# Mathematical Functions

Statistics 427: R Programming

Module 9

2020

## Quadratic function and the Golden Ratio

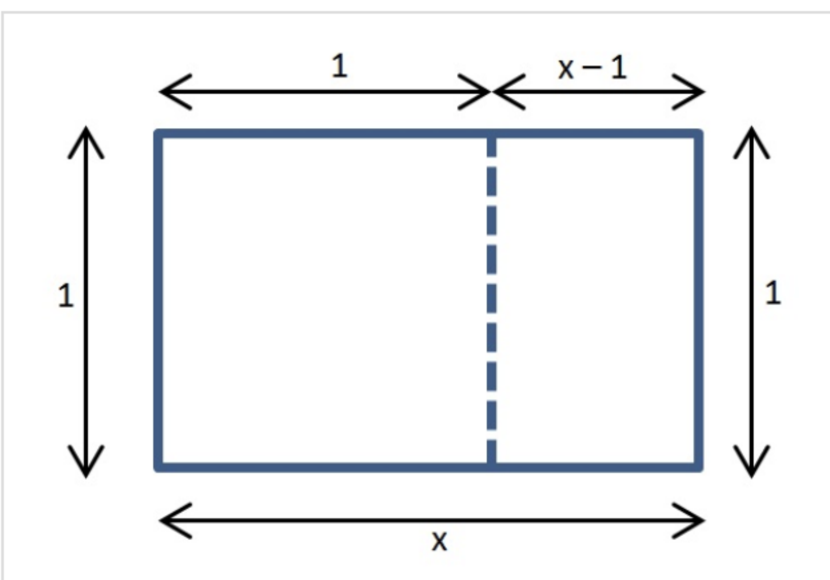
Since the time of the Mycenaean (ancient Greeks), architects have admired a certain proportion for the height and width of a building; mostly based on a particular rectangle produced to a certain rule. The rule states that if you cut the rectangle into a square and another smaller rectangle, the sides of the smaller rectangle have the same proportion as the original. This rectangle is known as the **Golden Rectangle**, and the ratio of the larger side to the smaller side is called the **Golden Ratio**.

## Golden Ratio I

The long length of a rectangle is $x$, and the square inside will be defined to have sides of length 1 (which is the length of the rectangle's short side). Thus *the ratio* of the long side to the short side, the Golden Ratio, is the quantity $x$. The quantity $x$ will be a real number greater than 1.

The Golden Ratio $x$ obeys the rule that the proportions of the newly formed small rectangle are the same as the proportions of the original rectangle. The long side of the original is $x$ and the short side is 1. The long side of the small rectangle is 1 and its short side is $x - 1$. If the sides have the same proportions, then the ratios of the long side to the short side are the same for both rectangles.

## Golden rectangle



## Golden Ratio II

$$\frac{x}{1} = \frac{1}{x-1}$$

Solve for $x$ with some algebra (yay!)

$$x(x-1) = 1$$

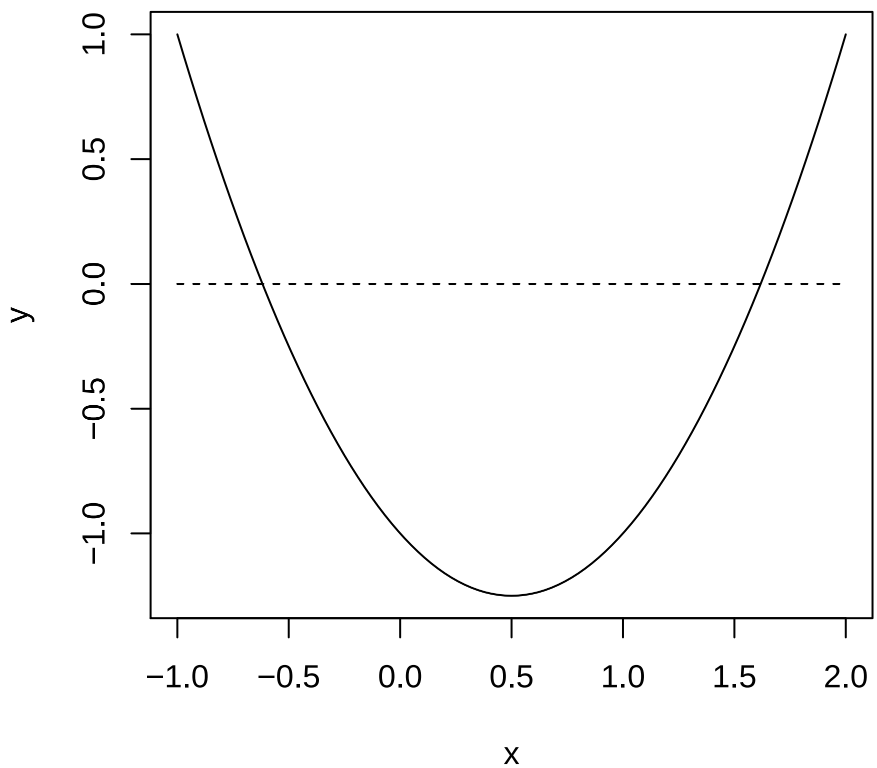
$$x^2 - x - 1 = 0$$

To solve this, factoring will not work but we can use the quadratic formula to solve.

## Golden Ratio III

But first before going there, let us graph this function.

```
xlo=-1; xhi=2
x=xlo+(xhi-xlo)*(0:100)/100 # ranges of x values
y=x^2-x-1 # values of y from range of x
plot(x,y,type='l')
y2=numeric(length(x)) # y=0 line
points(x,y2,type='l',lty='dashed') # y=0 on plot
```




## Golden Ratio IV

From the graph, you can estimate where the zeros of the function $y = x^2 - x - 1$ are. Looks to be around $x = -0.5$ and $x = 1.5$. Of course, using the quadratic formula will give us more accurate answers.

