$$3 \cdot \frac{4}{25} + 3 \cdot \frac{4 \cdot 2}{25 \cdot 5} + 3 \cdot \frac{4 \cdot 2^2}{25 \cdot 5^2}$$

$$= \frac{12}{25} \left( 1 + \frac{2}{5} + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \cdots \right)$$

$$= \frac{12}{25} \cdot \frac{1}{1-\frac{2}{5}} = \frac{12}{25} \cdot \frac{5}{3} = \frac{4}{5}$$
6. (12 pts) Use the Integral Test to determine if \( \sum_{k=1}^{\infty} \frac{1}{(2k+5)^{3/2}} \) the series converges or diverges.

\[
\int_{1}^{\infty} \frac{1}{(2x+5)^{3/2}} \, dx = \lim_{c \to \infty} \int_{1}^{c} (2x+5)^{-3/2} \, dx
\]

\[
= \lim_{c \to \infty} \left[ \left(-\frac{1}{\sqrt{2c+5}} \right) \right]_1^c = \lim_{c \to \infty} \left(-\frac{1}{\sqrt{2c+5}} + \frac{1}{\sqrt{7}} \right)
\]

\[
= \lim_{c \to \infty} \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{2c+5}} = \frac{1}{\sqrt{7}}
\]

\( \Rightarrow \) Series converges by the Integral Test.

7. (12 pts) Use the Ratio Test or Root Test to determine if \( \sum_{k=1}^{\infty} \frac{5k^2}{3^k} \) converges or diverges.

\[
\lim_{k \to \infty} \frac{5(k+1)^2/3^{k+1}}{5k^2/3^k} = \lim_{k \to \infty} \frac{5(k+1)^2 \cdot 3^k}{5k^2 \cdot 3^{k+1}} = \lim_{k \to \infty} \frac{k^2 + 2k + 1}{3k^2}
\]

\[
= \frac{1}{3} < 1
\]

\( \Rightarrow \) Series converges by the Ratio Test.

8. (12 pts) Use the Comparison Test or Limit Comparison Test to determine if \( \sum_{k=1}^{\infty} \frac{k^2}{k^3 + 4} \) converges or diverges.

\[
\frac{k^2}{k^3 + 4} < \frac{k^2}{k^3} = \frac{1}{k}
\]

but \( \sum \frac{1}{k} \) diverges and "less than divergent" doesn't tell us anything. So we need Limit Comparison Test!

\[
\lim_{k \to \infty} \frac{k^2}{k^3 + 4} \cdot \frac{1/k}{1/k} = \lim_{k \to \infty} \frac{k^3}{k^3 + 4} = 1 > 0,
\]

so since \( \sum \frac{1}{k} \) diverges,

\[
\sum \frac{k^2}{k^3 + 4} \text{ also diverges by the Limit Comparison Test.}
\]
9. (12 + 12 pts) Use any of the series convergence tests to determine whether the series below converge or diverge:

\[\sum_{k=1}^{\infty} \frac{1}{e^k + k} \quad \frac{1}{e^k + k} < \frac{1}{e^k} \text{ and } \sum \frac{1}{e^k} \text{ is a convergent geometric series } (\frac{1}{e} < 1)\]

So: Converges by the Comparison Test.

\[\sum_{k=1}^{\infty} \left(\frac{5k+1}{3k+5}\right)^k \quad \text{Root Test!}\]

\[\lim_{k \to \infty} \left(\left(\frac{5k+1}{3k+5}\right)^k\right)^{\frac{1}{k}} = \lim_{k \to \infty} \frac{5k+1}{3k+5} = \frac{5}{3} > 1\]

\[\Rightarrow \text{Series diverges by the Root Test}\]

Extra Credit. (up to 10 pts) What is the smallest value of n for which the nth partial sum of \(\sum_{k=1}^{\infty} \left(\frac{1}{2k+1} - \frac{1}{2k+3}\right)\) is within 10^{-3} of the actual sum of the series?

\[= \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{9}\right) + \cdots\]

So the nth partial sum is \(S_n = \frac{1}{3} - \frac{1}{2n+3}\)

So the sum is \(\lim_{n \to \infty} S_n = \frac{1}{3}\), and \(S_n\) is within 10^{-3} of this sum when \(\frac{1}{2n+3} < \frac{1}{1000} \Rightarrow 2n+3 > 1000\)

\[\Rightarrow 2n > 997 \quad \Rightarrow \quad n > \frac{997}{2} \quad \Rightarrow \quad n = 499 \text{ will do.}\]