If your quadratic function is in the form $ax^2 + bx + c$ ($a, b, c$ are real-valued constants, with $a \neq 0$), then using the Quadratic formula, the zeros are found by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac < 0$, then you will have non-real solution(s)

## Golden Ratio V

Using $a = 1, b = -1, c = -1$ from the function, use the quadratic formula to find its zeros (the measurements of the rectangles from earlier in lecture).

```
a=1; b=-1; c=-1
x1=(-b-sqrt(b^2-4*a*c))/(2*a)
x2=(-b+sqrt(b^2-4*a*c))/(2*a)
x1; x2
```

```
[1] -0.618034
```

```
[1] 1.618034
```

## Trigonometric functions

We ♡ trig! So do physicists and many other disciplines. Mathematical properties of triangles have played a crucial role in human scientific and technological development, from building of the pyramids to the modern understanding of the general theory of relativity.

Any three points not on a line determine a triangle, and triangles within their triangular essence have a large number of shapes. A general triangle has three sides and three interior angles, and the sum of the inside angles must sum to 180°.

## Right triangles

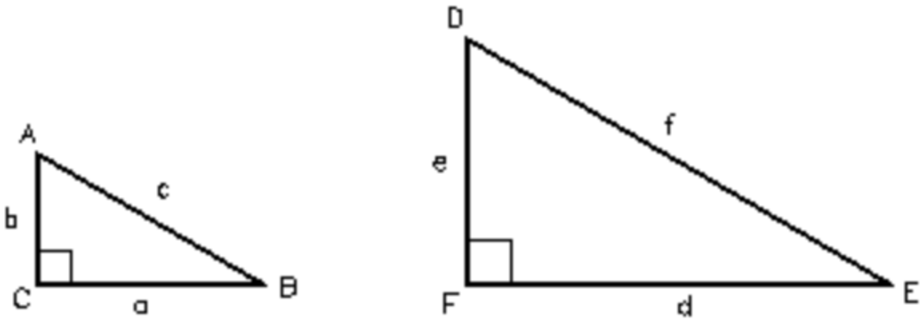
One type humans have singled out for its usefulness is the right triangle. A triangle with one of its angles measuring 90° is a right triangle. The angular measurements of the other two angles of a right triangle must themselves sum to 90° (so all three angles sum to 180°). One very useful property of a right triangle is given by Pythagorean Theorem, which we have reviewed already in Module 3.

Let us call the hypotenuse, the side opposite to the right angle, $r$ and the other two sides $x$ and $y$. Then we can say

$$r^2 = x^2 + y^2$$

## Similar triangles

If two right triangles have an additional angle with the same measure, then they are called *similar* triangles; their side lengths have the same proportions.




## Trig functions I

The usefulness of the similarity property of right triangles is that once one side length and one additional angle of a right triangle are measured, the length of the other sides can be calculated.

For an angle with angular measurement $\theta$ in a right triangle, there are six possible ratios for the side lengths. They are the basic trigonometric functions. (1) sine, (2) cosine, (3) tangent, (4) cotangent, (5) secant, and (6) cosecant.

`R` does have built-in functions `sin()`, `cos()`, and `tan()`; to use them you must think in radians rather than degrees

## Trig functions II

$$\sin\theta = \frac{y}{r} \quad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \quad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \quad \cot\theta = \frac{x}{y}$$

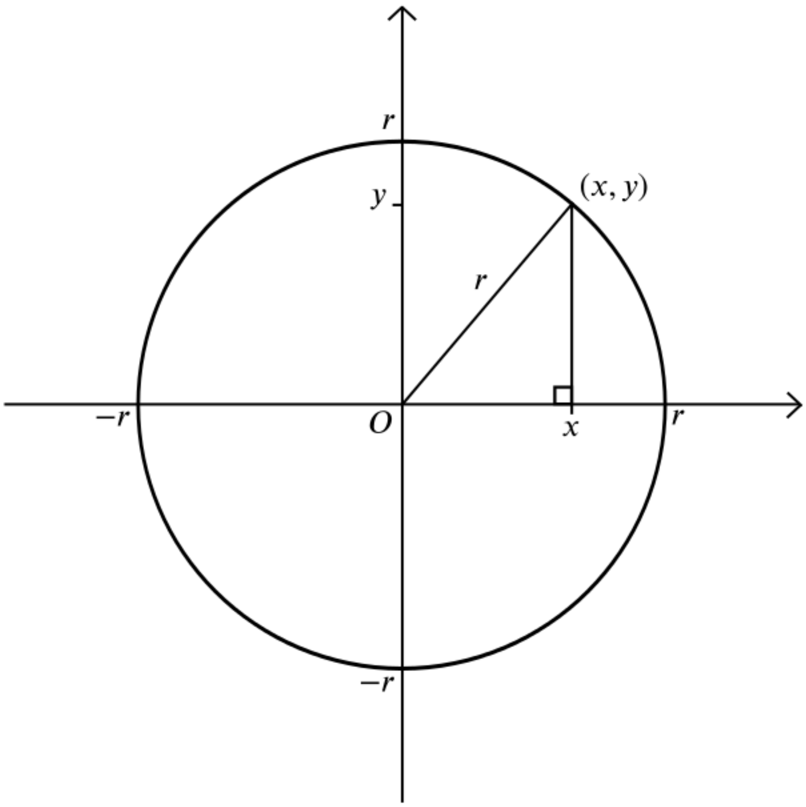
## Right triangles, circles, and radians I

Right triangles and circles are closely bound together. Take a Cartesian coordinate system and draw a circle centered at the origin with radius $r$. Pick any point $(x, y)$ in the positive quadrant. Drawing a line segment from the point to the origin is the hypotenuse of a right triangle with length $r = \sqrt{x^2 + y^2}$. The side with length $y$ is formed by a line segment drawn in a vertical direction to the horizontal axis, with the right angle being formed by the junction of the vertical line segment and the horizontal axis.

## Right triangles, circles, and radians II

Let $\theta$ be the measure of the angle formed by the horizontal segment and the horizontal axis (measured from the positive horizontal axis to the hypotenuse segment in an anti-clockwise direction). Now $\theta$ can have any value from near 0° to near 90°, and all of the values are possible for a right triangle. All the points $(x, y)$ in the positive quadrant that are at a fixed distance $r$ from the origin form a quarter arc of the circle. Every possible shape of a right triangle is represented by that quarter arc.

## Right triangles, circles, and radians III




## Right triangles, circles, and radians IV

Degrees to radians

| Degrees | 0 | 45 | 90 | 135 | 180 | 225 | 270 | 315 | 360 |
|---|---|---|---|---|---|---|---|---|---|
| Radians | 0 | $\pi/4$ | $\pi/2$ | $3\pi/4$ | $\pi$ | $5\pi/4$ | $3\pi/2$ | $7\pi/4$ | $2\pi$ |

## Right triangles, circles, and radians V

`pi` in `R` is reserved for the number $\pi$ so cannot be used for a name of an object you create.

Calculating values to create graphs of the trig functions by 45° increments:

```
theta=c(0,(1/4)*pi,(2/4)*pi,(3/4)*pi,pi,(5/4)*pi,(6/4)*pi,(7/4)*pi,2*pi)
sin(theta)
```

```
[1]  0.000000e+00  7.071068e-01  1.000000e+00  7.071068e-01  1.224647e-16
[6] -7.071068e-01 -1.000000e+00 -7.071068e-01 -2.449294e-16
```

```
cos(theta)
```

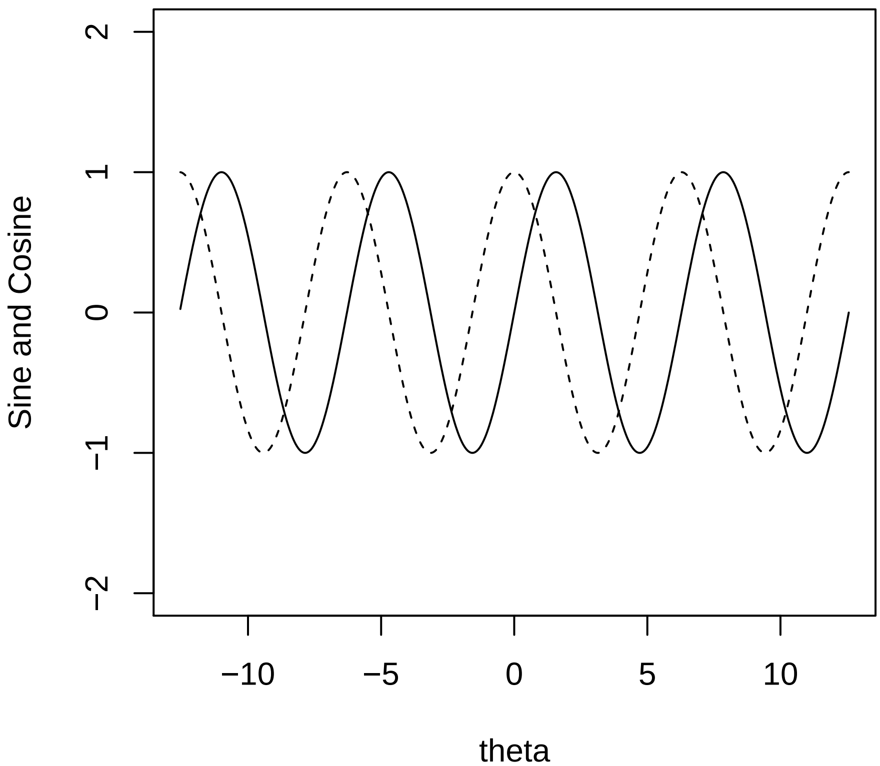
```
[1]  1.000000e+00  7.071068e-01  6.123234e-17 -7.071068e-01 -1.000000e+00
[6] -7.071068e-01 -1.836970e-16  7.071068e-01  1.000000e+00
```

```
tan(theta)
```

```
[1]  0.000000e+00  1.000000e+00  1.633124e+16 -1.000000e+00 -1.224647e-16
[6]  1.000000e+00  5.443746e+15 -1.000000e+00 -2.449294e-16
```

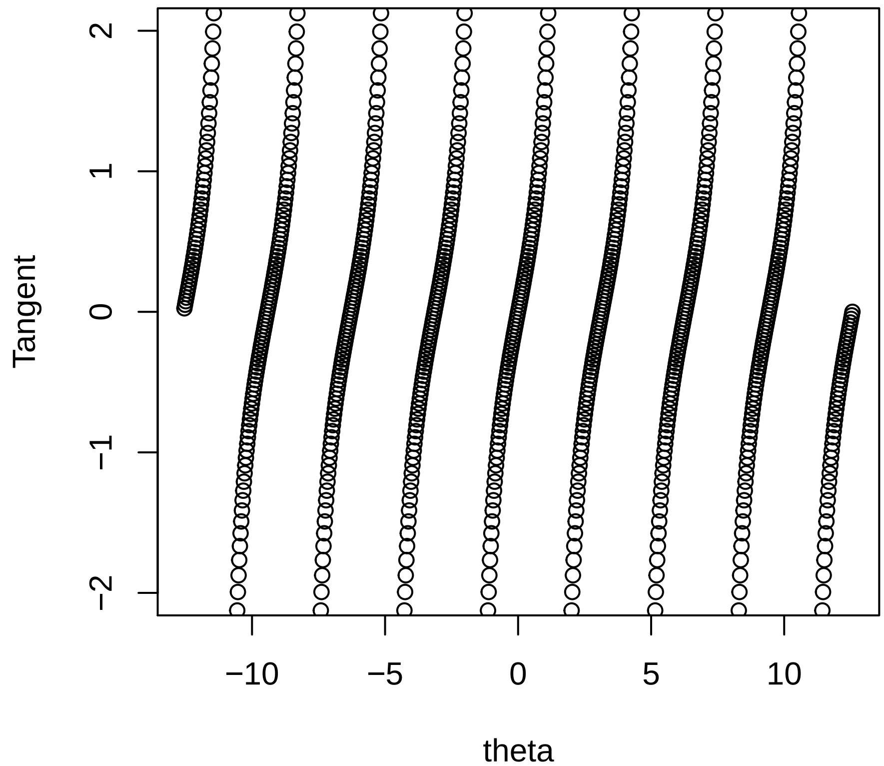
### Right triangles, circles, and radians VI

```
th.lo=-4*pi; th.hi=4*pi
theta=th.lo+(th.hi-th.lo)*(1:1000)/1000
y1=sin(theta); y2=cos(theta)
plot(theta,y1,type='l',ylab='Sine and Cosine',ylim=c(-2,2))
points(theta,y2,type='l',lty=2)
```




### Right triangles, circles, and radians VII

```
th.lo=-4*pi; th.hi=4*pi
theta=th.lo+(th.hi-th.lo)*(1:1000)/1000
y=tan(theta)
plot(theta,y,ylab='Tangent',ylim=c(-2,2))
```




### Properties of trigonometric functions I

From definitions of trigonometric functions as ratios, we can see various simple relationships among the functions.

$$\sin\theta = \frac{1}{\csc\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\vdots$$

### Properties of trigonometric functions II

Using Pythagorean theorem we can see that the equation of a circle is

$$x^2 + y^2 = r^2$$

the set of all points $(x, y)$ that satisfy the equation constitute a circle of radius $r$ centered at the origin. Divide both sides by $r^2$ to get

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

and by definition of sine and cosine,

$$(\sin\theta)^2 + (\cos\theta)^2 = 1$$

## Polar coordinates I

In Cartesian coordinates, each point on a plane is represented by an ordered pair of real numbers $(x, y)$. Any such point can also be represented by a different ordered pair of real numbers comprising the distance $r$ from the origin and the measure $\theta$ of the angle between the positive horizontal axis and the line segment between the origin and $(x, y)$. The distance $r$ must be non-negative and the angle $\theta$ is between 0 and $2\pi$, inclusive.

The ordered numbers $(r, \theta)$ are called "polar coordinates" of the point $(x, y)$. Polar coordinates help simplify many mathematical derivatives, such as the planetary orbits in physics. Given that the polar coordinates $(r, \theta)$ of a point, one obtains the corresponding Cartesian coordinates by applying simple trigonometric calculations.

## Polar coordinates II

$$x = r \cos \theta$$

$$y = r \sin \theta$$

Going the other way from Cartesian to polar, use the Pythagorean theorem

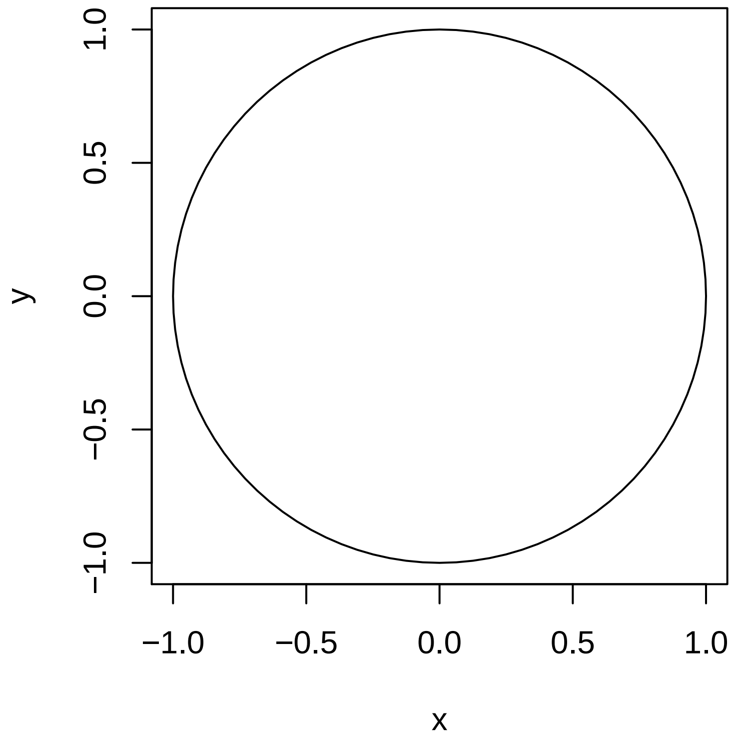
$$r = \sqrt{x^2 + y^2}$$

## Polar coordinates III

Getting $\theta$ from $x$ and $y$ using a formula is a bit more difficult, and will not be done here (it is a Calculus II or III concept). For this, a good use of polar coordinates is drawing circles and other forms that loop around some sort of center (kind of like a Spirograph). Here we will create a circle in `R` with radius $r$ centered at the origin, calculate a range of values of $\theta$ from 0 to $2\pi$ and calculate vectors of $x$ and $y$ values for plotting.

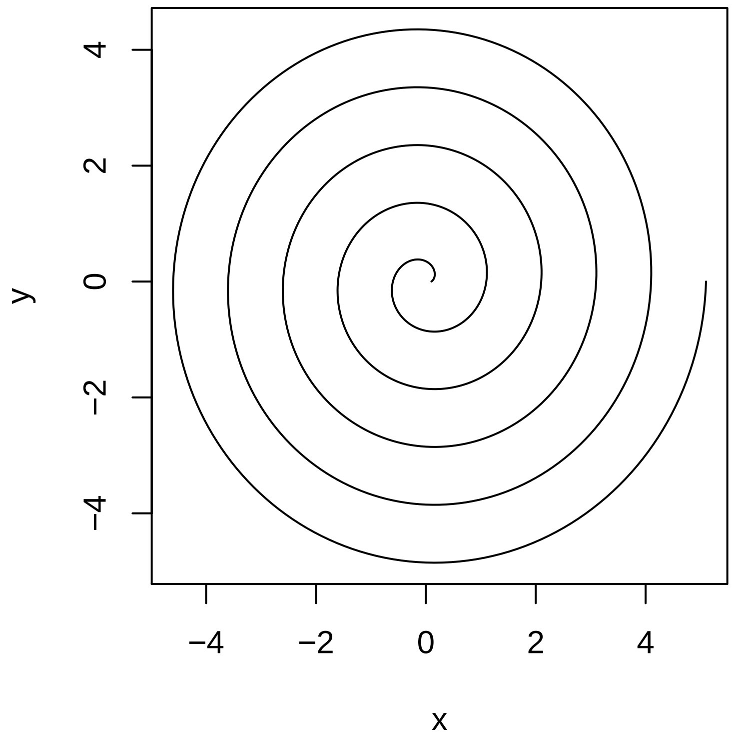
## Polar coordinates IV

```r
theta=2*pi*(0:100)/100; r=1
x=r*cos(theta); y=r*sin(theta)
par(pin=c(3,3)); plot(x,y,type='l')
```




### Polar coordinates V

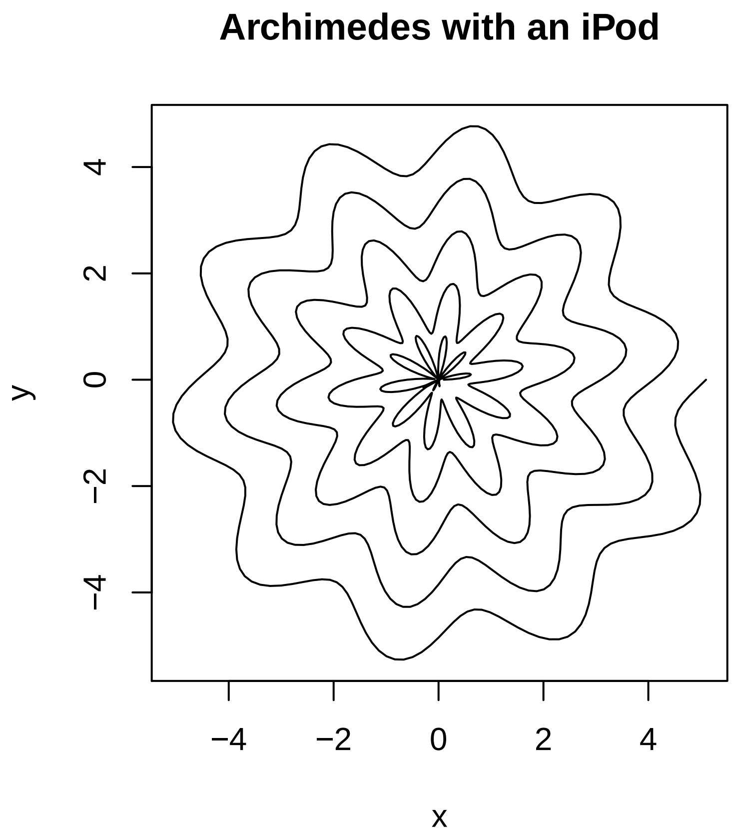
```
theta=10*pi*(0:1000)/1000
r=.1+theta/(2*pi)
x=r*cos(theta); y=r*sin(theta)
par(pin=c(3,3)); plot(x,y,type='l')
```



### Polar coordinates VI

There are all sorts of pretty designs that will come about with polar coordinates

```
theta=10*pi*(0:1000)/1000
r=.1+theta/(2*pi)+.5*sin(10*theta)
x=r*cos(theta); y=r*sin(theta)
par(pin=c(3,3)); plot(x,y,type='l',main='Archimedes with an iPod')
```




### Exponents I

Remember that the square root function $\sqrt{x}$ often represented as $x^{1/2}$, the cube root function $\sqrt[3]{x}$ represented as $x^{1/3}$, and so on. Similar to $x^1x^1 = x^{1+1} = x^2$, we can write $x^{1/2}x^{1/2} = x^{(1/2)+(1/2)} = x$. Further, $x^{2/3}$ can be interpreted as either taking the cube root of $x$ and squaring the result or squaring $x$ and then taking the cube root of the result: $x^{2/3} = (x^{1/3})^2 = (x^2)^{1/3}$.

### Exponents II

The definition is "raising $x$ to a rational power $m/n$" or $x^{m/n}$, where $m$ and $n$ are positive integers and $x$ is a positive real number, as the $n$th root of $x^m$ or, equivalently, $x^{1/n}$ raised to the $m$th power. Additionally, an exponent with a negative sign denotes reciprocal, $x^{-m/n} = \frac{1}{x^{m/n}}$.

The caret (^) is the exponentiation symbol in `R` such that the object left of the caret is the base and the object right of the caret is the exponent (the power).

### Exponents III

```
x=10
sqrt(x)
```

```
[1] 3.162278
```

```
x^(1/2)
```

```
[1] 3.162278
```

```
x^2*x^2
```

```
[1] 10000
```

```
x^(3/2)
```

```
[1] 31.62278
```

```
(x^3)^(1/2)
```

```
[1] 31.62278
```

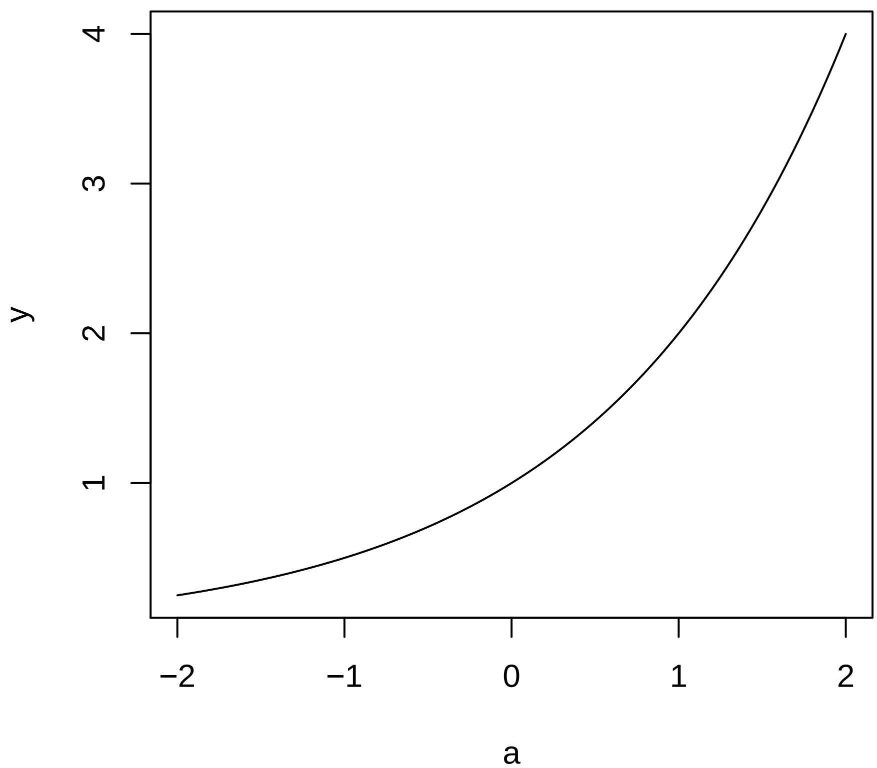
```
(x^(0:6))^(1/2)
```

```
[1]    1.000000    3.162278   10.000000   31.622777  100.000000  316.227766
[7] 1000.000000
```

### Exponents IV

If $a$ is a rational number, we can create a graph of $x^a$ for a range of values of $a$, fixing the value of $x$

```
a=4*(0:100)/100-2 # range of values of a from -2 to 2
x=2; y=x^a
plot(a,y,type='l')
```



## Exponents V

Raising $x$ to a real power obeys all the algebraic exponent laws learned for integer powers

$$x^0 = 1$$

$$x^1 = x$$

$$x^{-u} = \frac{1}{x^u}$$

$$x^u x^v = x^{(u+v)}$$

$$(x^u)^v = x^{uv}$$

## Exponents VI

```
0^0
```

```
[1] 1
```

```
pi
```

```
[1] 3.141593
```

```
x=2
x^pi
```

```
[1] 8.824978
```

```
(x^pi)*(x^pi)
```

```
[1] 77.88023
```

```
x^(pi+pi)
```

```
[1] 77.88023
```

```
x^sqrt(x)
```

```
[1] 2.665144
```

## Number $e$ I

The special (natural) number $e$ is derived from the following power function

$$y = \left(1 + \frac{1}{x}\right)^x$$

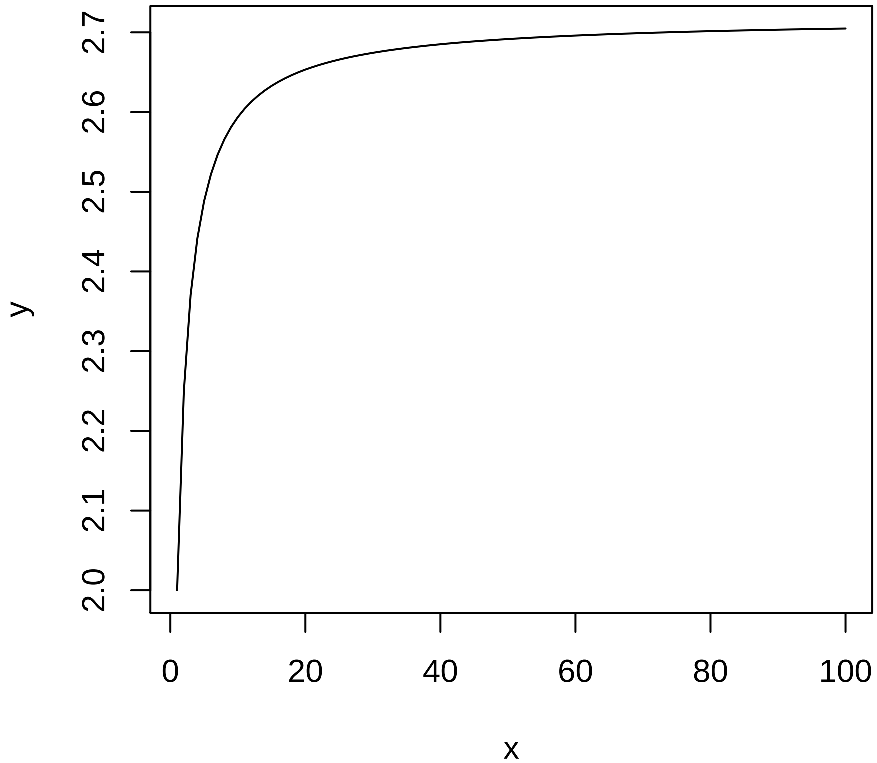
as $x$ approaches infinity ($x \to \infty$). When the value of $x$ is large, the term in $1/x$ inside the parentheses is small and the whole quantity in parentheses is close to the value of 1, but also being raised to a power of larger and larger values of $x$.

## Number $e$ II

As $x$ gets larger, $y$ approaches the value $2.71\ldots$, in fact it is:

$$e = 2.71828\ldots$$

```
x=1:100; y=(1+1/x)^x
plot(x,y,type='l')
```




## Exponential function I

The exponential function is defined as:

$$y = e^x$$

where $x$ is any real number. The resulting value of $y$ is a positive real number between 0 and 1 when $x$ is negative and greater than 1 when $x$ is positive. When $x$ is represented as a complicated calculation or function involving other quantities, is is often the following notation that is used as it is easier to read.

$$y = exp(x)$$

Where $x$ is the exponent and is any real number.

## Exponential function II

General form of `exp(x)` where `x` is a quantity or vector

```
x=1
exp(x)
```

```
[1] 2.718282
```

```r
x=0:5; exp(x)
```

```
[1]   1.000000   2.718282   7.389056  20.085537  54.598150 148.413159
```

```r
x=-5:0; exp(x)
```

```
[1] 0.006737947 0.018315639 0.049787068 0.135335283 0.367879441 1.000000000
```

## Exponential growth I

The interest rate example done in a previous module used an *annual* interest rate $r$. If we invest \$1 at an annual interest rate $r$ compounded *continuously* will yield $e^r$ dollars after one year. After 2 years, there would be $e^r e^r = e^{2r}$ dollars, after 3 years there would be $e^r e^r e^r = e^{3r}$ dollars, and so on. Eventually after $t$ years the initial \$1 would become $(e^r)^t = e^{rt}$ dollars.

## Exponential growth II

Using the same function we used for interest in Module 1 (for plotting), we will modify it slightly to reflect the continuous compounding of interest.

The original formula for compounding annually (and modified for monthly in our previous example):

$$P(t) = P_0(1 + r)^t$$

$P(t)$ is the final amount, $P_0$ is the initial investment, $r$ is the interest rate adjusted for period compounding, and $t$ is the time period.

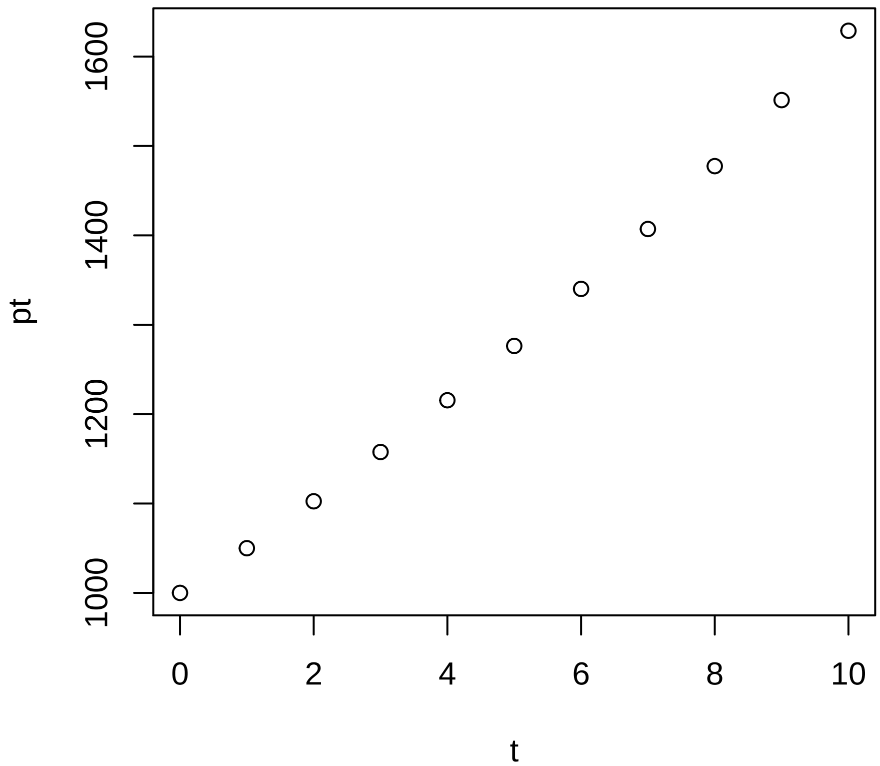
## Exponential growth III

Module 1 example:

```r
t=0:10; p0=1000; r=.05
pt=p0*(1+r)^t; pt
```

```
 [1] 1000.000 1050.000 1102.500 1157.625 1215.506 1276.282 1340.096 1407.100
 [9] 1477.455 1551.328 1628.895
```

```r
plot(t,pt,type='p')
```




### Exponential growth IV

Let's adjust for compounding continuously. $r$ will be divided by $n$ (the number of periods for compounding; i.e. 12 months in a year for monthly, etc.) and $t$ will be multiplied by $n$ and , which will look like:

$$(1 + r/n)^{nt} \approx \left(1 + \frac{1}{x}\right)^x = e^x$$

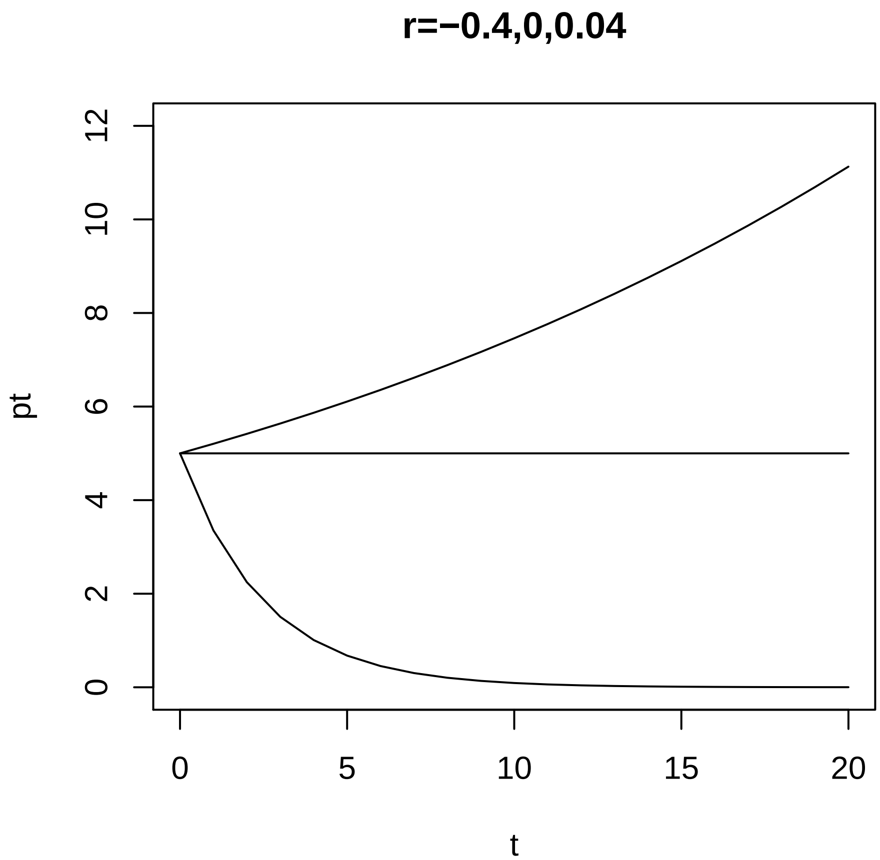
Substitute $e^{rt}$ for $(1 + r)^t$ and

$$P(t) = P_0 e^{rt}$$

### Exponential growth V

Now an example with R; example contains plot of exponential growth for positive, zero, and negative growth rates (negative growth rates are called decay rates). Use initial "investment" of 5 ($P_0$), values of time ($t$) from 0 to 20, and rates ($r$) of 0.04, 0, and -0.04.

```
p0=5; t=0:20; r1=0.04
pt=p0*exp(r1*t)
plot(t,pt,type='l',ylim=c(0,12)); title("r=-0.4,0,0.04")
r2=0; pt=p0*exp(r2*t)
points(t,pt,type='l')
r3=-0.4; pt=p0*exp(r3*t)
points(t,pt,type='l')
```




### Relationship between exponents and logarithms

The exponential function $y = e^x$ is a function of $x$ that is always increasing as $x$ increases. If we imagine flipping it over the $x$-axis, we now have the *logarithmic function*. Whatever power to which you raise $e$ to get $y$, that power is the "natural logarithm" of $y$ and it is written as follows:

$$x = \log(y)$$

One way of writing the logarithm definition is:

$$y = e^{\log(y)}$$

### Logarithmic function I

Logarithms were invented over 400 years ago as a way to reduce multiplication and division tasks to addition by taking advantage of the "adding exponents" rule. If $y = e^u$ and $z = e^v$ then $yz = e^u e^y = e^{(u+v)}$. Evidently $u + v = \log(yz)$; but because $u = \log(y)$ and $v = \log(z)$, the logarithm of a product is the sum of the logarithms of the numbers multiplied:

$$\log(yz) = \log(y) + \log(z)$$

Also $y/z = e^u/e^v = e^{(u-v)}$, so the logarithm of a quotient is the difference between the logarithms of the numbers undergoing division:

$$\log(y/z) = \log(y) - \log(z)$$

## Logarithmic function II

Any positive number besides $e$ can serve as the base for a system of logarithms. Logarithms with a base of 10 are taught in many precalculus classes and are encountered in some science and engineering applications (like the definition of $pH$ in chemistry and the Richter earthquake magnitude scale in geology).

We denote "log to the base 10" of a positive real number $y$ as $\log_{10}(y)$, with the following:

$$y = 10^{log_{10}(y)}$$

The natural logarithm is denoted as $ln$. In `R`, the function `log()` is the natural log. Other logs with differing bases like log base 10 is `log10()`, log base 2 is `log2()`, and so on.

## Logarithmic function III

```r
w=0; log(w)
```

```
[1] -Inf
```

```r
w=1:10; log(w)
```

```
 [1] 0.0000000 0.6931472 1.0986123 1.3862944 1.6094379 1.7917595 1.9459101
 [8] 2.0794415 2.1972246 2.3025851
```

```r
w=10000; log10(w)
```

```
[1] 4
```

```r
w=16; log2(w)
```

```
[1] 4
```

## Logarithm scales

In science, some phenomena are measured for convenience on a logarithmic scale. A logarithmic scale might be used for a quantity that has an enormous range of values or that varies multiplicatively. One common use is in the Richter scale that measures earthquake magnitudes.

The word magnitude should clue us in on the use of logarithms. Richter magnitude is defined as the base 10 logarithm of the amplitude of the quake waves recorded by a seismograph (amplitude is the distance of departures of the seismograph needle from its central reference point). Each whole number increase in magnitude represents a quake with waves measuring 10 times greater. A magnitude 6 has waves measuring 10 times greater than those of magnitude 5.